

A NOTE ON INTEGRAL VORTEX STRENGTH

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Controversial aspects of the conventional and widely used concept of the integral vortex strength are briefly discussed. The strength of a vortex is usually calculated as the circulation along the vortex boundary, or equivalently due to Green's theorem, as the surface integral of vorticity over the planar vortex cross section. However, the local effect of an arbitrary "superimposed shear" is fully absorbed by vorticity what makes the circulation a shear-biased vortex characteristic. The present paper shows that different vortex-strength models can be derived on the basis of different local vortex intensities proposed in the literature. The outcome of these models naturally differs, even for an ideally axisymmetric vortex. Three different vortex-strength models are compared and discussed by examining the unsteady Taylor vortex.

KEY WORDS: Circulation, Unsteady Taylor Vortex, Vortex Intensity, Vortex Strength, Vorticity, Vorticity Decomposition.

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V práci jsou stručně diskutovány sporné stránky konvenčního a široce užívaného pojetí integrální síly víru. Síla víru je obvykle počítána jako cirkulace podél hranice víru nebo ekvivalentně podle Greenovy věty jako plošný integrál výřivosti přes příčný rovinny řez vírem. Lokální efekt libovolného "superponovaného smyku" je však plně absorbován výřivosti, což činí z cirkulace smykově zkreslenou vírovou charakteristikou. Tento článek ukazuje, že lze odvodit různé modely síly víru na základě různých lokálních intenzit víru navržených v odborné literatuře. Výsledky těchto modelů se přirozeně liší, dokonce i pro ideálně osově symetrický vír. Na podkladě zkoumání nestacionárního Taylorova víru jsou porovnány a diskutovány tři různé modely síly víru.

KLÍČOVÁ SLOVA: cirkulace, nestacionární Taylorův vír, intenzita víru, síla víru, výřivost, rozklad výřivosti.

1. Introduction

The concept of the strength of a vortex tube – circulation Γ – is closely associated with Helmholtz's theorems and Kelvin's circulation theorem in basic fluid mechanics (*Batchelor*, 1967; *Panton*, 1984; *Saffman*, 1992; *Green*, 1995). Recall that a vortex tube is the set of all vortex lines passing through a simply-connected surface in space and a vortex line is everywhere tangent to the local vorticity vector. The concept of vortex strength as the circulation along the vortex boundary is widely employed to characterize the initial state, evolution, and decay of large-scale vortical structures. Here, the meaning of the term "vortex" and "vortical structure" is more general and not necessarily coinciding with a vortex tube (as defined above). Due to

Green's theorem, vortex strength is frequently determined as the surface integral of vorticity over the planar vortex cross section. The weak point of this well-established and widely used concept is that vorticity does not represent a true local vortex intensity. The controversy is caused by the well-known fact that vorticity cannot distinguish the effect of a "background shearing" from the "actual swirling motion of a vortex" (note that one obtains a net circulation for the region of a simple shear due to a net vorticity). Vorticity is always a more or less shear-biased local intensity of the actual swirling motion of a vortex. Consequently, the circulation along the vortex boundary is always a more or less shear-biased integral vortex strength. Moreover, the vortex circulation is dependent on the vortex-boundary definition adopted.

In the next two sections, it is shown, for simplicity in 2D, that different vortex-strength models can be derived on the basis of different local vortex intensities. Three different vortex-strength schemes are considered: conventional vorticity-based circulation, circulation based on the so-called residual vorticity (Kolář, 2007), and the integral of the local swirling-strength parameter (Zhou et al., 1999). The first two schemes are applied to the unsteady Taylor vortex for which they are easily analytically treatable. It is further shown that they provide natural bounds (upper and lower estimates) for the third scheme based on the local swirling-strength parameter.

2. Vortex-strength models

For an arbitrarily chosen vortex-intensity concept the vortex cross section is generally defined in Fig. 1. The conventional vortex circulation in 2D is given by

$$\Gamma = \oint_C b dC = \int_{A_1} \omega dA_1, \quad (1)$$

where b denotes a velocity component aligned along the vortex boundary C , vorticity ω is integrated over the vortex cross section A_1 bounded by C . The vortex boundary C can be defined by the requirement of zero vorticity contour – this is physically the most natural choice of threshold – inside which the vorticity magnitude is non-zero everywhere.

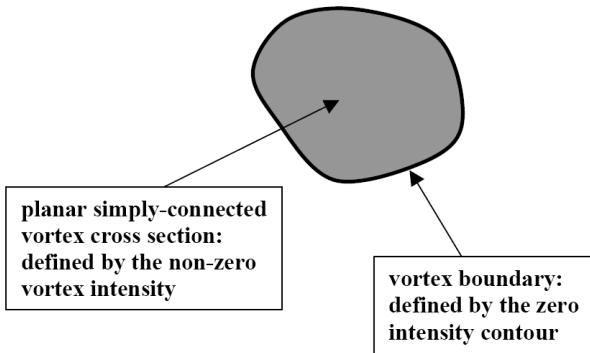


Fig. 1. Generally defined vortex cross section.

Obr. 1. Všeobecně definovaný řez vírem.

The fact that vorticity absorbs the local shearing effects has motivated a novel approach to vortex identification (Kolář, 2007) which is based on purely kinematic grounds, namely on the triple decomposition of the local relative motion near a

point (TDM). Vorticity is decomposed into two parts, shear vorticity and residual vorticity. The residual vorticity is associated with the local residual rigid-body rotation near a point obtained after the extraction of an effective pure shearing motion. Geometrical interpretation of the residual vorticity (denoted below as ω_{RES}) is particularly simple in 2D and quasi-2D problems.

Focusing on the flow kinematics near a point, the residual vorticity is interpreted in terms of the least-absolute-value angular velocity of all line segments, within the flow plane, going through the given point (Kolář, 2007, 2008). Consequently, the *global* motion of an ideal axisymmetric vortex in concentric shearing layers is interpreted in terms of the virtual superposition of distinguished *local* relative motions near a point as shown in Fig. 2 (for clarity including a circular translation). It should be emphasized that this superimposing construction is of a non-destructive nature (shear vorticity and residual vorticity are of the same sign) and is applicable to infinitesimal motional changes only.

The circulation based on the residual vorticity, labelled "residual circulation", can be introduced in a similar manner as the conventional circulation. The residual circulation reads

$$\Gamma_{\text{RES}} = \int_{A_2} \omega_{\text{RES}} dA_2, \quad (2)$$

where ω_{RES} is given by subtracting the shear vorticity ω_{SH} from the (total) vorticity ω . The (non-zero) residual vorticity is basically the simplest measure of the dominance of planar vorticity magnitude over planar strain-rate magnitude. For the planar velocity-gradient tensor of the form $\begin{pmatrix} u_x & u_y \\ v_x & -u_x \end{pmatrix}$ – alternatively represented by $\begin{pmatrix} s/2 & -\omega/2 \\ \omega/2 & -s/2 \end{pmatrix}$ ^{PRINCIPAL AXES} where s denotes (twice) the 2D principal rate of strain and ω is the 2D vorticity – it follows (Kolář, 2007)

$$\omega = v_x - u_y, \quad (3)$$

$$|s| = \sqrt{4u_x^2 + (u_y + v_x)^2}, \quad (4)$$

$$\omega_{\text{RES}} = \omega - \omega_{\text{SH}} = (\text{sgn } \omega) [|\omega| - |s|] \quad (5)$$

for $|\omega| \geq |s|$,

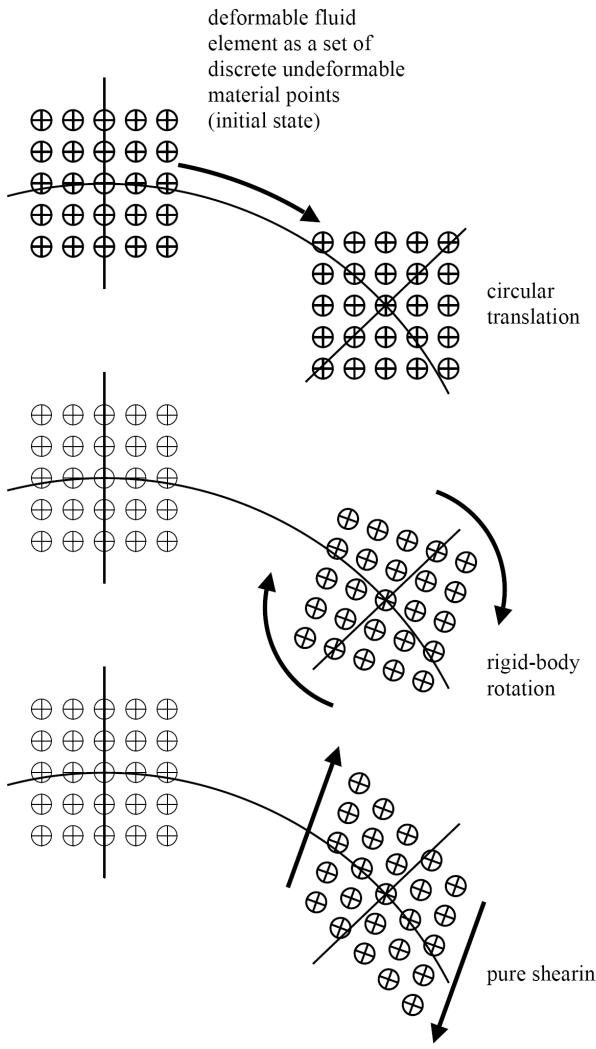


Fig. 2. Virtual infinitesimal motional changes of the fluid element within an ideally axisymmetric vortex (each change at the final stage), according to Kolář (2007).

Obr. 2. Virtuální infinitesimální pohybové změny elementu tekutiny uvnitř ideálně osově symetrického víru (každá změna v konečném stavu), podle lit. Kolář (2007).

$$\omega_{\text{RES}} = 0 \quad \text{for} \quad |\omega| \leq |s|. \quad (6)$$

The vortex cross sections in (1) and (2) are generally different, $A_1 \neq A_2$, as the vortex-boundary definition is naturally subjected to the local vortex intensity adopted: the condition of zero intensity at the boundary is required, see Fig. 1.

Note that, unlike Γ which is always non-zero for a simple shear, Γ_{RES} is always zero for a simple shear as desired for vortex-identification criterion.

A third model of the vortex strength is based on the local swirling-strength parameter of the vortex-identification method of Zhou et al. (1999). The local vortex intensity, called swirling strength, is

proposed in terms of the angular frequency of revolutions of spiralling streamlines in a local reference frame moving with the examined point. In 2D, this frequency is just a half of $\sqrt{\omega^2 - s^2}$ under the corresponding vortex-identification condition $|\omega| > |s|$. In view of the quantitatively same vortex-strength value for a rigid-body rotation evaluated by all the three considered vortex-strength models, it follows for the third model

$$\Gamma_Z = \int_{A_3} \sqrt{\omega^2 - s^2} \, dA_3. \quad (7)$$

The boundary of the second vortex-strength model Γ_{RES} coincides with that of the third model Γ_Z , and is found by the same condition $|s| = |\omega|$. Consequently, the corresponding vortex cross-sections are identical, $A_2 = A_3$.

3. Application of different vortex-strength models to the unsteady Taylor vortex

From the simple conversion between Cartesian and polar coordinates and the relations (3), (4) we obtain for an arbitrary unsteady axisymmetric distribution of tangential velocity $V \equiv V(r, t)$ (r is radius, $r = \sqrt{x^2 + y^2}$, and t denotes time)

$$\omega = \frac{1}{r} \frac{\partial}{\partial r} (rV) = \frac{\partial V}{\partial r} + \frac{V}{r}, \quad (8)$$

$$|s| = \left| \frac{\partial V}{\partial r} - \frac{V}{r} \right|. \quad (9)$$

The vortex region is identified in Kolář (2007) by the condition of non-zero residual vorticity, that is, in 2D, by the condition $|\omega| > |s|$ which can be rewritten as (monotonous decay of vorticity with increasing radius is naturally assumed, at least inside the vortex region)

$$\left| \frac{\partial V}{\partial r} + \frac{V}{r} \right| > \left| \frac{\partial V}{\partial r} - \frac{V}{r} \right|. \quad (10)$$

Consequently, $\partial V / \partial r$ and V/r must have the same sign inside the vortex region. The vortex boundary is found by $\omega_{\text{RES}} = 0$, or equivalently, by $\partial V / \partial r = 0$. Hence, the vortex radius is the radius at which the tangential velocity is a maximum what is just the conventional definition of the circular vortex core (Green, 1995).

From (5) and (8)–(10), the final expression for non-zero ω_{RES} (i.e. for the vortex region) reads

$$\omega_{\text{RES}} = 2(\text{sgn}\omega) \left| \frac{\partial V}{\partial r} \right| \quad \text{for } |\omega| > |s|. \quad (11)$$

The unsteady Taylor vortex (e.g. Panton, 1984) centred at (0,0) is described in polar coordinates by the tangential velocity distribution $V(r,t)$ of the following form

$$V = \frac{Ar}{t^2} \exp\left(-\frac{Br^2}{t}\right), \quad (12)$$

where A, B are positive physical parameters related to angular momentum and/or viscosity. Using (8), (11) and (12), the unsteady vortex boundary in terms of ω and ω_{RES} can be easily determined from the conditions $\omega = 0$ and $\omega_{\text{RES}} = 0$ (i.e. $\partial V / \partial r = 0$), respectively,

$$R_1 (\equiv \text{boundary radius at which } \omega = 0) = \sqrt{\frac{t}{B}}, \quad (13)$$

$$R_2 (\equiv \text{boundary radius at which } \omega_{\text{RES}} = 0) = \sqrt{\frac{t}{2B}}. \quad (14)$$

The circulation – the vortex strength based on ω – is obtained from (8), (12) and (13) after a short calculation (e denotes the base of the natural logarithm)

$$\Gamma = 2\pi R_1 \cdot V(R_1) = \frac{2\pi A}{B} \cdot \frac{1}{e} \cdot \frac{1}{t}. \quad (15)$$

The residual circulation – the vortex strength based on ω_{RES} – is determined from (11), (12) and (14) as a similar hyperbolic function of time as in the case of Γ

$$\begin{aligned} \Gamma_{\text{RES}} &= 4\pi R_2 \cdot V(R_2) - 4\pi \int_0^{R_2} V \, dr = \\ &= \frac{2\pi A}{B} \cdot \frac{2 - \sqrt{e}}{\sqrt{e}} \cdot \frac{1}{t}. \end{aligned} \quad (16)$$

From (15) and (16) it follows

$$\frac{\Gamma_{\text{RES}}}{\Gamma} = 2\sqrt{e} - e \approx 0.579. \quad (17)$$

This result that Γ_{RES} is less than sixty percent of Γ is valid for the unsteady Taylor vortex of the

velocity distribution (12). The same apparently holds for the ratio of the vortex-strength decay rates $d\Gamma_{\text{RES}}/dt$ and $d\Gamma/dt$. Recall that Γ_{RES} – unlike Γ – does not absorb any local shearing effects near a point and is based on the local rigid-body rotation only (cf. Fig. 2). It is easy to verify that for the limiting case of rigid-body rotation of a circular fluid region we obtain $\Gamma_{\text{RES}}/\Gamma = 1$, as expected.

The vortex region defined by ω_{RES} is the same as the region defined by the swirling-strength parameter $\sqrt{\omega^2 - s^2}$ (see Section 2). Both these regions are characterized by $|\omega| > |s|$ and bounded by the condition $|s| = |\omega|$. However, with the exception of the rigid-body rotation of a circular fluid region, both these regions are smaller than the ω -based vortex region bounded by the condition $\omega = 0$.

Denoting $\varepsilon = |s|/|\omega|$, we have

$$0 < 1 - \varepsilon < \sqrt{(1 - \varepsilon)(1 + \varepsilon)} < 1 \quad \text{for } 0 < \varepsilon < 1. \quad (18)$$

From here the following upper and lower estimates for the third vortex-strength model Γ_Z given by (7) can be drawn

$$\Gamma_{\text{RES}} < \Gamma_Z < \Gamma. \quad (19)$$

This result is not surprising as the swirling-strength parameter, though somewhat better for vortex-identification purposes than vorticity, still remains a strongly shear-absorbing quantity (Kolář, 2007). Consequently, at a given instant of time, the values of Γ_Z are higher than those of Γ_{RES} , though the vortex region is the same for both different underlying non-zero vortex intensities.

4. Concluding remarks

As shown above in 2D, different vortex-strength models can be derived on the basis of different local vortex intensities. The vortex region, vortex boundary and, consequently, the (integral) vortex strength are directly dependent on the choice of a local vortex intensity. The vortex strength is calculated for the planar simply-connected vortex cross section defined by the non-zero vortex intensity and bounded by the zero intensity contour (Fig. 1).

Three vortex-strength models are discussed and compared by examining the unsteady Taylor vortex. These models include the well-known conventional circulation Γ , the so-called residual circula-

tion Γ_{RES} based on the residual vorticity (Kolář, 2007), and the integral Γ_z of the local swirling-strength parameter (Zhou et al., 1999). The outcome of these models naturally differs, even for an ideally axisymmetric vortex. For the unsteady Taylor vortex flow, Γ and Γ_{RES} are analytically expressed by (15) and (16) as hyperbolic functions of time, and the integral Γ_z is estimated by (19).

Unlike Γ_{RES} , both Γ and Γ_z are shear-biased vortex-strength models. However, for the special case of a simple shear, Γ is always non-zero whereas Γ_{RES} and Γ_z are always zero as desired for true vortex characteristics. In this regard, Γ_z is certainly better than Γ . The integral Γ_z is derived from the local vortex intensity $\sqrt{\omega^2 - s^2}$. As pointed out in Kolář (2007), the quantity $\omega^2 - s^2$ exhibits a strong inherent bias towards $|\omega| + |s|$ while employed in vortex identification. This fact makes Γ_z a shear-biased vortex-strength model though generally having lower values than Γ which is apparently more absorbing a local shear than Γ_z (see also (19)). Summing up, in the description of vortex strength, there are strong reasons to prefer the residual circulation Γ_{RES} to the conventional circulation Γ and the integral Γ_z . Note that the quantity Γ_{RES} has already proved its usefulness for the description of single and twin jets in crossflow (Kolář and Savory, 2007). For different nozzle arrangements, an almost universal behaviour of Γ_{RES} has been revealed for the resulting secondary-flow counter-rotating vortex pair.

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List of symbols

A	– parameter of the unsteady Taylor vortex according to (12) [T],	B	– parameter of the unsteady Taylor vortex according to (12) [$L^{-2} T$],
A_1	– vortex region defined by ω [L^2],	C	– vortex boundary [L],
A_2	– vortex region defined by ω_{RES} [L^2],	e	– base of the natural logarithm, Euler's number [-],
A_3	– vortex region defined by $\sqrt{\omega^2 - s^2}$ [L^2],	r	– radius [L],
b	– velocity component aligned along the vortex boundary C [$L T^{-1}$],	R_1	– boundary radius at which $\omega = 0$ [L],
		R_2	– boundary radius at which $\omega_{\text{RES}} = 0$ [L],
		s	– (twice) principal rate of strain (in 2D) [T^{-1}],
		t	– time [T],
		u, v	– components of the velocity vector (in 2D) [$L T^{-1}$],
		V	– tangential velocity [$L T^{-1}$],
		x, y	– Cartesian coordinates (in 2D) [L],
		x, y	– subscripts stand for spatial partial derivatives,
		Γ	– surface integral of ω [$L^2 T^{-1}$],
		Γ_{RES}	– surface integral of ω_{RES} [$L^2 T^{-1}$],
		Γ_z	– surface integral of $\sqrt{\omega^2 - s^2}$ [$L^2 T^{-1}$],
		ε	– ratio, $\varepsilon = s / \omega $ [-],
		ω	– vorticity (in 2D) [T^{-1}],
		ω_{RES}	– residual vorticity (in 2D) [T^{-1}],
		ω_{SH}	– shear vorticity (in 2D) [T^{-1}].

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POZNÁMKA O INTEGRÁLNÍ SÍLE VÍRU

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Integrální síla víru je počítána obvykle jako cirkulace podél hranice víru nebo, ekvivalentně podle Greenovy věty, jako plošný integrál výřivosti přes příčný roviný řez vírem. V případě interakce víru se smykkem je však lokální efekt libovolného "superponovaného smyku" plně absorbován výřivostí, což činí z cirkulace smykově zkreslenou vírovou charakteristikou. Ve 2D je ukázáno, že různé modely síly víru lze odvodit na základě různých

lokálních intenzit víru. Tyto integrální charakteristiky jsou počítány přes příčný rovinný řez vírem, který je na hranici definován podmínkou nulové intenzity víru. Oblast víru a hranice víru přímo podléhají volbě lokální intenzity víru.

Na podkladě zkoumání nestacionárního Taylorova víru (12) jsou porovnány a diskutovány tři různé modely sily víru. Tyto modely zahrnují konvenční cirkulaci Γ , tzv. reziduální cirkulaci Γ_{RES} , která je založena na reziduální výřivosti (Kolář, 2007), a integrál Γ_Z lokálního parametru výřivé sily (Zhou et al., 1999). Výsledky těchto modelů se přirozeně liší, dokonce i pro ideálně osově symetrický vír.

Cirkulace Γ a reziduální cirkulace Γ_{RES} jsou analyticky vyjádřeny pro nestacionární Taylorův vír (12) jako hyperbolické funkce času vztahy (15) a (16). Pro velikost integrálu Γ_Z je stanoven odhad (19).

Na rozdíl od Γ_{RES} , jak Γ tak Γ_Z jsou smykově zkreslené modely sily víru. Pro speciální případ prostého smyku je však Γ vždycky nenulové, zatímco Γ_{RES} a Γ_Z jsou vždycky nulové, jak je žádoucí pro správné vírové charakteristiky. V tomto ohledu je Γ_Z lepší než Γ . Integrál Γ_Z je odvozen z lokální intenzity víru $\sqrt{\omega^2 - s^2}$. Jak je zdůrazněno v práci Kolář (2007), veličina $\omega^2 - s^2$ vykazuje při užití pro vírovou identifikaci silné zkreslení směrem k $[\omega + |s|]$. Tato skutečnost nevyhnutelně činí z integrálu Γ_Z smykově zkreslený model sily víru, ačkoliv obecně dosahuje nižších hodnot než cirkulace Γ , která absorbuje lokální smyk výrazněji než Γ_Z (viz také (19)). Můžeme shrnout, že existují pádné důvody preferovat při

popisu integrální síly víru reziduální cirkulaci Γ_{RES} před konvenční cirkulací Γ a integrálem Γ_Z .

Seznam symbolů

A	– parametr nestacionárního Taylorova víru podle (12) [T],
A_1	– oblast víru definovaná pomocí ω [L^2],
A_2	– oblast víru definovaná pomocí ω_{RES} [L^2],
A_3	– oblast víru definovaná pomocí $\sqrt{\omega^2 - s^2}$ [L^2],
b	– složka rychlosti podél hranice víru C [$L T^{-1}$],
B	– parametr nestacionárního Taylorova víru podle (12) [$L^{-2} T$],
C	– hranice víru [L],
e	– základ přirozených logaritmů, Eulerovo číslo [-],
r	– rádius [L],
R_1	– hraniční rádius, pro který $\omega = 0$ [L],
R_2	– hraniční rádius, pro který $\omega_{\text{RES}} = 0$ [L],
s	– (dvakrát) hlavní rychlosť deformace (ve 2D) [T^{-1}],
t	– čas [T],
u, v	– složky vektoru rychlosti (ve 2D) [$L T^{-1}$],
V	– tangenciální rychlosť [$L T^{-1}$],
x, y	– kartézske souřadnice (ve 2D) [L],
x, y	– dolní indexy označující prostorové parciální derivace,
Γ	– plošný integrál přes ω [$L^2 T^{-1}$],
Γ_{RES}	– plošný integrál přes ω_{RES} [$L^2 T^{-1}$],
Γ_Z	– plošný integrál přes $\sqrt{\omega^2 - s^2}$ [$L^2 T^{-1}$],
ε	– poměr, $\varepsilon = s / \omega $ [-],
ω	– výřivost (ve 2D) [T^{-1}],
ω_{RES}	– reziduální výřivost (ve 2D) [T^{-1}],
ω_{SH}	– smyková výřivost (ve 2D) [T^{-1}].