DETERMINING A PARTICULAR SOLUTION FOR THE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract: *As with the n-th order linear differential equations with constant coefficients, the problem to be solved is related to determining a particular solution, and then, using the general solution of the attached homogeneous system of linear differential equations with constant coefficients, to write the general solution of the initially given system. For homogeneous systems of linear differential equations with constant coefficients, the determination of the general solution is the method of eliminating or reducing which make the system a linear differential equation of the same order as that of the system, and its methods of solving it applies or the method of own values and vectors. If the system is non-homogeneous, then we also have to determine a particular solution that can be done in the same way as in the case of n-th order differential equations with constant coefficients, if the method of reduction or elimination was used, or the method of variation of constants, regardless of the method used to determine the general solution of the attached homogenous system of linear differential equations with constant coefficients. Whichever method is used, determining a particular solution for a system of linear differential equations with constant coefficients is difficult, in this study being proposed a method similar to that of n-th order linear differential equations with constant coefficients.*

Keywords: systems of linear differential equations with constant coefficients, homogeneous systems of linear differential equations with constant coefficients, particular solution, general solution

1. Introduction

This study presents a simple and practical method to determine a particular solution for a system of linear differential equations with constant coefficients, and starts from the observation of how you can get a particular solution to an *n*-th order linear differential equation with constant coefficients.

The free terms of the system can be part of the same class of functions that the free term can be part of for the *n*-th order linear differential equations with constant coefficients, namely a combination such as:

$$P(x) + P_1(x) \cdot e^{\alpha_1 \cdot x} + P_2(x) \cdot e^{\alpha_2 \cdot x} + ... + P_n(x) \cdot e^{\alpha_n \cdot x} +$$
$$+ e^{\omega_1 \cdot x} \cdot (Q_{11}(x) \cdot cos(\beta_1 \cdot x) + Q_{12}(x) \cdot sin(\beta_1 \cdot x)) +$$
$$+ e^{\omega_2 \cdot x} \cdot (Q_{21}(x) \cdot cos(\beta_2 \cdot x) + Q_{22}(x) \cdot sin(\beta_2 \cdot x)) + ... +$$
$$+ e^{\omega_m \cdot x} \cdot (Q_{m1}(x) \cdot cos(\beta_m \cdot x) + Q_{m2}(x) \cdot sin(\beta_m \cdot x)) +$$
$$+ Q_1(x) \cdot cos(\gamma_1 \cdot x) + Q_2(x) \cdot cos(\gamma_2 \cdot x) + ... + Q_k(x) \cdot cos(\gamma_k \cdot x) +$$
$$+ R_1(x) \cdot sin(\delta_1 \cdot x) + R_2(x) \cdot sin(\delta_2 \cdot x) + ... + R_l(x) \cdot sin(\delta_l \cdot x) +$$
$$+ e^{\varepsilon_1 \cdot x} S_1(x) \cdot cos(\lambda_1 \cdot x) + e^{\varepsilon_2 \cdot x} S_2(x) \cdot cos(\lambda_2 \cdot x) + ... + e^{\varepsilon_r \cdot x} S_r(x) \cdot cos(\lambda_r \cdot x) +$$
$$+ e^{\rho_1 \cdot x} T_1(x) \cdot sin(\theta_1 \cdot x) + e^{\rho_2 \cdot x} T_2(x) \cdot sin(\theta_2 \cdot x) + ... + e^{\rho_s \cdot x} T_s(x) \cdot sin(\theta_s \cdot x)$$

$$(1)$$

where $\beta_i \neq \gamma_j$, $(\forall) i \in \overline{1,m}$ and $j \in \overline{1,k}$,

$\beta_i \neq \delta_j$, $(\forall) i \in \overline{1,m}$ and $j \in \overline{1,l}$,

$\beta_i \neq \lambda_j$, $(\forall) i \in \overline{1,m}$ and $j \in \overline{1,r}$,

$\beta_i \neq \theta_j$, $(\forall) i \in \overline{1,m}$ and $j \in \overline{1,s}$,

$\gamma_i \neq \delta_j$, $(\forall) i \in \overline{1,k}$ and $j \in \overline{1,l}$,

$\gamma_i \neq \lambda_j$, $(\forall) i \in \overline{1,k}$ and $j \in \overline{1,r}$,

$\gamma_i \neq \theta_j$, $(\forall) i \in \overline{1,k}$ and $j \in \overline{1,s}$,

$\delta_i \neq \lambda_j$, $(\forall) i \in \overline{1,l}$ and $j \in \overline{1,r}$,

$\delta_i \neq \theta_j$, $(\forall) i \in \overline{1,l}$ and $j \in \overline{1,s}$,

$\lambda_i \neq \theta_j$, $(\forall) i \in \overline{1,r}$ and $j \in \overline{1,s}$,

and P, P_i, $i \in \overline{1,n}$, Q_{ij}, $i \in \overline{1,m}$ and $j \in \overline{1,2}$, Q_i, $i \in \overline{1,k}$, S_i, $i \in \overline{1,r}$, T_i, $i \in \overline{1,s}$, are polynomials.

In the same way it can be shown that, without restricting the generality degree, the free terms of the system can be in the form:

$$P_1(x) \cdot e^{\alpha_1 \cdot x} + P_2(x) \cdot e^{\alpha_2 \cdot x} + ... + P_n(x) \cdot e^{\alpha_n \cdot x} +$$
$$+ e^{\gamma_1 \cdot x} \cdot (Q_1(x) \cdot cos(\beta_1 \cdot x) + R_1(x) \cdot sin(\beta_1 \cdot x)) +$$
$$+ e^{\gamma_2 \cdot x} \cdot (Q_2(x) \cdot cos(\beta_2 \cdot x) + R_2(x) \cdot sin(\beta_2 \cdot x)) + ... +$$
$$+ e^{\gamma_m \cdot x} \cdot (Q_m(x) \cdot cos(\beta_m \cdot x) + R_m(x) \cdot sin(\beta_m \cdot x))$$

$$(2)$$

This will be the form used for the free terms of the system we want to determine a particular solution to.

2. Description of the problem

We will consider that we have a system of linear differential equations with constant coefficients of the form

$$\begin{cases} y_1'(x) = a_{11} \cdot y_1(x) + a_{12} \cdot y_2(x) + ... + a_{1n} \cdot y_n(x) + f_1(x) \\ y_2'(x) = a_{21} \cdot y_1(x) + a_{22} \cdot y_2(x) + ... + a_{2n} \cdot y_n(x) + f_2(x) \\ \vdots \\ y_n'(x) = a_{n1} \cdot y_1(x) + a_{n2} \cdot y_2(x) + ... + a_{nn} \cdot y_n(x) + f_n(x) \end{cases}$$

$$(3)$$

where $x \in I \subset \mathbf{R}$ is an interval of $\mathbf{R}$, and f_i, $i \in \overline{1,n}$, are functions like (2).

The attached homogeneous system of linear differential equations with constant coefficients will be like

$$\begin{cases} y_1'(x) = a_{11} \cdot y_1(x) + a_{12} \cdot y_2(x) + ... + a_{1n} \cdot y_n(x) \\ y_2'(x) = a_{21} \cdot y_1(x) + a_{22} \cdot y_2(x) + ... + a_{2n} \cdot y_n(x) \\ \vdots \\ y_n'(x) = a_{n1} \cdot y_1(x) + a_{n2} \cdot y_2(x) + ... + a_{nn} \cdot y_n(x) \end{cases}, \qquad (4)$$

where $x \in I \subset \mathbf{R}$ is an interval of $\mathbf{R}$.

The particular solution of the system of linear differential equations with constant coefficients (3), in which free terms $f_i(x)$, $i \in \overline{1,n}$ are functions in the form of (2) is constructed like the that of the n-th order linear differential equations with constant coefficients.

3. The form of the particular solution and its determination exemplification

As in the case of n-th order linear differential equations with constant coefficients, the construction of a particular solution is based on the characteristic equation attached to the homogeneous system (4).

The characteristic equation attached to the homogeneous system of linear differential equations with constant coefficients (4) is of the form:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} - \lambda & ... & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & ... & a_{nn} - \lambda \end{vmatrix} = 0. \qquad (5)$$

We will assume that it has the following expansion:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} - \lambda & ... & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & ... & a_{nn} - \lambda \end{vmatrix} = \begin{matrix} (-1)^n \cdot (\lambda - a_1)^{n_1} \cdot (\lambda - a_2)^{n_2} \cdot ... \cdot (\lambda - a_q)^{n_q} \cdot \\ \cdot [(\lambda - (b_1 + c_1 \cdot i)) \cdot (\lambda - (b_1 - c_1 \cdot i))]^{m_1} \cdot \\ \cdot [(\lambda - (b_2 + c_2 \cdot i)) \cdot (\lambda - (b_2 - c_2 \cdot i))]^{m_2} \cdot ... \cdot \\ \cdot [(\lambda - (b_p + c_p \cdot i)) \cdot (\lambda - (b_p - c_p \cdot i))]^{m_p} \end{matrix}, \qquad (6)$$

where $n_1 + n_2 + ... + n_q + 2 \cdot m_1 + 2 \cdot m_2 + ... + 2 \cdot m_p = n$. In other words, we will assume that the characteristic equation (6) attached to the system (4) has q real and distinct solutions with multiplicity orders $n_1, n_2, ...,$ respectively n_q and $2 \cdot p$ distinct conjugate complex solutions with multiplicity orders $m_1, m_2, ...,$ respectively m_p.

We will assume that the free terms $f_i(x)$, $i \in \overline{1,n}$ are functions in the form of

$$f_i(x) = \begin{matrix} P_{i1}(x) \cdot e^{\alpha_{i1} \cdot x} + P_{i2}(x) \cdot e^{\alpha_{i2} \cdot x} + ... + P_{ik_i}(x) \cdot e^{\alpha_{ik_i} \cdot x} + \\ + e^{\gamma_{i1} \cdot x} \cdot (Q_{i1}(x) \cdot cos(\beta_{i1} \cdot x) + R_{i1}(x) \cdot sin(\beta_{i1} \cdot x)) + \\ + e^{\gamma_{i2} \cdot x} \cdot (Q_{i2}(x) \cdot cos(\beta_{i2} \cdot x) + R_{i2}(x) \cdot sin(\beta_{i2} \cdot x)) + ... + \\ + e^{\gamma_{il_i} \cdot x} \cdot (Q_{il_i}(x) \cdot cos(\beta_{il_i} \cdot x) + R_{il_i}(x) \cdot sin(\beta_{il_i} \cdot x)) \end{matrix}, \qquad (7)$$

where k_i and l_i, $i \in \overline{1,n}$ are natural numbers (positive integers), α_{ij}, $i \in \overline{1,n}$ and $j \in \overline{1,k_i}$ and β_{ij}, γ_{ij}, $i \in \overline{1,n}$ and $j \in \overline{1,r_i}$ are real numbers, and P_{ij}, $i \in \overline{1,n}$ and $j \in \overline{1,k_i}$ and Q_{ij}, R_{ij}, $i \in \overline{1,n}$ and $j \in \overline{1,r_i}$ are polynomials with real coefficients.

We also assume that $\alpha_{i_1 j_1}$, $\alpha_{i_1 j_2}$, ..., $\alpha_{i_1 j_{r_1}}$, $i_a \in \overline{1,n}$ and $j_a \in \overline{1,k_{i_a}}$, with $a \in \overline{1,r}$, and

$\gamma_{p_1 q_1} + \beta_{p_1 q_1} \cdot i$, $\quad \gamma_{p_2 q_2} + \beta_{p_2 q_2} \cdot i$, $\quad$ ...,

$\gamma_{p_s q_s} + \beta_{p_s q_s} \cdot i$, $\quad p_b \in \overline{1,n}$ and $\quad q_b \in \overline{1,l_{p_b}}$,

with $b \in \overline{1,s}$ are among the solutions of the

characteristic equation (5) which has the expansion (6).

Under these circumstances, a particular solution of the linear differential equation system with constant coefficients (3), where the free terms $f_i(x)$, $i \in \overline{1,n}$, are like (7) will be as

$$y_{ip}(x) = +\begin{aligned}&\sum_{i\in\overline{1,n}\setminus\{i_1,i_2,...,i_r\}}\sum_{j\in\overline{1,k_i}}S_{ij}(x)\cdot e^{\alpha_{ij}\cdot x} + \sum_{a\in\overline{1,r}}\left(S_{i_a j_a}(x)\cdot x^{n_a}\cdot e^{\alpha_{i_a j_a}\cdot x} + \sum_{j\in\overline{1,k_a}\setminus\{j_a / a\in\overline{1,r}\}}S_{i_a j}(x)\cdot e^{\alpha_{i_k j_k}\cdot x}\right)+\\&+\sum_{i\in\overline{1,n}\setminus\{p_1,p_2,...,p_l\}}\sum_{j\in\overline{1,l_i}}e^{\gamma_{ij}\cdot x}\cdot\left[U_{ij}(x)\cdot \cos(\beta_{ij}\cdot x)+V_{ij}(x)\cdot \sin(\beta_{ij}\cdot x)\right]+\\&+\sum_{b\in\overline{1,s}}\Big\{x^{n_b}\cdot e^{\gamma_{p_b q_b}\cdot x}\cdot\left[U_{p_b q_b}(x)\cdot \cos(\beta_{p_b q_b}\cdot x)+V_{p_b q_b}(x)\cdot \sin(\beta_{p_b q_b}\cdot x)\right]+\\&+\sum_{j\in\overline{1,l_b}\setminus\{q_b / b\in\overline{1,s}\}}e^{\gamma_{p_b q_b}\cdot x}\cdot\left[U_{p_b q_b}(x)\cdot \cos(\beta_{p_b q_b}\cdot x)+V_{p_b q_b}(x)\cdot \sin(\beta_{p_b q_b}\cdot x)\right]\Big\}\end{aligned}, \quad i \in \overline{1,n} \quad (7)$$

where S_{ij}, $i \in \overline{1,n}$ and $j \in \overline{1,k_i}$ and U_{ij}, V_{ij}, $i \in \overline{1,n}$ and $j \in \overline{1,l_i}$ are polynomials with degree S_{ij} = degree P_{ij}, $i \in \overline{1,n}$ and $j \in \overline{1,k_i}$ and degree U_{ij} = degree V_{ij} = max {degree Q_{ij}, degree R_{ij}}, $i \in \overline{1,n}$ and $j \in \overline{1,l_i}$, and n_a is the multiplicity order of the radical $\alpha_{i_a j_a}$, $a \in \overline{1,r}$ and n_b is the multiplicity order of the radical $\gamma_{p_b q_b} + \beta_{p_b q_b} \cdot i$, $b \in \overline{1,s}$.

The form of the solution (7) considers the form of functions $f_i(x)$, $i \in \overline{1,n}$ and the form of the characteristic equation (6).

Further on, we will illustrate the way in which the previously described method should be applied.

Let there be the linear differential equation system with constant coefficients

$$\begin{cases} y_1'(x) = & -y_2(x) + \cos x \\ y_2'(x) = y_1(x) + & \sin 2\cdot x \end{cases} \quad (8)$$

We will first determine the solutions of the attached characteristic equation:

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0. \quad (9)$$

The solutions of these equations are $\lambda_1, \lambda_2 = \pm i$. Given the form of the constant terms and radicals of the characteristic equation, we will search for a particular solution such as

$$\begin{cases} y_{1p}(x) = C_1 \cdot \sin x + C_2 \cdot \cos x + C_3 \cdot x \cdot \sin x + C_4 \cdot x \cdot \cos x + C_5 \cdot \sin 2\cdot x + C_6 \cdot \cos 2\cdot x \\ y_{2p}(x) = C_7 \cdot \sin x + C_8 \cdot \cos x + C_9 \cdot x \cdot \sin x + C_{10} \cdot x \cdot \cos x + C_{11} \cdot \sin 2\cdot x + C_{12} \cdot \cos 2\cdot x \end{cases}$$

We have:

$$\begin{cases} y_{1p}'(x) = C_1 \cdot \cos x - C_2 \cdot \sin x + C_3 \cdot \sin x + C_3 \cdot x \cdot \cos x + C_4 \cdot \cos x - C_4 \cdot x \cdot \sin x + \\ \quad + 2\cdot C_5 \cdot \cos 2\cdot x - 2\cdot C_6 \cdot \sin 2\cdot x \\ y_{2p}'(x) = C_7 \cdot \cos x - C_8 \cdot \sin x + C_9 \cdot \sin x + C_9 \cdot x \cdot \cos x + C_{10} \cdot \cos x - C_{10} \cdot x \cdot \sin x + \\ \quad + 2\cdot C_{11} \cdot \cos 2\cdot x - 2\cdot C_{12} \cdot \sin 2\cdot x \end{cases}$$

Introducing the functions y_{1p} and y_{2p} and their derivatives in the system (8) we get:

$$\begin{cases} C_1 \cdot \cos x - C_2 \cdot \sin x + C_3 \cdot \sin x + C_3 \cdot x \cdot \cos x + C_4 \cdot \cos x - C_4 \cdot x \cdot \sin x + 2 \cdot C_5 \cdot \cos 2 \cdot x - 2 \cdot C_6 \cdot \sin 2 \cdot x = \\ = -(C_7 \cdot \sin x + C_8 \cdot \cos x + C_9 \cdot x \cdot \sin x + C_{10} \cdot x \cdot \cos x + C_{11} \cdot \sin 2 \cdot x + C_{12} \cdot \cos 2 \cdot x) + \cos x \\ C_7 \cdot \cos x - C_8 \cdot \sin x + C_9 \cdot \sin x + C_9 \cdot x \cdot \cos x + C_{10} \cdot \cos x - C_{10} \cdot x \cdot \sin x + 2 \cdot C_{11} \cdot \cos 2 \cdot x - 2 \cdot C_{12} \cdot \sin 2 \cdot x = \\ = C_1 \cdot \sin x + C_2 \cdot \cos x + C_3 \cdot x \cdot \sin x + C_4 \cdot x \cdot \cos x + C_5 \cdot \sin 2 \cdot x + C_6 \cdot \cos 2 \cdot x + \sin 2 \cdot x \end{cases}$$

By identifying the coefficients, we obtain the follwing system of linear equations:

$$\begin{cases} C_1 + C_4 = -C_8 + 1 \\ -C_2 + C_3 = -C_7 \\ C_3 = -C_{10} \end{cases}, \begin{cases} -C_4 = -C_9 \\ 2 \cdot C_5 = -C_{12} \\ -2 \cdot C_6 = -C_{11} \end{cases}, \begin{cases} C_7 + C_{10} = C_2 \\ -C_8 + C_9 = C_1 \\ C_9 = C_4 \end{cases} \text{ and } \begin{cases} -C_{10} = C_3 \\ 2 \cdot C_{11} = C_6 \\ -2 \cdot C_{12} = C_5 + 1 \end{cases}.$$

The solution of the above system is:

$$\begin{cases} C_1 = 1/2 \\ C_2 = C_7 = 0 \\ C_3 = C_{10} = 0 \\ C_4 = C_9 = 1/2 \end{cases} \text{ and } \begin{cases} C_5 = 1/3 \\ C_6 = C_{11} = 0 \\ C_8 = 0 \\ C_{12} = -2/3 \end{cases}.$$

In conclusion, a particular solution of the given system (8) is

$$\begin{cases} y_{1p}(x) = 1/2 \cdot \sin x + 1/2 \cdot x \cdot \cos x + 1/3 \cdot \sin 2 \cdot x \\ y_{2p}(x) = 1/2 \cdot x \cdot \sin x - 2/3 \cdot \cos 2 \cdot x \end{cases}.$$

After determining the particular solution, we may write the general solution of the given system.

Further on, we will assume that we have a system of linear differential equations with constant coefficients such as:

$$\begin{cases} y_1'(x) = \quad\quad - y_2(x) + e^{2 \cdot x} \cdot (2 \cdot x + 1) \\ y_2'(x) = y_1(x) + \quad\quad e^{3 \cdot x} \cdot (4 \cdot x^2 - 3 \cdot x + 2) \end{cases}. \quad (10)$$

Given that system (10) has the same characteristic equation (9) and that $\lambda = 2$ and $\lambda = 3$ are not its solutions, we will try to find a particular solution of the system, such as:

$$\begin{cases} y_{1p}(x) = e^{2 \cdot x} \cdot (A_1 \cdot x + B_1) + e^{3 \cdot x} \cdot (C_1 \cdot x^2 + D_1 \cdot x + E_1) \\ y_{2p}(x) = e^{2 \cdot x} \cdot (A_2 \cdot x + B_2) + e^{3 \cdot x} \cdot (C_2 \cdot x^2 + D_2 \cdot x + E_2) \end{cases}.$$

We have:

$$\begin{cases} y_{1p}'(x) = e^{2 \cdot x} \cdot (2 \cdot A_1 \cdot x + A_1 + 2 \cdot B_1) + e^{3 \cdot x} \cdot [3 \cdot C_1 \cdot x^2 + (2 \cdot C_1 + 3 \cdot D_1) \cdot x + D_1 + 3 \cdot E_1] \\ y_{2p}'(x) = e^{2 \cdot x} \cdot (2 \cdot A_2 \cdot x + A_2 + 2 \cdot B_2) + e^{3 \cdot x} \cdot [3 \cdot C_2 \cdot x^2 + (2 \cdot C_2 + 3 \cdot D_2) \cdot x + D_2 + 3 \cdot E_2] \end{cases}.$$

Introducing functions y_{1p} and y_{2p} and their derivatives in the system (10) we get:

$$\begin{cases} e^{2 \cdot x} \cdot (2 \cdot A_1 \cdot x + A_1 + 2 \cdot B_1) + e^{3 \cdot x} \cdot [3 \cdot C_1 \cdot x^2 + (2 \cdot C_1 + 3 \cdot D_1) \cdot x + D_1 + 3 \cdot E_1] = \\ = -[e^{2 \cdot x} \cdot (A_2 \cdot x + B_2) + e^{3 \cdot x} \cdot (C_2 \cdot x^2 + D_2 \cdot x + E_2)] + e^{2 \cdot x} \cdot (2 \cdot x + 1) \\ e^{2 \cdot x} \cdot (2 \cdot A_2 \cdot x + A_2 + 2 \cdot B_2) + e^{3 \cdot x} \cdot [3 \cdot C_2 \cdot x^2 + (2 \cdot C_2 + 3 \cdot D_2) \cdot x + D_2 + 3 \cdot E_2] = \\ = e^{2 \cdot x} \cdot (A_1 \cdot x + B_1) + e^{3 \cdot x} \cdot (C_1 \cdot x^2 + D_1 \cdot x + E_1) + e^{3 \cdot x} \cdot (4 \cdot x^2 - 3 \cdot x + 2) \end{cases}$$

By identifying the coefficients, we obtain the following linear equation system:

$$\begin{cases} 2\cdot A_1 = -A_2 + 2 \\ A_1 + 2\cdot B_1 = -B_2 + 1 \\ 3\cdot C_1 = -C_2 \\ 2\cdot C_1 + 3\cdot D_1 = -D_2 \\ D_1 + 3\cdot E_1 = -E_2 \end{cases} \text{and} \begin{cases} 2\cdot A_2 = A_1 \\ A_2 + 2\cdot B_2 = B_1 \\ 3\cdot C_2 = C_1 + 4 \\ 2\cdot C_2 + 3\cdot D_2 = D_1 - 3 \\ D_2 + 3\cdot E_2 = E_1 + 2 \end{cases}.$$

The solution of the above system is:

$$\begin{cases} A_1 = 4/5 \\ A_2 = 2/5 \end{cases}, \begin{cases} B_1 = 4/25 \\ B_2 = -3/25 \end{cases}, \begin{cases} C_1 = -2/5 \\ C_2 = 6/5 \end{cases}, \begin{cases} D_1 = 39/50 \\ D_2 = -77/50 \end{cases} \text{and} \begin{cases} E_1 = -294/500 \\ E_2 = 492/500 \end{cases}.$$

In conclusion, a particular solution of the given system (10) is

$$\begin{cases} y_{1p}(x) = (4/5\cdot x + 4/25)\cdot e^{2\cdot x} + (-2/5\cdot x^2 + 39/50\cdot x - 294/500)\cdot e^{3\cdot x} \\ y_{2p}(x) = (2/5\cdot x - 3/25)\cdot e^{2\cdot x} + (6/5\cdot x^2 - 77/50\cdot x + 492/500)\cdot e^{3\cdot x} \end{cases}.$$

After determining the particular solution, we may write the general solution of the given system.

Finally, let us assume that we now have the following system of linear differential equations with constant coefficients:

$$\begin{cases} y_1'(x) = \qquad -y_2(x) + e^x\cdot(2\cdot x + 1) \\ y_2'(x) = y_1(x) + \qquad 4\cdot x^2 - 3\cdot x + 2 \end{cases}. \quad (11)$$

Since the system (11) has the same characteristic equation (9) and that $\lambda = 0$ and $\lambda = 1$ are not its solutions, we will try to find a particular solution of the system in the form of:

$$\begin{cases} y_{1p}(x) = e^x\cdot(A_1\cdot x + B_1) + C_1\cdot x^2 + D_1\cdot x + E_1 \\ y_{2p}(x) = e^x\cdot(A_2\cdot x + B_2) + C_2\cdot x^2 + D_2\cdot x + E_2 \end{cases}.$$

We have:

$$\begin{cases} y_{1p}'(x) = e^x\cdot(A_1\cdot x + A_1 + B_1) + 2\cdot C_1\cdot x + D_1 \\ y_{2p}'(x) = e^x\cdot(A_2\cdot x + A_2 + B_2) + 2\cdot C_2\cdot x + D_2 \end{cases}.$$

Introducing the functions y_{1p} and y_{2p} and their derivatives in the system (11) we get:

$$\begin{cases} e^x\cdot(A_1\cdot x + A_1 + B_1) + 2\cdot C_1\cdot x + D_1 = -\left[e^x\cdot(A_2\cdot x + B_2) + C_2\cdot x^2 + D_2\cdot x + E_2\right] + e^x\cdot(2\cdot x + 1) \\ e^x\cdot(A_2\cdot x + A_2 + B_2) + 2\cdot C_2\cdot x + D_2 = e^x\cdot(A_1\cdot x + B_1) + C_1\cdot x^2 + D_1\cdot x + E_1 + 4\cdot x^2 - 3\cdot x + 2 \end{cases}$$

By identifying the coefficients, we obtain the following linear equations system:

$$\begin{cases} A_1 = -A_2 + 2 \\ A_1 + B_1 = -B_2 + 1 \\ 2\cdot C_1 = -D_2 \\ -C_2 = 0 \\ D_1 = -E_2 \end{cases} \text{and} \begin{cases} A_2 = A_1 \\ A_2 + B_2 = B_1 \\ 2\cdot C_2 = D_1 - 3 \\ D_2 = E_1 + 2 \\ C_1 + 4 = 0 \end{cases}.$$

The solution of the above system is:

$$\begin{cases} A_1 = 1 \\ A_2 = 1 \\ B_1 = 1/2 \\ B_2 = -1/2 \\ C_1 = -4 \end{cases} \text{and} \begin{cases} C_2 = 0 \\ D_1 = 3 \\ D_2 = 8 \\ E_1 = 6 \\ E_2 = -3 \end{cases}.$$

In conclusion, a particular solution of the given system (10) is

$$\begin{cases} y_{1p}(x) = e^x\cdot(x + 1/2) - 4\cdot x^2 + 3\cdot x + 6 \\ y_{2p}(x) = e^x\cdot(x - 1/2) + 8\cdot x - 3 \end{cases}.$$

After determining the particular solution, we may write the general solution of the given system.

Conclusions and disscussions

Determining a particular solution for a system of linear differential equations with constant coefficients like (3), having the free terms $f_i(x)$, $i \in \overline{1,n}$, such as (2), can be done generally, regardless of the form of the characteristic equation (5), as we have seen in the case of n-th order linear differential equations with constant coefficients.

It is a simple and efficient method for determining a particular solution without neither having to determine all solutions to the characteristic equation - in the case of the variation of constants method, nor a series of calculations that involve successive elimination as in the case of the cancellation method to solving n-th order linear differential equations with constant coefficients.

Moreover, for the variation of constants method, determining all solutions of the characteristic equation is generally very complicated, and determining a particular solution additionally requires replacing the general solution in the system and then determining the functions generated by constants, which involves, besides solving an n-th order system, integrating n functions. The method presented actually builds the particular solution and ultimately resumes only to determining the solution of a system of linear equations.