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INTERPOLATION OPERATORS ON A TRIANGLE WITH ALL CURVED SIDES

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Abstract: *This paper contains a survey regarding interpolation and Cheney-Sharma type operators defined on a triangle with all curved sides; we considers as well some of the product and Boolean sum operators. We study their interpolation properties and the degree of exactness.*

Keywords: interpolation operators, triangle with curved sides, Cheney-Sharma type operators, products and Boolean sum operators

1. Introduction

The aim of this survey is to present some interpolation and Cheney-Sharma type operators for functions defined on a triangle with all curved side (see [1], [2]). They came as an extension of the corresponding operators for functions defined on triangles with all straight sides (see, e.g.,[3], [4], [9]).

We study these operators especially from the theoretical point of view.

We study two main aspects of the constructed operators: the interpolation properties and the degree of exactness.

Recall that dex(P)=r (where P is an interpolation operator) if P $f=f$, for $f\in P_r$, and there exists $g\in P_{r+1}$ such that $Pg\neq g$, where P_m denote the space of the polynomials in two variables of global degree at most m.

In Section 2 we study Lagrange, Hermite and Birkhoff interpolation operators and in Section 3 we present some Cheney-Sharma type operators together with their product and Boolean sum for the triangle with all curved sides.

Given $h>0$, denote by $\widetilde{T}_h$ the triangle having the vertices $V_1=(0,h), V_2=(h,0)$ and $V_3=(0,0)$, and the three curved sides γ_1,γ_2 (along the coordinates axes) and γ_3 (opposite to the vertex V_3). We define γ_1 by $(x,f_1(x))$ with $f_1(0)=f_1(h)=0, f_1(x)\leq 0,$ for $x\in[0,h]$; γ_2 defined by $(g_2(y),y)$, with $g_2(0)=g_2(h)=0, g_2(y)\leq 0,$ for $y\in[0,h]$ and γ_3 defined by one-to-one functions f_3 and g_3, where g_3 is the inverse of the function f_3, i.e., $y=f_3(x)$ and $x=g_3(y)$, with $x,y\in[0,h]$ and $f_3(0)=g_3(0)=h$, $h\in R_+$, (see Figure 1).

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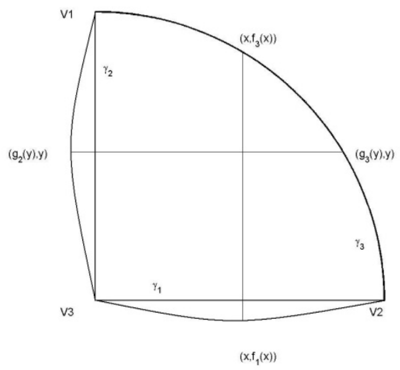

Figure 1. Triangle $\widetilde{T}_h$

2. Interpolation operators

2.1. Lagrange-type operators

Suppose that F is a real-valued function defined on $\widetilde{T}_h$. Let L_1 and L_2 be the interpolation operators defined by

$$(L_1F)(x,y) = \frac{x-g_3(y)}{g_2(y)-g_3(y)}F(g_2(y),y) + \frac{x-g_2(y)}{g_3(y)-g_2(y)}F(g_3(y),y) \tag{1}$$

$$(L_2F)(x,y) = \frac{y-f_3(x)}{f_1(x)-f_3(x)}F(x,f_1(x)) + \frac{y-f_1(x)}{f_3(x)-f_1(x)}F(x,f_3(x)) \tag{2}$$

Theorem 1.([1]) If $F:\widetilde{T}_h \to R$, then we get
(1) the interpolation properties: $L_1F = F$, on $\gamma_2 \cup \gamma_3$, $L_2F = F$, on $\gamma_1 \cup \gamma_3$.
(2) the degree of exactness: $\mathrm{dex}(L_i)=1$, $i=\overline{1,2}$.

Proof. (1) $(L_1F)(g_2(y),y) = F(g_2(y),y)$,
$(L_1F)(g_3(y),y) = F(g_3(y),y)$,
$(L_2F)(x,f_1(x)) = F(x,f_1(x))$,
$(L_2F)(x,f_3(x)) = F(x,f_3(x))$.
So, the interpolation properties are verified.
(2) $L_1e_{ij} = e_{ij}$ for $i,j \in N, i,j \le 1$ and $L_1e_{20} \ne e_{20}$, where $e_{ij}(x,y) = x^iy^j$. So. It follows that $\mathrm{dex}(L_i)=1$.

2.2. Hermite-type operators

Suppose that F has the partial derivatives $F^{(1,0)}$ on γ_2 and γ_3 respectively $F^{(0,1)}$ on γ_1 and γ_3. We consider the operators H_1 and H_2 defined by

$$\begin{aligned}(H_1F)(x,y) &= \frac{[x-g_3(y)]^2[3g_2(y)-g_3(y)-2x]}{[g_2(y)-g_3(y)]^3}\cdot F(g_2(y),y) + \frac{[x-g_2(y)]^2[3g_3(y)-g_2(y)-2x]}{[g_3(y)-g_2(y)]^3}\cdot \\ &F(g_3(y),y) + \frac{[x-g_2(y)][x-g_3(y)]^2}{[g_2(y)-g_3(y)]^2}\cdot F^{(1,0)}(g_2(y),y) + \frac{[x-g_3(y)][x-g_2(y)]^2}{[g_3(y)-g_2(y)]^2}\cdot F^{(1,0)}(g_3(y),y)\end{aligned} \tag{3}$$

and

$$(H_2F)(x,y)=\frac{[y-f_3(x)]^2[3f_1(x)-f_3(x)-2y]}{[f_1(x)-f_3(x)]^3}\cdot$$
$$F(x,f_1(x))+\frac{[y-f_1(x)]^2[3f_3(x)-f_1(x)-2y]}{[f_3(x)-f_1(x)]^3}\cdot$$
$$F(x,f_3(x))+\frac{[y-f_1(x)][y-f_3(x)]^2}{[f_1(x)-f_3(x)]^2}\cdot$$
$$F^{(0,1)}(x,f_1(x))+\frac{[y-f_3(x)][y-f_1(x)]^2}{[f_3(x)-f_1(x)]^2}\cdot \qquad (4)$$
$$F^{(0,1)}(x,f_3(x))$$

Theorem 2.([1]) If $F:\widetilde{T}_h\to R$, then we get (1) the interpolation properties: $H_1F=F$, on $\gamma_2\cup\gamma_3$, $H_1F^{(1,0)}=F^{(1,0)}$, on $\gamma_2\cup\gamma_3$, $H_2F=F$, on $\gamma_1\cup\gamma_3$, $H_2F^{(0,1)}=F^{(0,1)}$, on $\gamma_1\cup\gamma_3$.
(2) the degree of exactness:
$\mathrm{dex}(H_1)=\mathrm{dex}(H_2)=2$.

Proof. (1) $(H_1F)(g_2(y),y)=F(g_2(y),y)$,
$(H_1F)(g_3(y),y)=F(g_3(y),y)$,

$$(H_1F)^{(1,0)}(x,y)=$$
$$\frac{6[x-g_3(y)][g_2(y)-x]}{[g_2(y)-g_3(y)]^3}\cdot F(g_2(y),y)$$
$$+\frac{6[x-g_2(y)][g_3(y)-x]}{[g_3(y)-g_2(y)]^3}\cdot F(g_3(y),y)$$
$$+\frac{[x-g_3(y)][3x-2g_2(y)-g_3(y)]}{[g_2(y)-g_3(y)]^2}\cdot$$
$$F^{(1,0)}(g_2(y),y)$$
$$+\frac{[x-g_2(y)][3x-2g_3(y)-g_2(y)]}{[g_3(y)-g_2(y)]^2}\cdot$$
$$F^{(1,0)}(g_3(y),y).$$

We have:
$(H_1F)^{(1,0)}(g_2(y),y)=F^{(1,0)}(g_2(y),y)$,
$(H_1F)^{(1,0)}(g_3(y),y)=F^{(1,0)}(g_3(y),y)$.
Also for the interpolation properties of H_2.

(2) We obtain $H_1e_{ij}=e_{ij}$ for $i,j<2$ and $H_1e_{30}\neq e_{30}$, where $e_{ij}(x,y)=x^iy^j$. So, it follows that $\mathrm{dex}(H_1)=2$. Similar for $\mathrm{dex}(H_2)=2$.

2.2. Birkhoff-type operators

We give some examples of operators which interpolate the given function $F:\widetilde{T}_h\to R$ on a side of triangle, respectively, its partial derivatives on the others side.
We suppose that the function $F:\widetilde{T}_h\to R$ has the partial derivatives $F^{(0,1)}$ on γ_3 and γ_1 and $F^{(1,0)}$ on γ_3.
We consider the Birkhoff-type operators B_1 and B_2 defined by

$$(B_1F)(x,y)=F(g_2(y),y)+(x-g_2(y))F^{(1,0)}(g_3(y),y) \qquad (5)$$

$$(B_2F)(x,y)=F(x,f_1(x))+(y-f_1(x))F^{(0,1)}(x,f_3(x)) \qquad (6)$$

Theorem 3.([1]) If $F:\widetilde{T}_h\to R$, then we get (1) the interpolation properties: $B_1F=F$ on γ_2, $(B_1F)^{(1,0)}=F^{(1,0)}$ on γ_3, $B_2F=F$ on γ_1, $(B_2F)^{(0,1)}=F^{(0,1)}$ on γ_3.
(2) the degree of exactness:
$\mathrm{dex}(B_1)=\mathrm{dex}(B_2)=1$.

Proof. (1) $(B_1F)(g_2(y),y)=F(g_2(y),y)$,
$(B_1F)^{(1,0)}(g_3(y),y)=F^{(1,0)}(g_3(y),y)$,
$(B_1F)(x,f_1(x))=F(x,f_1(x))$,
$(B_1F)^{(0,1)}(x,f_3(x))=F^{(0,1)}(x,f_3(x))$.
(2) $B_1e_{ij}=e_{ij}$ for $i,j\le 1$ and $B_1e_{20}\neq e_{20}$, where $e_{ij}(x,y)=x^iy^j$.

So, it follows that $\mathrm{dex}(B_1)=1$. Similar for $\mathrm{dex}(B_2)=1$.

3. Cheney-Sharma type operators

The Cheney-Sharma type operators on a triangle with curved sides are extension of the Cheney-Sharma type operators of second type, given by E.W.Cheney and A.Sharma in [6].

Let $m \in N$ and β a nonnegative parameter. The Cheney-Sharma operators of second kind $Q_m : C([0,1]) \to C([0,1])$, introduced in [6], are given by

$$(Q_m f)(x) = \sum_{i=0}^{m} q_{m,i}(x) f\left(\frac{i}{m}\right), \tag{7}$$

$$q_{m,j}(x) = \binom{m}{i} \frac{x(x+i\beta)^{i-1}(1-x)[1-x+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}}$$

For $m,n \in N, \alpha, \beta \in R_+$, we consider the following extensions of the Cheney-Sharma operator given in (7):

Let F be a real-valued function defined on $\widetilde{T}_h$ and $(x, f_1(x)), (x, f_3(x))$, respectively, $(g_2(y), y), (g_3(y), y)$ the points in which the parallel lines to the coordinates axes, passing through the point $(x,y) \in \widetilde{T}_h$, intersect the sides γ_1, γ_2 and γ_3 (see Figure 1). We consider the uniform partitions of the intervals $[g_2(y), g_3(y)]$ and $[f_1(x), f_3(x)], x, y \in [0,h]$:

$$\Delta_m^x = \left\{ g_2(y) + i\frac{g_3(y)-g_2(y)}{m} \middle| i = \overline{0,m} \right\},$$

respectively,

$$\Delta_n^y = \left\{ f_1(x) + j\frac{f_3(x)-f_1(x)}{n} \middle| j = \overline{0,n} \right\}.$$

$$q_{n,j}(x,y) = \binom{n}{j} \frac{\dfrac{y-f_1(x)}{f_3(x)-f_1(x)}\left(\dfrac{y-f_1(x)}{f_3(x)-f_1(x)} + j\alpha\right)}{(1+n\alpha)^{n-1}}$$

$$(Q^x{}_m F)(x) = \sum_{i=0}^{m} q_{m,i}(x,y) F\left(g_2(y) + i\frac{g_3(y)-g_2(y)}{m}, y \right),$$

$$q_{m,i}(x,y) = \binom{m}{i} \frac{\dfrac{x-g_2(y)}{g_3(y)-g_2(y)}\left(\dfrac{x-g_2(y)}{g_3(y)-g_2(y)} + i\beta\right)}{(1+m\beta)^{m-1}} \cdot$$

$$\left[1 - \frac{x-g_2(y)}{g_3(y)-g_2(y)}\right] \cdot \left[1 - \frac{x-g_2(y)}{g_3(y)-g_2(y)} + (m-i)\beta\right]^{m-i-1}$$

respectively,

$$(Q^y{}_n F)(x) = \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, f_1(x) + j\frac{f_3(x)-f_1(x)}{n} \right),$$

with

$$\cdot \left[1 - \frac{y-f_1(x)}{f_3(x)-f_1(x)}\right]\left[1 - \frac{y-f_1(x)}{f_3(x)-f_1(x)} + (n-j)\alpha\right]^{n-j-1}.$$

Theorem 4.([2]) If F is a real-valued function defined on $\widetilde{T}_h$ then

(1) $Q_m^x F = F$ on $\gamma_2 \cup \gamma_3$,

(2) $Q_n^y F = F$ on $\gamma_1 \cup \gamma_3$,

(3) $(Q_n^y e_{ij})(x,y) = x^i y^j, i = 0,1; j \in N$,

(4) $(Q_n^y e_{ij})(x,y) = x^i y^j, i \in N, j = 0,1$.

Proof. 1) We write

$$(Q_m^x F)(x,y) = \frac{1}{(1+m\beta)^{m-1}}\Bigg\{\left[1-\frac{x-g_2(y)}{g_3(y)-g_2(y)}\right]\cdot \left[1-\frac{x-g_2(y)}{g_3(y)-g_2(y)}+m\beta\right]^{m-1}\cdot F(g_2(y),y)$$
$$+\frac{x-g_2(y)}{g_3(y)-g_2(y)}\left[1-\frac{x-g_2(y)}{g_3(y)-g_2(y)}\right]\cdot \sum_{i=0}^{m}\binom{m}{i}\left(\frac{x-g_2(y)}{g_3(y)-g_2(y)}+i\beta\right)^{i-1}\cdot \left[1-\frac{x-g_2(y)}{g_3(y)-g_2(y)}+(m-i)\beta\right]^{m-i-1}\cdot F\left(g_2(y)+i\frac{g_3(y)-g_2(y)}{m},y\right)$$
$$+\frac{x-g_2(y)}{g_3(y)-g_2(y)}\cdot\left[\frac{x-g_2(y)}{g_3(y)-g_2(y)}+m\beta\right]^{m-1}\cdot F(g_3(y),y)\Bigg\}$$

So,

$$(Q_m^x F)(g_2(y),y) = F(g_2(y),y),$$
$$(Q_m^x F)(g_3(y),y) = F(g_3(y),y).$$

The 2) is proved in a similar way with 1). The proof for 3) and 4) follows by the property $dex(Q_m)=1$ (proved in [6]).

Let $P_{mn}^1 = Q_m^x Q_n^y$, respectively, $P_{nm}^2 = Q_n^y Q_m^x$ be the product of the operators Q_m^x and Q_n^y. We have

$$\left(P_{mn}^1 F\right)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}(x_i,y)\cdot F\left(x_i, f_i(x_i)+j\frac{f_3(x_i)-f_1(x_i)}{n}\right),$$
$$x_i = g_2(y)+i\frac{g_3(y)-g_2(y)}{m},$$

and

$$\left(P_{nm}^2 F\right)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y_j)q_{n,j}(x,y)\cdot F\left(g_2(y_j)+i\frac{g_3(y_j)-g_2(y_j)}{m}, y_j\right),$$
$$y_j = f_1(y)+j\frac{f_3(x)-f_1(x)}{n}.$$

Theorem 5.([2]) If F is a real-valued function defined on $\widetilde{T}_h$ then:

1) $(P_{mn}^1\, F)(V_3) = F(V_3),\ (P_{mn}^1\, F) = F,$ on Γ_3
2) $(P_{nm}^2\, F)(V_3) = F(V_3),\ (P_{nm}^2\, F) = F,$ on Γ_3

Proof. The proof follows from the properties:

$$(P_{mn}^1\, F)(x,0) = (Q_m^x\, F)(x,0),$$
$$(P_{mn}^1\, F)(0,y) = (Q_n^y\, F)(0,y),$$
$$(P_{mn}^1\, F)(x,f_3(x)) = F(x,f_3(x)), x,y\in[0,h]$$

and

$$(P_{nm}^2\, F)(x,0) = (Q_m^x\, F)(x,0),$$
$$(P_{nm}^2\, F)(0,y) = (Q_n^y\, F)(0,y),$$
$$(P_{nm}^2\, F)(g_3(x),y) = F(g_3(y),y), x,y\in[0,h]$$

which can be verified by a straightforward computation.

We consider the Boolean sums of the operators Q_m^x and Q_n^y, i.e.,

$$S_{mn}^1 = Q_m^x \oplus Q_n^y = Q_m^x + Q_n^y - Q_m^x Q_n^y,$$

respectively

$$S_{nm}^2 = Q_n^y \oplus Q_m^x = Q_n^y + Q_m^x - Q_n^y Q_m^x$$

Theorem 6.([2]) If F is a real-valued function defined on $\widetilde{T}_h$ then:

$$S_{mn}^1\Big|_{\partial\widetilde{T}} = F\Big|_{\partial\widetilde{T}}$$

and

$$S^2_{nm}\big|_{\partial\tilde{T}} = F\big|_{\partial\tilde{T}}.$$

Proof.

As

$$(S^1_{mn}F)(x, f_1(x)) = (Q^x_m F)(x, f_1(x)),$$

$$(S^1_{mn}F)(g_2(y), y) = (Q^y_n F)(g_2(y), y),$$

$$(S^1_{mn}F)(x, f_3(x)) = F(x, f_3(x)),$$

the proof follows.

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