Linear stability analysis reveals exclusion zone for sliding bed transport

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Abstract: A bend or any another pipe component disturbs solids transport in pipes. Longitudinal pressure profiles downstream of such a component may show a stationary transient harmonic wave, as revealed by a recent settling slurry laboratory experiment. Therefore the fundamental transient response of the two-layer model for fully stratified flow is investigated as a first approach. A linear stability analysis of the sliding bed configuration is conducted. No stationary transient harmonic waves are found in this analysis, but adaptation lengths for exponential recovery are quantified. An example calculation is given for a 0.1 m diameter pipeline.

Also consequences for long stretches of pipe line emerged. A so far undiscovered exclusion zone is found in the I-V diagram. This exclusion zone is situated adjacent to the deposit limit velocity locus curve. This simplified physical system reveals that flow velocities should be taken about 10% greater than the calculated maximum deposit limit velocity for stable converging flow.

Keywords: Stratification; Hydrotransport; Transients; Bed layer.

INTRODUCTION

It is good practice in hydraulic laboratories to situate test sections at a safe distance from upstream bends and inlets. For hydraulic transport of solids there are however virtually no guidelines on appropriate distance. Colwell and Shook (1988) mention a distance of 50D based on measurements in the pseudo-homogeneous hydrotransport regime. In the literature, on many occasions the distance to the inlet is not mentioned and authors assume that equilibrium conditions are being measured, i.e. the results are assumed independent of downstream position at their measuring section.

A bend or any other component disturbs the flow, and the flow has to revert again to an asymptotic equilibrium situation via transient development. This may lead to problems in the interpretation of existing data, the risk being that the flow is not fully developed, or worse, that it is being measured in the trough or at the crest of stationary harmonic waves. Pressure gradient measurements over adjacent intervals at a 0.1 meter diameter pipe, Matoušek and Krupička (2013), point at a possibility that stationary harmonic waves could be present. Motivated by the data presented in Matoušek and Krupička (2013), longitudinal pressure profile measurements commenced in a horizontal 0.04 m diameter pipe at Delft University of Technology, Figure 1. A straight pipe section situated downstream of a 180 degree r/D = 5 bend is equipped with a series of 9 pressure taps distributed over a distance of 3 m. The measured pressure profile and associated local pressure gradient profile are shown in Figure 2. These are 300 s averages, equivalent to about 20 mixture circulations. It shows that the pressure gradient varies harmonically with distance, and that within a distance of 70D no asymptotic equilibrium condition is reached. Silica sand is employed. The spatial volumetric solids concentration in the flow loop is 20%, the median grain size is 0.325 mm. The flow velocity is 1.5 m/s, whereas the deposit limit velocity is at about 1 m/s. For this condition the delivered concentration is 18% by volume.

Talmon (1999) drew attention to the availability of linear stability and normal mode analysis to investigate the dynamics of stratified slurry flow in pipes. In this case it is no longer assumed that the flow is independent of streamwise distance.

In the current paper the same technique is applied, but now transient stationary behaviour is analysed. A macroscopic system approach is followed. A simplified schematisation of the physics is applied and it is not ventured into detailed descriptions of physical processes at the interface or in the suspension, such as in for example Matoušek (2009), Krupička and Matoušek (2010), Matoušek and Krupička (2014), Matoušek et al. (2014), Kaushal et al. (2005) and Kaushal and Tomita (2013). In fact the analysis pertains to the very basic behaviour of the two-layer structure and is a first step in a process to develop theory for longitudinal development of stratified sand-water flows in pipes.

The methodology is to first quantify the asymptotic equilibrium condition and next to quantify transient behaviour towards this equilibrium. Externally imposed are the flow rate, equilibrium bed level and friction coefficients. In this case the velocities of bed and suspension are calculated explicitly via the solution of a quadratic equation in the flow velocity of the suspension U_{s} , a method similar to that described by Jones (2011). Transient development is described by the same equations, but now including advection and mass acceleration terms.



Fig. 1. Line drawing essential components of test set-up.



Fig. 2. Measured fluid pressures and associated pressure gradient of slurry as a function of distance from the upstream bend, flow velocity 1.5 m/s. Whiskers denote standard deviation of time series and are plotted at locations of upstream pressure taps.



Fig. 3. Definition sketch of a sliding bed layer in a two-layer schematisation, showing bed layer, suspension, flow velocities, shear stresses and fluid pressure profile.

The present analysis, which reveals an exclusion zone but not the sought harmonic variation, forms a basis for ongoing analysis considering more physical processes; suspension transport and mass-exchange between the bed layer and the suspension.

Governing equations

Assumptions in the present two-layer model (2-LM) for horizontal stratified solids transport are: - sliding bed regime, solids transport by sliding bed only, - uniform velocity distribution per layer, - vertical accelerations are neglected (1-D equations), - constant friction coefficients, as for fully developed uni-directional flow, - en-bloc sliding of the bed (no shear in bed layer), loose porous bed layer (constant porosity), - timeindependent.

The cross section of the 2-LM configuration is sketched in Figure 3. The overriding fluid flow is referred to as suspension to comply with past and future nomenclature when suspended load transport is considered. The total cross-sectional area (A) and the flow rate (Q) are constant with distance x:

$$A = A_b + A_s$$
, $Q = UA = A_b u_b + A_s u_s$ (1), (2)

Flow rates of bed and suspension do not vary along the length of the pipeline in the present analysis because there is no mass exchange through the bed-suspension interface:

$$A_b u_b = \text{constant}, \quad A_s u_s = \text{constant}$$
 (3), (4)

The bed layer is modelled as a granular layer subject to Coulomb grain friction, pore fluid friction, longitudinal hydraulic gradient, shear stresses exerted by the overlain fluid flow, mass acceleration and effects from a longitudinal bed level slope. The momentum balance of the bed layer is:

$$\rho_b A_b u_b \frac{\partial u_b}{\partial x} = -A_b \frac{\partial P}{\partial x} + \tau_{int} W - \tau_{bot} P_{bot}$$

$$-\mu_s \frac{F_n}{F_w} A_b (\rho_b - \rho_w) g + A_b (\rho_s - \rho_w) g \frac{\partial z_b}{\partial x}$$
(5)

The last term on the right hand side accounts for a driving force which originates from density differences between the suspended load layer and the pore water in the bed layer when the interface is non-horizontal. In this term the granular character of the bed layer is preserved: the grain skeleton transfers forces from bed level variations to the pipe wall. The density of the bed layer is: $\rho_b = n \rho_w + (1 - n) \rho_{sand}$. The momentum balance of the suspension is:

$$\rho_s A_s u_s \frac{\partial u_s}{\partial x} = -A_s \frac{\partial P}{\partial x} - \tau_{\text{int}} W - \tau_{top} P_{top}$$
(6)

Elimination of the fluid pressure gradient between both momentum equations gives:

$$\rho_{s}u_{s}\frac{\partial u_{s}}{\partial x} - \rho_{b}u_{b}\frac{\partial u_{b}}{\partial x} + \tau_{int}\left(\frac{W}{A_{s}} + \frac{W}{A_{b}}\right) + \tau_{top}\frac{P_{top}}{A_{s}}$$

$$-\tau_{bot}\frac{P_{bot}}{A_{b}} - \mu_{s}\frac{F_{n}}{F_{w}}(\rho_{b} - \rho_{w})g + (\rho_{s} - \rho_{w})g\frac{\partial z_{b}}{\partial x} = 0$$
(7)

The shear stresses are modelled by:

$$\tau_{top} = \frac{\lambda_{top}}{8} \rho_s u_s^2, \quad \tau_{int} = \frac{\lambda_{int}}{8} \rho_s (u_s - u_b)^2$$
(8), (9)

$$\tau_{bot} = \alpha_{bot} \rho_b u_b^2 \tag{10}$$

Linearization

Harmonic variations of the variables are substituted in the continuity equation (Eq. 2) and the combined momentum balance of both layers, Eq. (7):

$$u_b = U_b + u_b', \quad u_s = U_s + u_s'$$
 (11), (12)

where U_s and U_b , are the asymptotic mean velocities. The asymptotic mean velocities U_s and U_b , are determined first. Bed level height z_b is inputted and U_s and U_b are calculated by a 2-LM model, see Section A of the Supplementary material. The harmonic variations are modelled mathematically by the product

of an exponential function and a cosine function:

$$\left[\frac{u_{s}'}{U}, \frac{u_{b}'}{U}\right] = \left[\tilde{u}_{s}, \tilde{u}_{b}\right] e^{ikx} + complex \ conjugate \ (13)$$

This is a short-hand notation for:

$$\left[\frac{u_s'}{U}, \frac{u_b'}{U}\right] = \left[\tilde{u}_s, \tilde{u}_b \right] e^{-k_i x} \cos(k_r x)$$
(14)

where
$$k = k_{\rm r} + {\rm i}k_{\rm i}$$
 (15)

The real part k_r of the wave number k represents the wave number of harmonic variations. The imaginary part k_i of k indicates whether the perturbations amplify with downstream distance $(k_i < 0)$, or are damped $(k_i > 0)$. Differences in amplitude and phase of the harmonic waves are accommodated in the complex amplitudes \tilde{u}_s , \tilde{u}_b . Fundamental relations for differentiation are:

$$\frac{\partial u_s \,' U}{\partial x} = -ik \frac{u_s'}{U}, \quad \frac{\partial u_b \,' U}{\partial x} = -ik \frac{u_b'}{U}, \quad (16), (17)$$

The characteristic adaptation length of perturbations is $x_0 = 1/k_i$. Four typical responses are shown in Figure 4. Linearization of the equations and mathematical relations for calculating the wave number *k* are given in Section B and C of the Supplementary material.

CALCULATIONS Linear stability analysis (normal mode analysis)

A series of calculations is conducted where the bed height is fixed and the superficial velocity is varied. For each superficial velocity, the velocity of the bed layer and suspension layer for the equilibrium of the two-layer configuration are calculated first by the 2-LM model (Section A of the Supplementary material). Next the transient response according to the normal mode analysis is calculated. Parameter values of a typical calculation are listed in Table 1. The listed values for wall friction coefficients (λ_{top} , α_{bot} and μ_s) are more or less established values.



Fig. 4. Schematisation of typical responses to a perturbation: course of bed level height with distance.

Model parameters	Value
diameter pipe, D [m]	0.1
density sand [kg/m ³]	2650
density suspension [kg/m ³]	1000
density water [kg/m ³]	1000
porosity bed, n [-]	0.45
friction factor Darcy-Weisbach pipe wall, λ_{top} [–]	0.015
friction factor Darcy-Weisbach bed shear stress, λ_{int} [–]	0.07
Coulomb friction coefficient, μ_{s} [–]	0.4
friction pore fluid against pipe wall, $\alpha_{\rm bot}$ [-]	0.004

Table 1. Parameters in linear stability analysis.

The selected value for interface friction coefficient yields realistic deposit limit velocities in the 0.1 m diameter pipe. The calculated complex component of the wavenumber k is graphically displayed as a function of flow velocity in Figure 5. For the real part it is calculated $k_r = 0$ for all circumstances, indicating no harmonic sinusoidal wave development. The equilibrium calculation, also shown in Figure 5, gives a bed velocity U_b which increases approximately linearly with superficial flow velocity. The calculated deposit limit velocity is 1.9 m/s (at z/D = 0.2). The calculations show that the imaginary part of the complex wave number k_iD changes sign at about 2.1 m/s. For lower velocities, downstream amplifying perturbations are found (divergence). For higher velocities, downstream decaying perturbations are found (convergence).

I-V diagram

Wilson (1979) created an I-V diagram showing a deposit limit velocity locus curve (= a curve describing the edge of the stationary bed layer regime: $U_b = 0$). Hydraulic gradients are again calculated with the 2-LM equilibrium calculation method given in Section A of the Supplementary material, and the deposit limit velocity is reached when a bed velocity of $U_b = 0$ is calculated. Figure 6 shows the I-V diagram (bed layer only configuration, parameters according to Table 1). The hydraulic gradient *I* is calculated by considering the momentum balance of the upper layer:

$$I = \frac{A}{A_s} \frac{4}{\pi} \frac{\rho_s}{\rho_w} \frac{U^2}{gD} \left(\frac{P_{top}}{D} \frac{\lambda_{top}}{8} \left(\frac{U_s}{U} \right)^2 + \frac{W}{D} \frac{\lambda_{int}}{8} \left(\frac{U_s}{U} - \frac{U_b}{U} \right)^2 \right)$$
(18)

The linear stability analysis shows that there exists a region of amplifying diverging solutions, see Figure 6. This region is bounded by a marginal stability curve ($k_i = 0$) and the deposit limit curve. Curves of iso-decay rates are inscribed in this I-V diagram. Indicated are conditions with characteristic lengths quantifying 63% decay at $x_0 = 20D$ and 50D ($k_iD = 0.05, 0.02$). A graphical representation of these transient responses is shown in Figure 7. The type of analysis earlier depicted in Figure 5 is along one curve of z/D = constant. From Figure 6 it is concluded that for example at U = 2.25 m/s and z/D = 0.2, 63% adaptation of the two-layer configuration is reached at a distance of 20 pipe diameters downstream of the bend (20D). An adaptation of 95% is than reached at x = 40D.

Multi-valued delivered concentration near deposit limit velocity

A common approach is to quantify the hydraulic gradient as a function of flow velocity and delivered concentration. Figure 8 shows that close to the deposit limit velocity, multiple bed heights are possible at a given delivered solids concentration. The ones that are situated in the amplification region ($k_i < 0$) are unattainable, and the solution will diverge with respect to the calculated asymptotic equilibrium.

Without resorting to linear stability analysis, the presence of the exclusion zone can also be inferred in hindsight from a series of equilibrium calculations with a standard 2-LM model. The boundary condition is the delivered concentration entering the pipe. If the sliding bed conditions are outside the exclusion zone, an increase in bed level leads to an increase in solids transport rate (greater C_{vd} , see Figure 8). In that case more solids are discharged from the pipe than are entering, and as a consequence the bed level will drop again. This is a self-stabilising



Fig. 5. Outcome of stability calculations at a relative bed level of z/D = 0.2: flow velocity of the bed layer U_b at asymptotic equilibrium and calculated complex wave number k_i of perturbations indicating amplification, resonance or damping.



Fig. 6. Curves of iso-adaptation lengths inscribed in an I-V diagram quantifying the hydraulic gradient and deposit limit velocity as a function of flow velocity for a pipe line diameter of 0.1 m (for further parameters: Table 1).



Fig. 7. Exponential adaptation length profiles of overdamped system. Characteristic 63% and 95% adaptation lengths for $k_i D = 0.02$ and 0.05 are indicated.



Fig. 8. Calculated volumetric delivered concentration C_{vd} for sliding bed layer transport mode. Down sloping branches (divergence) are situated in the exclusion zone of Figure 6.

mechanism. If on the other hand, the conditions are in the exclusion zone, an increase in bed level at the exit of the pipe leads to a lowering of the exiting solids transport rate. The amount of solids in the pipe increases, increasing the bed level further. This is an unstable situation.

The delivered concentrations shown in Figure 8 are low because bed velocities are small under these conditions. In reality there will be a sheet flow layer at the bed surface providing the majority of solids transport, and the delivered concentration will be significantly higher.

As a consequence of the presence of this exclusion zone, it is not unreasonable to design transportation systems at a velocity about 10% higher than the calculated maximum V_{dl} . This corroborates with a 10% safety margin for deposit velocity which is applied in case studies presented in Matoušek (2004). This stems from experience with laboratory tests; the specific energy consumption is low and the flow is stable at velocities slightly above the deposition-limit velocity, Matoušek (2014).

CONCLUSION

In a quest to quantify adaption lengths of two-layer fully stratified flow, an exclusion zone is discovered where no downstream converging 2-LM solution can develop. This zone is situated adjacent to the deposit limit velocity locus curve.

For stable conditions outside this exclusion zone, adaptation lengths for the two-layer structure are quantified. Adaptation lengths shorten with increasing superficial flow velocity.

As a consequence of the presence of this exclusion zone, it is not unreasonable to design transportation systems at a velocity about 10% higher than the calculated maximum V_{dl} .

Similar analyses including more physical processes are being conducted to capture the mechanism of wave development.

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Nomenclature

A =cross-sectional area of pipe,

 $A_{\rm b}$ = cross-sectional area of bed layer,

 $A_{\rm s}$ = cross-sectional area of suspension,

 $C_{\rm vd}$ = volumetric delivered solids concentration,

D = pipe diameter,

 Fr_b = densimetric Froude number of bed layer,

- $F_{\rm n}$ = normal granular force of bed on pipe wall,
- $F_{\rm w}$ = immersed weight of bed layer,

g = gravity,

I = hydraulic gradient,

 $i = \sqrt{-1}$,

 $k = k_r + i k_i =$ complex wave number,

n = porosity of bed layer,

- P = pressure at the crest of the pipe line's cross-section,
- $P_{\rm bot}$ = circumferential contact length of bed layer and pipe,
- P_{top} = circumferential contact length of suspension and pipe,
- Q = flow rate,
- r = bend radius
- U = superficial flow velocity,
- $U_{\rm b}$ = asymptotic velocity of bed layer,
- $U_{\rm s}$ = asymptotic velocity of the suspension,
- $u_{\rm b}$ = velocity of bed layer,
- $u_{\rm s}$ = velocity of the suspension,

W = width of bed surface,

x =length-coordinate,

 $x_0 = 63\%$ adaptation length,

 $z_b = bed$ surface level,

 $u_{\rm s}$ ', $u_{\rm b}$ ' = perturbations of the velocities of suspension and bed layer,

 \tilde{u}_s, \tilde{u}_b = complex dimensionless amplitudes,

 $\alpha_{\rm bot}$ = friction coefficient of pore fluid with pipe,

- λ_{top} = Darcy-Weisbach friction factor of pipe wall suspension,
- λ_{int} = Darcy-Weisbach friction factor of bed-layer suspension,

 $\mu_{\rm s}$ = Coulomb friction coefficient of solids bed against pipe,

 $\rho_{\rm b}$ = density of the bed-layer (including pore fluid),

 $\rho_{\rm s}$ = average density of suspension,

 $\rho_{\rm sand}$ = sand density,

- $\rho_{\rm w}$ = density of pore fluid (water),
- τ_{bot} = hydrodynamic shear stress at bottom of pipeline,
- τ_{int} = shear stress at interface between both layers,
- τ_{top} = shear stress at top section of pipeline.

SUPPLEMENTARY MATERIAL

Supplementary nomenclature

a_{ij}, b_{ij}	= linearisation coefficients,
c_0	= volumetric solids concentration of the suspension,
α_1, α_2	= transfer coefficients,
$\beta_1, \beta_2, \beta_3$	= varying shape parameters,
η	= shape coefficient,
ζ	= varying shape parameter.

A. Solution of 2-LM equilibrium model

Asymptotic equilibrium of forces, from Eq. (7), is described by:

$$\frac{\tau_{\text{int}}}{\rho_s U^2} \left(\frac{WD}{A_s} + \frac{WD}{A_b} \right) + \frac{\tau_{top}}{\rho_s U^2} \left(\frac{P_{top}D}{A_s} \right) - \frac{\tau_{bot}}{\rho_s U^2} \left(\frac{P_{bot}D}{A_b} \right) - \mu_s \left(\frac{F_n}{F_w} \right) \frac{(\rho_b - \rho_w)gD}{\rho_s U^2} = 0$$
(S1)

Relations for shear stresses, Eq. (8), (9) and (10), are substituted. The velocity of the bed layer is next eliminated between Eq. (S1) and Eq. (2). For fixed z_b/D the solution is given by:

$$a\left(\frac{U_s}{U}\right)^2 + b\frac{U_s}{U} + c = 0 \tag{S2}$$

with:
$$a = 1 + \frac{\lambda_{top}}{\lambda_{int}} \left(\frac{A_b}{A}\right)^3 \frac{P_{top}}{W} - \frac{8\alpha_{bot}}{\lambda_{int}} \frac{\rho_b}{\rho_s} \left(\frac{A_s}{A}\right)^3 \frac{P_{bot}}{W}$$
 (S3)

$$b = -2 + 2\frac{8\alpha_{bot}}{\lambda_{int}}\frac{\rho_b}{\rho_s} \left(\frac{A_s}{A}\right)^2 \frac{P_{bot}}{W}$$
(S4)

$$c = 1 - \mu_s \frac{F_n}{F_w} \frac{(\rho_b - \rho_w)gD}{\rho_s U^2} \frac{8}{\lambda_{\text{int}}} \left(\frac{A_b}{A}\right)^3 \frac{A_s}{WD}$$
(S5)

$$-\frac{\delta \alpha_{bot}}{\lambda_{\rm int}} \frac{\rho_b}{\rho_s} \frac{A_s}{A} \frac{P_{bot}}{W}$$

$$\frac{U_s}{U} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{S6}$$

Geometrical dimensions of the 2-LM configuration are quantified as a function of bed height:

$$W = 2\sqrt{z_b(D - z_b)}$$
, $P_{bot} = \arccos(1 - 2\frac{z_b}{D})D$ (S7), (S8)

$$P_{top} = \pi D - P_{bot} \tag{S9}$$

$$A_b = \frac{P_{bot}D}{4} + \frac{WD}{4}(-1 + 2\frac{z_b}{D})$$
(S10)

$$\frac{F_n}{F_w} = 2 \frac{W + (-1 + 2\frac{z_b}{D})P_{bot}}{P_{bot} + (-1 + 2\frac{z_b}{D})W}$$
(S11)

B. Linearisation of equations and calculation of wave number *k*

The continuity equation, Eq. (2), is written as:

$$\frac{A_b}{A} = \frac{U_s - U}{U_s - U_b} \quad \text{and} \quad \frac{A_s}{A} = \frac{U - U_b}{U_s - U_b} \tag{S12}, (S13)$$

Subsequent linearisation gives:

$$\frac{A_b'}{A} = -\frac{A_s'}{A} = -\alpha_2 \frac{u_b}{U} + \alpha_1 \frac{u_s'}{U}$$
(S14)

where A_b ' and A_s ' are fluctuations of the cross-sectional area, and:

$$\alpha_1 = \frac{U(U - U_b)}{(U_s - U_b)^2}, \ \alpha_2 = \frac{U(U - U_s)}{(U_s - U_b)^2}$$
(S15), (S16)

and
$$\alpha_2 = -\alpha_1 \frac{A_b}{A_s}$$
 (S17)

The suspension layer flows faster, therefore: α_1 = positive and α_2 = negative. At small bed velocities: α_1 = O(1). Linearisation of the continuity equation of the bottom layer, Eq. (3), gives:

$$\frac{U_b}{U}\frac{\dot{A_b}}{A} + \frac{A_b}{A}\frac{\dot{u_b}}{U} = 0$$
(S18)

Linearisation of the combined momentum equation, Eq. (7), is accomplished by multiplication of Eq. (7) with $D/(\rho_s U^2)$ and substitution of decomposed velocities Eq. (11) and Eq. (12).

The *x*-derivatives of the asymptotic velocities U_b and U_s are zero. The derivatives of the fluctuating velocities are given by Eq. (16) and Eq. (17). The internal two-layer geometry varies and shear stresses vary. For these variables similar decompositions are substituted as for the other variables. This produces contributions consisting only of asymptotic mean values of geometry, shear stress and Coulomb bed friction plus contributions containing fluctuating variables. The former collection of terms defines the asymptotic solution, Eq. (S1), and is by definition zero. These terms are therefore removed from the equation. Products of two or more fluctuating variables are discarded because only linear terms are retained in the linearisation process. The resulting linearised combined momentum equation is:

$$ikD\left(\frac{U_s}{U}\frac{u'_s}{U} - \frac{\rho_b}{\rho_s}\frac{U_b}{U}\frac{u'_b}{U} + \frac{1}{Fr_b^2}\frac{c_0}{1-n}\frac{A}{WD}\frac{A'_b}{A}\right) + \frac{\tau_{int}}{\rho_s U^2}\left(\frac{WD}{A_s} + \frac{WD}{A_b}\right) + \frac{\tau_{top}}{\rho_s U^2}\frac{P_{top}D}{A_s} - \frac{\tau_{bot}}{\rho_s U^2}\frac{P_{bot}D}{A_b} + \frac{\tau_{int}}{\rho_s U^2}\left(\frac{WD}{A_s} + \frac{WD}{A_b}\right) + \frac{\tau_{top}}{\rho_s U^2}\left(\frac{P_{top}D}{A_s}\right) - \frac{\tau_{bot}}{\rho_s U^2}\left(\frac{P_{bot}D}{A_b}\right) - \frac{\mu_s}{Fr_b^2}\left(\frac{F_n}{F_w}\right) = 0$$
(S19)

with:
$$Fr_b^2 = \frac{\rho_s U^2}{(\rho_b - \rho_w)gD}$$
 (S20)

The geometrical dimensions A_b , W, P_{bot} and P_{top} are a function of bed level height z_b and pipe diameter D. Analytical expressions for these are given in Section A. The velocities U_b and U_s in the stratified asymptotic configuration are calculated with the 2-LM model given in Section A.

Solving the linearised equation for complex wave number

The linearised equations are represented in a matrix, Eq. (S21). The first row represents the continuity equation of the bed layer, which is a linearisation of Eq. (3). The second row represents the momentum balance of both layers, which is a linearisation of Eq. (7). The a_{ij} components represent the mass and momentum equations without variation of frictional forces. The b_{ij} components represent the variation of these.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \begin{bmatrix} \dot{u'_s} \\ U \\ \dot{u'_b} \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
(S21)

in which:

$$a_{11} = \alpha_1 \frac{U_b}{U} \tag{S22}$$

$$a_{12} = -\alpha_2 \frac{U_s}{U}$$
 (or $a_{12} = -\alpha_2 \frac{U_b}{U} + \frac{A_b}{A}$) (S23)

$$a_{21} = \frac{U_s}{U}ikD + \frac{\alpha_1}{Fr_b^2} \frac{c_0}{1 - n} \frac{A}{WD}ikD$$
 (S24)

$$a_{22} = -\frac{\rho_b}{\rho_s} \frac{U_b}{U} ikD - \frac{\alpha_2}{Fr_b^2} \frac{c_0}{1 - n} \frac{A}{WD} ikD$$
(S25)

$$b_{21} = 2\left(\frac{\lambda_{\text{int}}}{8}\eta\left(\frac{U_s}{U} - \frac{U_b}{U}\right) + \frac{\lambda_{top}}{8}\frac{P_{top}D}{A_s}\frac{U_s}{U}\right) + \zeta\frac{\alpha_1}{Fr_b^2}\frac{A}{WD}$$
(S26)

$$b_{22} = -2\left(\frac{\lambda_{\text{int}}}{8}\eta\left(\frac{U_s}{U} - \frac{U_b}{U}\right)\right) + \alpha_{bot}\frac{\rho_b}{\rho_s}\frac{P_{bot}D}{A_b}\frac{U_b}{U} - \zeta\frac{\alpha_2}{Fr_b^2}\frac{A}{WD}$$
(S27)

where:
$$\eta = \frac{WD}{A_s} + \frac{WD}{A_b}$$
 (S28)

and:

$$\zeta = -\mu_s \frac{F_n}{F_w} \frac{4}{\pi} \frac{W}{D} \beta_1 + Fr_b^2 \frac{16}{\pi^2} \frac{W}{D} \left\{ \frac{\tau_{top}}{\rho_s U^2} \frac{P_{top}}{D} \frac{A}{A_s} \beta_2 - \frac{\tau_{bot}}{\rho_s U^2} \frac{P_{bot}}{D} \frac{A}{A_b} \beta_3 \right\}$$
(S29)

The parameter ζ follows from the last four terms in Eq. (S19). These terms represent the influence of variations of the

shape of the two-layer geometry (i.e. variation of combinations of P_{bot} , P_{top} , W, A_b , A_s and F_n/F_w with variation in bed level z_b). Expressions for the β_i -coefficients in Eq. (S29) are given in Section C. To solve for kD the determinant is set to zero:

$$DET = a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21}) = 0$$
(S30)

The products a_{ij} , a_{ij} exclusively lead to a *ikD* term representing stream wise influences, being: accelerations and variations in bed level. The products a_{ij} , b_{ij} represent the influence of friction forces. The determinant reads:

$$DET = 0 = \begin{cases} -\alpha_1 \frac{\rho_b}{\rho_s} \left(\frac{U_b}{U}\right)^2 + \alpha_2 \left(\frac{U_s}{U}\right)^2 + \\ \frac{\alpha_1 \alpha_2}{Fr_b^2} \frac{c_0}{1 - n} \frac{A}{WD} \left(\frac{U_s}{U} - \frac{U_b}{U}\right) \end{cases} \text{ikD} \\ -2 \left(\frac{U_s}{U} + \frac{U_b}{U} - 1\right) \frac{\lambda_{\text{int}}}{8} \eta + \alpha_2 \frac{\lambda_{top}}{8} \left(\frac{U_s}{U}\right)^2 \frac{P_{top}D}{A_s} \left(2 + \frac{A_s}{A}\beta_2\right) \\ -\alpha_1 \alpha_{bot} \frac{\rho_b}{\rho_s} \left(\frac{U_b}{U}\right)^2 \frac{P_{bot}D}{A_b} \left(2 - \frac{A_b}{A}\beta_3\right) + \mu_s \frac{\alpha_1}{Fr_b^2} \frac{F_n}{F_w} \frac{A_b}{A}\beta_1 \end{cases}$$
(S31)

This equation gives $k_r = 0$ (= no harmonic waves) and k_i as representative for the damping length. Considering the sign of α_1 and α_2 , the coefficient of the *ikD*-term is always negative. The sign of the sum of the last six terms determines whether perturbations amplify or are damped in downstream direction.

C. Quantification of geometrical variations

The linearised expression for the friction force terms in the combined momentum equation is, see Eq. (S19):

$$\frac{\tau_{\text{int}}}{\rho_s U^2} \eta + \frac{\tau_{top}}{\rho_s U^2} \left(\frac{P_{top}D}{A_s}\right) - \frac{\tau_{bot}}{\rho_s U^2} \left(\frac{P_{bot}D}{A_b}\right) + \frac{\tau_{\text{int}}}{\rho_s U^2} \eta' + \frac{\tau_{top}}{\rho_s U^2} \left(\frac{P_{top}D}{A_s}\right) - \frac{\tau_{bot}}{\rho_s U^2} \left(\frac{P_{bot}D}{A_b}\right) - \frac{\mu_s}{Fr_b^2} \left(\frac{F_n}{F_w}\right)'$$
(S32)

Elimination of τ_{int} between Eq. (S13) and Eq. (S32) and expressing geometric fluctuations into the fluctuation of the bed area A_b gives:

$$\frac{\tau_{\text{int}}}{\rho_s U^2} \eta + \frac{\tau_{top}}{\rho_s U^2} \left(\frac{P_{top}D}{A_s}\right) - \frac{\tau_{bot}}{\rho_s U^2} \left(\frac{P_{bot}D}{A_b}\right) + \left\{-\frac{\mu_s}{Fr_b^2} \frac{F_n}{F_w} \beta_1 + \frac{\tau_{top}}{\rho_s U^2} \frac{P_{top}D}{A_s} \beta_2 - \frac{\tau_{bot}}{\rho_s U^2} \frac{P_{bot}D}{A_b} \beta_3\right\} \frac{A_b}{A}$$
(S33)

where:

$$\beta_{1} = \left\{ \frac{1}{F_{n} / F_{w}} \frac{\partial F_{n} / F_{w}}{\partial A_{b} / A} - \frac{1}{\eta} \frac{\partial \eta}{\partial A_{b} / A} \right\}$$

$$= \frac{A}{A_{b}} \frac{P_{bot}}{W} \frac{F_{w}}{F_{n}} - \frac{A}{A_{s}} - \frac{\pi}{4} (\frac{D}{W})^{3} 2(1 - 2\frac{z_{b}}{D})$$
(S34)

$$\beta_{2} = \left\{ \frac{1}{P_{top} / A_{s}} \frac{\partial P_{top} / A_{s}}{\partial A_{b} / A} - \frac{1}{\eta} \frac{\partial \eta}{\partial A_{b} / A} \right\}$$

$$= -\frac{\pi}{4} \frac{2D^{3}}{P_{top}W^{2}} + \frac{A}{A_{b}} - \frac{\pi}{4} (\frac{D}{W})^{3} 2(1 - 2\frac{z_{b}}{D})$$
(S35)
$$\beta_{3} = \left\{ \frac{1}{P_{bot} / A_{b}} \frac{\partial P_{bot} / A_{b}}{\partial A_{b} / A} - \frac{1}{\eta} \frac{\partial \eta}{\partial A_{b} / A} \right\}$$

$$= \frac{\pi}{4} \frac{2D^{3}}{P_{bot}W^{2}} - \frac{A}{A_{s}} - \frac{\pi}{4} (\frac{D}{W})^{3} 2(1 - 2\frac{z_{b}}{D})$$
(S36)

The coefficients β_1 , β_2 , β_3 basically depend on the relative bed level. Their functional dependency is depicted in Figure S1.



Fig. S1. Graphical depiction of the coefficients β_1 , β_2 , β_3 which quantify the influence of bed-level variations on the momentum balance through geometric variation of the cross-sectional area.

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