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Nonlinear power system model reduction based on empirical Gramians

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An effective nonlinear model reduction approach, empirical Gramians balanced reduction approach, is studied, to reduce the computation complexity in nonlinear power system model application. The realization procedure is: firstly, computing the empirical controllable and observable Gramians matrices of nonlinear power system model, secondly, by these two matrices, computing the balance transformation matrix to obtain the balanced system model of the original model, then, computing the controllable and observable matrices of the balanced system to obtain the diagonal Hankel singular matrix. Finally, deciding the lower-order subspace to obtain the reduced power system model. A 15-machine power system model is taken as an example to perform the reduction simulation analysis.

K e y w o r d s: power system, nonlinear model reduction, empirical Gramians, balanced truncation

1 Introduction

In power system analysis and control, nonlinear higher order models are often chosen for a better description of systems dynamic behavior, however, with a pay-off of an increased computation complexity. Especially with the increasing of the scale of the interconnected modern power system, the order of power system dynamic model is increased rapidly, making it almost impossible for the model-based analysis and control to be implemented in real time.

Model reduction is one of the effective methods to solve the above-mentioned obstacle. Many researches have been conducted in this field, such as dynamic equivalents [1–3], model analysis [4, 5] and identification [6, 7], singular perturbation techniques [8], Hankel norm approximation [9], balanced realization approach [10], *etc.* There are already some effective theories and applications published on linear power system model reduction. Chaniotis [11] uses Krylov subspace method for model reduction, and gives the mapping of various subspace modes to coherent generators. Chow [12] uses Gramians method for model reduction to simplify the design of damping controllers for interarea oscillations. He proves that the controller based on the reduced model maintains the dynamic characteristics of the original system. Martin [13] applies Gramians method in a large power system for the suppression of low frequency oscillation. Especially, Antoulas and his coworkers [14] provide a general method for linear model reduction by using Lyapunov equations. However, in power system application, nonlinear models are usually preferred because it can better describe the systems overall dynamic behavior and the designed controller based on it is more robust. Unfortunately, the reduction of nonlinear model has not been very well developed as that of linear system model. Since 19 century, it has been known that nonlinear system cannot be solved by general analytic method, and the intricate behavior of the nonlinear system could be very complicated even only a couple of state variables are involved. Under this context, researchers must use simulation method to study the dynamic behavior of nonlinear system. In recent years, some elementary studies have been implemented via Gramians method for nonlinear system model reduction by researchers as by Sirovich in [15], Lall in [16], Hahn in [17–19], *etc.*

In this paper, we apply the empirical Gramians and balanced realization methods to accomplish the reduction of nonlinear power system model for the design of its controller. The application of the method in a 15-generators test system is provided and the effectiveness of the proposed method in nonlinear system model reduction is verified.

2 Nonlinear System Model Reduction

2. 1 The Karhunen-Loeve (KL) decomposition

The dynamic model of a nonlinear autonomous system is given by

$$\dot{\mathbf{x}}(t) = \boldsymbol{f}(\mathbf{x}(t)) \tag{1}$$

where, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector of system. The KL decomposition can be described as the following: for a given set in $\mathbb{R}^n$, the core idea is to find an r-dimensional subspace $\mathbb{R}^r$, which makes the error of projecting set in $\mathbb{R}^n$ onto $\mathbb{R}^r$ the minimum. In another word, in time interval $(0, T)$, there exists a data set $\{\mathbf{x}(t); \mathbf{x}(t) \in \mathbb{R}^n\}$ decided by the states $\mathbf{x}(t)$. By finding a projection $\boldsymbol{H}$ projecting all data in the set onto the subspace spanned

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by eigenfunction $\{\boldsymbol{\varphi}_j(\mathbf{x})\}_{j=1}^{r}$. The optimal solution can be obtained by minimizing the error

$$\min \int_0^T \|\boldsymbol{x}(t) - \boldsymbol{H}\boldsymbol{x}(t)\|^2 \mathrm{d}t \tag{2}$$

In order to find the projection $\boldsymbol{P}_r$, an $n \times n$ correlation matrix $\boldsymbol{R}$ is defined by

$$\boldsymbol{R} = \int_0^T \mathbf{x}(t)\mathbf{x}^\top(t)\mathrm{d}t\,. \tag{3}$$

The eigenvalues λ_k of $\boldsymbol{R}$ and corresponding eigenfunctions $\boldsymbol{\varphi}_k$ can be evaluated as

$$\boldsymbol{R}\boldsymbol{\varphi}_k = \lambda_k \boldsymbol{\varphi}_k\,, \quad \lambda_1 \geq \cdots \geq \lambda_n \geq 0\,. \tag{4}$$

The eigenvalues obtained through (4) are nonnegative real number, as the correlation matrix $\boldsymbol{R}$ is symmetric and positive semi definite. eigenfunctions could be chosen to be orthogonal. Therefore, each state $x_i(t)$ can be expanded in the following form

$$x_i(t) = \sum_{k=1}^{n} a_{ik}\boldsymbol{\varphi}_k$$

where, $a_{ik} = \langle \mathbf{x}_i(t), \boldsymbol{\varphi}_k \rangle$, $\langle \boldsymbol{\varphi}_i, \boldsymbol{\varphi}_k \rangle = \delta_{ik}$.

By ignoring eigenfunctions with zero and nearly zero eigenvalues, a subspace $\mathbb{R}^r$ characterized by $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_r\}$ can be obtained to approximate $\mathbb{R}^n$, and this r-dimensional approximation is given by

$$\tilde{\mathbf{x}}_i(t) = \sum_{k=1}^{r} a_{ik}\boldsymbol{\varphi}_k\,.$$

Therefore, projector $\boldsymbol{P}_r$ can be expressed by

$$\boldsymbol{P}_r = \sum_{k=1}^{r} \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^\top \tag{5}$$

The physical meaning of the corresponding eigenvalue of the eigenfunction is that they could maximize the average energy of the projection of data sets on the subspace spanned by the eigenfunction

$$\begin{aligned} &\arg\max_{\{\boldsymbol{\varphi}_k\}} \langle \|\boldsymbol{P}_k \boldsymbol{x}(t)\|^2 \rangle = \\ &\arg\min_{\{\boldsymbol{\varphi}_k\}} \langle \|\boldsymbol{x}(t) - \boldsymbol{H}\boldsymbol{x}(t)\|^2 \rangle \end{aligned} \tag{6}$$

where, $\langle \cdot \rangle$ represents the average on the data set. The matrix $\boldsymbol{P}_k$ is introduced below, (6) has the same meaning as (2). Especially, the average energy of the data sets (or the dynamic system) projected onto the subspace can be described by

$$\int_0^T \|\boldsymbol{P}_k \boldsymbol{x}(t)\|^2 \mathrm{d}t = \sum_{k=1}^{r} \lambda_k \tag{7}$$

Generally, the order r of the dynamic system projected onto the subspace can be decided by the ratio of the average energy of dynamic system on the subspace and that on the original space, that is,

$$\frac{\sum_{k=1}^{r} \lambda_k}{\sum_{k=1}^{n} \lambda_k} \geq \varepsilon \tag{8}$$

where, ε is a real number smaller than 1. When ε is chosen very close to 1, the original dynamic system can be very well approximated by the subspace spanned by the eigenfunction $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_r$. In the back of analysis, through properly selecting ε, we determine the orders of the reducted model of nonlinear power system.

2.2 *Galerkin projection*

Once the eigenfunctions of the subspace are obtained, the system described by (1) can be projected onto a subspace by Galerkin projection. Galerkin projection matrix can be expressed by

$$\boldsymbol{P}_k = [\boldsymbol{P}_r \;\; \boldsymbol{0}_{n-r}] \tag{9}$$

where, $\boldsymbol{0}_{n-r}$ is an $r \times (n-r)$ zero matrix.

By defining a state transformation $\tilde{\mathbf{x}}(t) = \boldsymbol{P}_k \boldsymbol{x}(t)$, the reduced-order system is given by

$$\dot{\tilde{\mathbf{x}}}(t) = \boldsymbol{P}_k \boldsymbol{f}\left(\boldsymbol{P}_k^\top \tilde{\mathbf{x}}(t)\right) \tag{10}$$

where, $\tilde{\mathbf{x}}(t) \in \mathbb{R}^r \subset \mathbb{R}^n$, $\boldsymbol{P}_k^\top$ is the transposition of $\boldsymbol{P}_k$. The initial condition of this reduced-order system is given by $\tilde{\mathbf{x}}(0) = \boldsymbol{P}_k \boldsymbol{x}(0)$.

3 Balanced reduction of nonlinear power system

3.1 *Empirical Gramians matrix of nonlinear power system*

In this section, a synchronous generator connected to an infinite bus is taken as an example to explain the procedure of the reduction of power system model. The generator is described by the six-order utility dynamic model [20], and the impedance of transmission line is X_Σ between the generator and the infinite system, so the nonlinear dynamic model of this simple system can be described as follows

$$\begin{aligned} \dot{\delta} &= \omega - \omega_0\,, \\ H\dot{\omega} &= P_m - P_e - D(\omega - \omega_0)\,, \\ T'_{d0}\dot{E}'_q &= E_f - E'_q + (x_d + x'_d)i_d\,, \\ T'_{q0}\dot{E}'_d &= -E'_d - (x_q - x'_q)i_q\,, \\ T''_{d0}\dot{E}''_q &= E'_q - E''_q + (x_d - x''_d)i_d\,, \\ T''_{q0}\dot{E}''_d &= E'_d - E''_d - (x_q - x''_q)i_q\,. \end{aligned} \tag{11}$$

The output $y(t)$ of the system is defined by

$$y(t) = [\delta \ \ \omega \ \ V]^\top \tag{12}$$

In (11) and (12),

$$\begin{aligned}
P_e &= E_d''i_d + E_q''i_q + (x_d'' - x_q'')i_d i_q\,,\\
i_d &= E_q'' - V_0\cos\delta/(x_d'' + x_\Sigma)\,,\\
i_q &= -E_d'' - V_0\sin\delta/(x_d'' + x_\Sigma)\,,\\
u_d &= -E_d'' - x_q''i_q - r_a i_d\,,\\
u_q &= E_q'' + x_d''i_d - r_a i_q\,,\\
V &= (u_d^2 + u_q^2)^{1/2}\,,
\end{aligned}$$

where, δ and ω are the power angle and angular velocity of the synchronous generator respectively. E_d', E_d'', E_q', and E_q'' are transient and sub-transient voltages of the dq coordinates of the generator respectively. x_d, x_d', x_d'', x_q, x_q', and x_q'' are the synchronous, transient, sub-transient reactance of the dq coordinates of the generator respectively. T_{d0}', T_{d0}'', T_{q0}', and T_{q0}'' are respectively transient and sub-transient open-circuit time constant of the dq coordinates. H is the inertia coefficient. D is the constant damping coefficient of the generator. X_Σ is the sum of the inductance of transformer and transmission line. u_d, u_q, i_d, and i_q are voltages and currents of dq coordinates respectively. V is the generator terminal voltage. V_0 is the voltage of infinite bus. E_f is the output of excitation controller. P_m is power of the prime motor.

The power system given above can be expressed in a form of a general nonlinear power system model

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right)\,, \\ \boldsymbol{y}(t) = \boldsymbol{h}\left(\boldsymbol{x}(t)\right) \end{cases} \tag{13}$$

where, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ describes the dynamic behavior of the single-machine-to-infinite bus system, and $\boldsymbol{x}(t) \in \mathbb{R}^6$, and $\boldsymbol{u}(t) \in \mathbb{R}^1$, and $\boldsymbol{u}(t) = \boldsymbol{E}_f$, the output of the system, $\boldsymbol{y}(t) = \boldsymbol{h}\big(\boldsymbol{x}(t)\big) \in \mathbb{R}^3$, is the nonlinear function of states $\boldsymbol{x}(t)$, as given in (12). Since we are now considering a controlled system with inputs, we can make the assumption that the initial state of the system is zero, and parametrize the trajectories for principal component analysis with respect to the system input $\boldsymbol{u}$.

From the viewpoint of system control theory, the systems controllability and observability reveal the relations between systems states, inputs and outputs of the system. In general, the controllability describes how the control input affects systems states, whereas the observability means that the change of states can be observed through the output of system. Therefore, the controllable and observable Gramians matrices can tell how the inputs and outputs affect the system states.

For a linear system, when doing model reduction, the controllable and observable Gramians matrices can be obtained by evaluating the Lyapunov equation. Then, by analyzing the eigenvalue of Gramians matrix, the order of the reduced model can be decided [21]. However, for the nonlinear system, on the one hand the Lyapunov equation cannot be acquired in general, and on the other hand there is still no explicit method of directly calculating the controllable and observable Gramians matrices.

Therefore, in the following, the empirical method proposed by Sirovich is taken. The samples of states and outputs of the system are obtained by numerical simulation first, then be used to construct the empirical controllable and observable Gramians matrices for the nonlinear system.

First, the correlation matrix $\boldsymbol{R}$ is constructed as

$$\boldsymbol{R} = \int_0^\infty \big(\mathbf{x}(t) - \mathbf{x}^m\big)\big(\mathbf{x}(t) - \mathbf{x}^m\big)^\top \mathrm{d}t \tag{14}$$

where, $\mathbf{x}(t)$ is the state at time t, and $\mathbf{x}^m$ is the sample average of the steady state of the nonlinear system.

Next, the input is applied through impulses $u_1, \dots, u_p$ to the system in (13), and the corresponding states $x_1(t), \dots, x_p(t)$ and outputs $y_1(t), \dots, y_q(t)$ can be obtained, and then they will be expanded by the KL expansion to calculate the empirical controllable Gramians matrix and empirical observable Gramians matrix.

For the sake of the description of the system practical dynamic behavior, when calculating the empirical controllable and observable Gramians matrices, the uncertainty introduced by the changes of the size and direction of each input excitation should be considered. So, several sets need to be defined in the following [16]

$$\begin{aligned}
\mathcal{T}^n = \left\{T_1, \dots, T_r; T_i \in \mathbb{R}^{n\times n}, T_i^\top T_i = I, i = 1, \dots, r\right\}\\
\mathcal{M} = \{c_1, \dots, c_s; c_i \in \mathbb{R}, c_i > 0, i = 1, \dots, s\}\\
\mathcal{E} = \{e_1, \dots, e_n\}
\end{aligned}$$

where, $\mathcal{T}^n$ is any set of r orthogonal matrices, T_i is the direction of excitation input, and $\mathcal{M}$ is the set of s positive constants, c_i represents the set of the size of excitation inputs, and $\mathcal{E}^n$ is the set of n unit vectors. e_i is a unit vector in $\mathbb{R}^n$.

Definition 1. Assume that $\mathcal{T}^n$, $\mathcal{M}$, and $\mathcal{E}^n$ are data sets given above. For a nonlinear system given by (13), the empirical controllable Gramians matrix $\boldsymbol{W}_c$ is defined by

$$\boldsymbol{W}_c = \sum_{l=1}^{r}\sum_{m=1}^{s}\sum_{i=1}^{p}\frac{1}{rsc_m^2}\int_0^\infty \boldsymbol{\Phi}^{ilm}(t)\mathrm{d}t \tag{15}$$

where, $\boldsymbol{\Phi}^{ilm}(t) = \big(\boldsymbol{x}^{ilm}(t) - \boldsymbol{x}^m\big)\big(\boldsymbol{x}^{ilm}(t) - \boldsymbol{x}^m\big)^\top \in \mathbb{R}^{n\times n}$ and $\boldsymbol{x}^{ilm}(t)$ is the system state when the input is applied through impulse $\boldsymbol{u}(t) = \boldsymbol{u}^m + c_m \boldsymbol{T}_l \boldsymbol{e}_i \tau(t)$. $\boldsymbol{u}^m$ is the steady input of the controller and $\tau(t)$ is impulse function.

Due to the dynamic behavior of system depends on its structure, its state change can be reflected through applying external excitation, here, we use impulsive signal

$\tau(t)$ to simulate the input of external controllers to obtain the sample data of system state. By performing KL expansion to state variables in (15), the empirical controllable Gramians matrix $\boldsymbol{W}_c$ can be constructed, the corresponding eigenfunction of the non-zero eigenvalue of which can span subspace containing all possible states of the system, so the nonlinear system given in (13) can be projected onto the subspace by the projection matrix. Through this approach, the order of the nonlinear power system model can be reduced.

For a controllable nonlinear system with inputs and outputs, it is not enough only studying how the inputs affect the systems states, at the same time the influence of system state to the output, *ie* the observability of system, should also be considered. The following definition gives the method to construct the observable empirical Gramians matrix.

For a controllable nonlinear system with inputs and outputs, it is not enough only studying how the inputs affect the systems states, at the same time the influence of system state to the output, i.e. the observability of system, should also be considered. The following definition gives the method to construct the observable empirical Gramians matrix.

Definition 2. Assume that T^n, M, and E^n are data sets given above. For a nonlinear system given in (13), the empirical observable Gramians matrix W_0 is defined by

$$\boldsymbol{W}_0 = \sum_{l=1}^{r} \sum_{m-1}^{s} \frac{1}{rsc_m^2} \int_0^{\infty} \boldsymbol{T}_l \boldsymbol{\Psi}^{lm}(t) \boldsymbol{T}_l^{\top} \mathrm{d}t \tag{16}$$

where, $\boldsymbol{\Psi}_{ij}^{lm}(t) = \left(\boldsymbol{y}^{ilm}(t) - \boldsymbol{y}^m\right)^{\top} \left(y^{jlm}(t) - \boldsymbol{y}^m\right) \in \mathbb{R}^{n \times n}$. $\boldsymbol{y}^{ilm}(t)$ is the output of the system given in (13) under the initial condition $\mathbf{x}_0 = \boldsymbol{x}^m + c_m \boldsymbol{T}_l \boldsymbol{e}_i$, $\boldsymbol{x}^m$ is the sample average of the steady state of system, $\boldsymbol{y}^m$ is the average of system output.

Similarly, the corresponding eigenfunction to the nonzero eigenvalue of $\boldsymbol{W}_0$ can span a new subspace, on which the nonlinear system given in (13) can then be projected to obtain the reduced nonlinear system model, which still maintains the observable characteristic of the original system. The corresponding subspaces can respectively be decided according to empirical controllable Gramians matrix $\boldsymbol{W}_c$ and empirical observable Gramians matrix $\boldsymbol{W}_0$ from the view of controllability or observability. However, in general these two subspaces obtained are different with each other. It means that the influence of states on inputs and outputs of system is not consistent, *ie* certain states have a great influence on the inputs of the system, but have little influence on the outputs of the system, or vice versa. Therefore, for a given nonlinear system, if any one of the controllable and observable characteristics is emphasized, the other one will weaken.

In this paper, in order to maintain both the controllability and observability of the system, the balanced realization approach is used to decide which subspace the original system should be projected onto.

3.2 Balanced approach for model reduction

A system is a balance system if the systems controllable and observable Gramians matrices are the same and diagonal. By transferring the non-balance system to its corresponding balanced system, the two different subspaces can be transformed into one subspace, where the original system can be projected for model reduction with maintaining both the controllability and observability of the system.

The corresponding balanced system of a nonlinear power system can be obtained through a non-singular linear transformation matrix $\boldsymbol{T}$, without affecting the systems input-output characteristics. By the coordinate transformation $\overline{\mathbf{x}}(t) = \boldsymbol{T}\mathbf{x}(t)$, the balanced system model is then obtained

$$\begin{cases} \dot{\mathbf{x}}(t) = \boldsymbol{T}\boldsymbol{f}\big(\boldsymbol{T}^{-1}\overline{\mathbf{x}}(t), \boldsymbol{u}(t)\big), \\ \boldsymbol{y}(t) = \boldsymbol{h}\big(\boldsymbol{T}^{-1}\overline{\mathbf{x}}(t)\big) \end{cases} \tag{17}$$

where, $\overline{\mathbf{x}}(t) \in \mathbb{R}^n$ is the state of the balanced power system model. Note that, from the viewpoint of linear algebra theory, the states of the balanced system $\overline{\mathbf{x}}(t)$ are the linear combination of the states $\boldsymbol{x}(t)$ of the original system, so they are equivalent.

Balanced transformation matrix $\boldsymbol{T}$ can be computed by the empirical controllable and observable Gramians matrices. The computation procedure is listed below:

1) Performing Cholesky decomposition to the empirical Gramians matrices $\boldsymbol{W}_c$ and $\boldsymbol{W}_0$ to obtain matrices $\boldsymbol{L}_c$ and $\boldsymbol{L}_0$.

$$\boldsymbol{W}_c = \boldsymbol{L}_c \boldsymbol{L}_c^{\top}, \qquad \boldsymbol{W}_0 = \boldsymbol{L}_0 \boldsymbol{L}_0^{\top}$$

2) Multiplying matrices $\boldsymbol{L}_0^{\top} \boldsymbol{L}_c$, then performing singular value decomposition to $\boldsymbol{L}_0^{\top} \boldsymbol{L}_c$ to obtain the diagonal matrix $\boldsymbol{\Sigma}_1$, orthogonal matrices $\boldsymbol{U}$ and $\boldsymbol{V}$,

$$\boldsymbol{L}_0^{\top} \boldsymbol{L}_c, = \boldsymbol{U} \boldsymbol{\Sigma}_1 \boldsymbol{V}.$$

3) Computing the nonsingular transformation matrix

$$\boldsymbol{T} = \boldsymbol{L}_c \boldsymbol{V} \boldsymbol{\Sigma}_1^{-1/2}$$

The balanced system in (17) has exactly the same inputs and outputs behavior as the original system described in (13), and its controllable Gramians matrix $\overline{\boldsymbol{W}}_c$ and observable Gramians matrix $\overline{\boldsymbol{W}}_0$ are the same, and can be computed by the balanced transformation matrix $\boldsymbol{T}$ as listed below

$$\begin{aligned} \overline{\boldsymbol{W}}_c &= \boldsymbol{T} \boldsymbol{W}_c \boldsymbol{T}^{\top}, \\ \overline{\boldsymbol{W}}_0 &= (\boldsymbol{T}^{-1})^{\top} \boldsymbol{W}_0 \boldsymbol{T}^{-1}, \\ \overline{\boldsymbol{W}}_c &= \overline{\boldsymbol{W}}_0 = \boldsymbol{\Sigma} \end{aligned}$$

where, $\boldsymbol{\Sigma}$ is diagonal matrix, and λ_i, the elements of $\boldsymbol{\Sigma}$, are the Hankel singular value, satisfying $\lambda_1 \geq \ldots \lambda_n \geq 0$.

The zero λ_i is then dumped, as the corresponding input and output does not affect the state of system. Also, λ_i with very small value may also be omitted, therefore reducing the order of the system model even further. Finally, the states of the balanced system can be classified as important r states and unimportant $(n-r)$ ones, that is,

$$\overline{\mathbf{x}}(t) = \begin{pmatrix} \tilde{\mathbf{x}}_1(t) \\ \tilde{\mathbf{x}}_2(t) \end{pmatrix}$$

where, $\tilde{\mathbf{x}}_1(t) = \boldsymbol{P}\tilde{\mathbf{x}}(t) \in \mathbb{R}^r$ are important state variables preserved in the reduced model, whereas $\tilde{\mathbf{x}}_2(t) = \boldsymbol{Q}\tilde{\mathbf{x}}(t) \in \mathbb{R}^{n-r}$ are unimportant ones being omitted. Then the balanced model of system in (13) is described by

$$\begin{cases} \dot{\tilde{\mathbf{x}}}_1(t) = \boldsymbol{PTf}\big(\boldsymbol{T}^{-1}\left[\tilde{\mathbf{x}}_1(t) \;\; \tilde{\mathbf{x}}_2(t)\right]^\top, \boldsymbol{u}(t)\big), \\ \dot{\tilde{\mathbf{x}}}_2(t) = \boldsymbol{QTf}\big(\boldsymbol{T}^{-1}\left[\tilde{\mathbf{x}}_1(t) \;\; \tilde{\mathbf{x}}_2(t)\right]^\top, \boldsymbol{u}(t)\big), \\ \boldsymbol{y}(t) = \boldsymbol{h}\big(\boldsymbol{T}^{-1}\left[\tilde{\mathbf{x}}_1(t) \;\; \tilde{\mathbf{x}}_2(t)\right]^\top\big), \end{cases}$$

where $\boldsymbol{P} = [I_r \;\; 0_{r,(n-r)}] \in \mathbb{R}^{r\times n}$, $\boldsymbol{Q} = [0_{(n-r),r} \;\; I_{(n-r)}] \in \mathbb{R}^{(n-r)\times n}$, $\boldsymbol{I}_r$ is the identical matrix, and $0_{n1,n2}$ is the 0-matrix.

By using the truncation method, equations with $\dot{\tilde{\mathbf{x}}}_2(t)$ can be eliminated. After rewriting $\tilde{\mathbf{x}}_1(t)$, $\dot{\tilde{\mathbf{x}}}_1(t)$ as $\mathbf{x}(t)$ and $\dot{\tilde{\mathbf{x}}}(t)$, the balanced reduced model of the nonlinear power system is then given by

$$\begin{cases} \dot{\tilde{\mathbf{x}}}_1(t) = \boldsymbol{PTf}\big(\boldsymbol{T}^{-1}\boldsymbol{P}^\top\tilde{\mathbf{x}}(t), \boldsymbol{u}(t)\big), \\ \boldsymbol{y}(t) = \boldsymbol{h}\left(\boldsymbol{T}^{-1}\boldsymbol{P}^\top\tilde{\mathbf{x}}(t)\right), \end{cases} \tag{18}$$

where $\tilde{\mathbf{x}}(t) \in \mathbb{R}^r$.

4 Balanced reduction of nonlinear power system model

4.1 Computation of empirical Gramians matrix

Because of involving integral calculation, the computation burden is heavy for computing empirical controllable Gramians matrix in (15) and observable Gramians matrix in (16). We use data samples of systems states and outputs under various excitations obtained by simulation method to reduce the computation complexity. Assume that all these data samples are obtained at discrete time interval $t_1, \ldots, t_q$, then the correlation matrix of states or outputs in (14) can be expressed in the following discrete manner,

$$\boldsymbol{R} = \sum_{k=1}^{q}(\boldsymbol{x}_k - \boldsymbol{x}^m)(\boldsymbol{x}_k - \boldsymbol{x}^m)^\top. \tag{19}$$

therefore the discrete form of $\boldsymbol{W}_c$ and $\boldsymbol{W}_0$ can be expressed as

$$\boldsymbol{W}_c = \sum_{l=1}^{r}\sum_{m=1}^{s}\sum_{i=1}^{p}\frac{1}{rsc_m^2}\sum_{k=0}^{q}\boldsymbol{\Phi}_k^{ilm}, \tag{20}$$

$$\boldsymbol{W}_0 = \sum_{l=1}^{r}\sum_{m=1}^{s}\frac{1}{rsc_m^2}\sum_{k=0}^{q}\boldsymbol{T}_l\boldsymbol{\Psi}_{kij}^{lm}\boldsymbol{T}_l^\top, \tag{21}$$

where $\boldsymbol{\Phi}_k^{ilm} = \big(\boldsymbol{x}_k^{ilm} - \boldsymbol{x}^m\big)\big(\boldsymbol{x}_k^{ilm} - \boldsymbol{x}^m\big)^\top$, $\boldsymbol{\Psi}_{kij}^{lm} = \big(\boldsymbol{y}_k^{ilm} - \boldsymbol{y}^m\big)^\top\big(\boldsymbol{y}_k^{jlm} - \boldsymbol{y}^m\big)$.

$\boldsymbol{W}_c$ and $\boldsymbol{W}_0$ can be obtained by (20), (21) with empirical data samples. The computation accuracy of $\boldsymbol{W}_c$ and $\boldsymbol{W}_0$ depends on the number of data samples and parameters such as r, s, and c_m. The more the data samples, the higher the accuracy, although with a cost of heavier computation burden.

4.2 Balanced reduction algorithm of nonlinear power system model (BRNPS)

After obtaining the $\boldsymbol{W}_c$ and $\boldsymbol{W}_0$ by the method in the last section, we can calculate the transformation matrix $\boldsymbol{T}$, then the calculated empirical controllable Gramians matrix $\overline{\boldsymbol{W}}_c$ and empirical observable Gramians matrix $\overline{\boldsymbol{W}}_0$ of the balanced system, which, according to Section 2.2, are the same and a diagonal Hankel singular matrix. The optimal value of rank of the subspace can be determined by the elements of the Hankel matrix. Furthermore, the Galerkin projection matrix $\boldsymbol{P}$ is decided. And the reduction model of nonlinear power system is finally obtained. The procedure is described by the following algorithm.

Algorithm: Balanced Reduction of Nonlinear Power System(BRNPS)

Input: Empirical controllable Gramians matrix $\boldsymbol{W}_c$ and observable Gramians matrix $\boldsymbol{W}_0$.

Output: Reduction model of non-linear power system.

it Initialization: The error threshold ε.

- Computing balanced transformation matrix $\boldsymbol{T}$

(1) STEP 1: Applying Cholesky decomposition method to W_c and $\boldsymbol{W}_0$ to obtain $\boldsymbol{L}_c$ and $\boldsymbol{L}_0$.

(2) Computing the diagonal matrix $\boldsymbol{\Sigma}_1$, orthogonal $\boldsymbol{U}$ and $\boldsymbol{V}$.

(3) Computing the balanced transformation matrix $\boldsymbol{T}$.

(4) Obtaining the balanced system (17).

- STEP 2: Computing balanced controllable Gramian matrix $\boldsymbol{W}_c$ and balanced observable Gramian matrix $\boldsymbol{W}_0$
- STEP 3: Computing Hankel singular matrix $\boldsymbol{\Sigma}$.
- STEP 4: Computing non-linear reduction model (18) via T and P.
- STEP 5: Stop

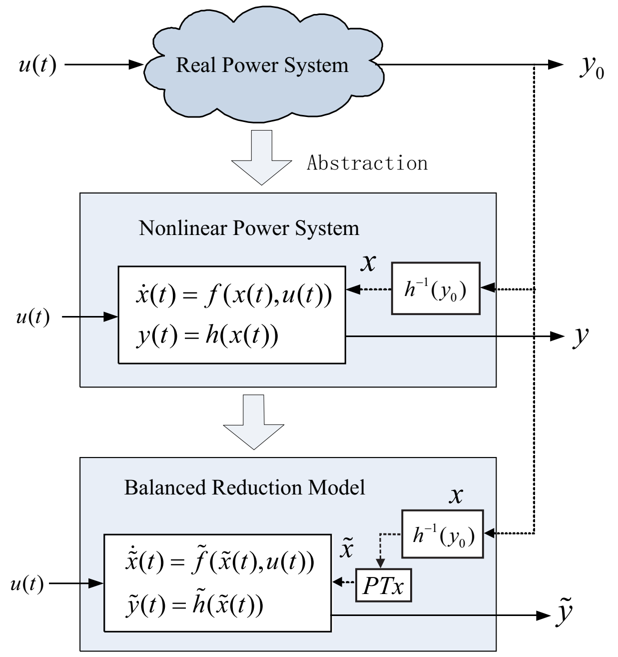

Fig. 1. The relation between the balanced and the original

4.3 The relation between the balanced reduction model and original system model

Figure 1 explains the relation between the abstract nonlinear power system model and its balanced reduction model to help understand how the method of model reduction above can be used in various power system applications.

For a real power system, $\boldsymbol{u}(t)$ is the input of external controller, and $\boldsymbol{y}_0$ is the output of the system,*ie* measurements of sensors.

The object of model reduction refers to power system model, hence for its controllers, such as excitation controllers, power system stabilizers, static var compensators and so on, should not be considered in the process of power system model order reduction. The reason is that when the power system model includes its controllers, the reduced order model obtained is difficult to use in practice, especially for the controller design.

Nonlinear model in (13) can be used to describe a power system with input $\boldsymbol{u}(t)$ and output $\boldsymbol{y}(t)$. The input of the model, which is the same as the real power system, can be obtained from the output of various power system controllers, while the initial values of system state $\boldsymbol{x}(t)$ can be obtained by suitable transformation of outputs of sensors, that is,

$$\boldsymbol{x}(0) = \boldsymbol{h}^{-1}(\boldsymbol{y}_0)\,. \tag{22}$$

Therefore, the output of power system model $\boldsymbol{y}(t)$ should be similar to the output of real measurements $\boldsymbol{y}_0$, meaning that the nonlinear model could be used to describe the real power system.

The balanced reduction model in (18) has the same inputs as those of the original system model in (13), and the outputs of controllers can be directly applied to the inputs of the reduction model. Although the states $\tilde{\mathbf{x}}(t)$ of the balanced model do not have clear physical meaning, its application in power system is not affected. Because the information the states contains is the same as the physically meaningful states $\mathbf{x}(t)$ of the original system model, and every state in $\tilde{\mathbf{x}}$ contains much dynamic information of all states of original system, and there exists a transformation between $\tilde{\mathbf{x}}(t)$ and $\mathbf{x}(t)$ by balanced transformation matrix $\boldsymbol{T}$ and Galerkin projection matrix $\boldsymbol{P}$ as

$$\mathbf{x}(t) = \boldsymbol{T}^{-1}\boldsymbol{P}^{\top}\tilde{\mathbf{x}}(t)\,. \tag{23}$$

Therefore, the output of the balanced reduction system is also similar to those of the measuring sensors, and the reduction model by the empirical Gramians balanced reduction method can approximate the original system with an acceptable error.

When doing disturbance analysis by the reduced system model, the disturbance can also be transformed to the equivalent balanced model by matrix $\boldsymbol{T}$ and $\boldsymbol{P}$, as described in (18). Therefore, different failures in real power system can also be simulated in the reduced system model.

5 Case study

In this section, two different test systems are used to demonstrate the effectiveness of the proposed method of model order reduction.

The balanced reduction method firstly makes the original system model transform onto the balanced one whose the model orders are the same as the original system through the transformation matrix, and then the dynamic model of balanced system is projected onto the low-dimensional space from the high one, *ie* some balanced state variables which have little influence on the input and output of system are truncated. However, it is not meaning that some states of the original power system are vanished. The reason is that the states of balanced reduction system are still the principal components of all states of original system even though the balanced system model is reduced to a 1-order differential equation. The balanced reduction method is not simplified or ignored the certain physical parameters of original system model. Its reduced principal is different in essence with the equivalent method usually used in power system. Because any parameters of original system model is not simplified in the process of the balanced reduction, and it is not suitable for comparing the balanced reduction method with the traditional equivalent simplified method, so it is more reasonable to compare the various-order reduced model with the original system model in the following simulation and analysis.

In power system steady state, the simulation results of all balancing reduced order models compared with the original system model are completely overlapped due to the same initial state, and the effectiveness of the model order reduction can not be demonstrated. Therefore, power system behavior under failures is considered to compare the dynamics of the reduced power system model with the original power system model to verify the proposed model reduction method.

5.1 Single-machine-to-infinite system model analysis

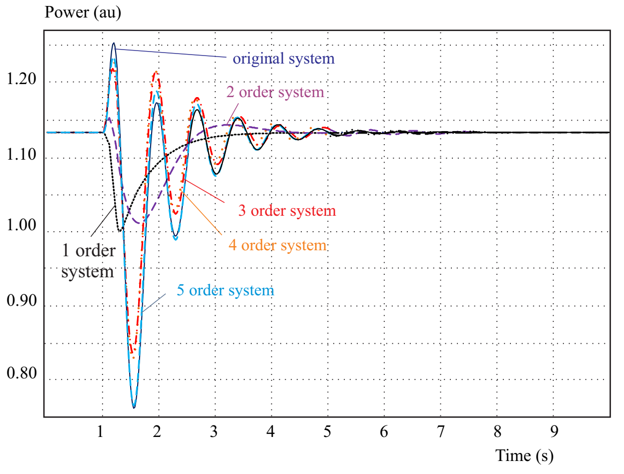

Fig. 2. Power angles of reduced systems

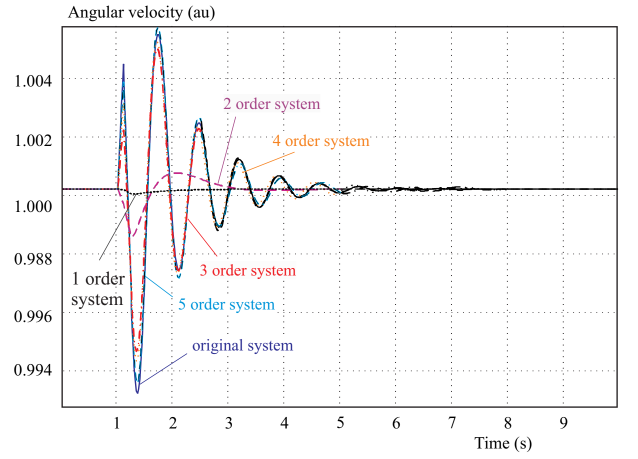

Fig. 3. Angle velocities of reduced systems

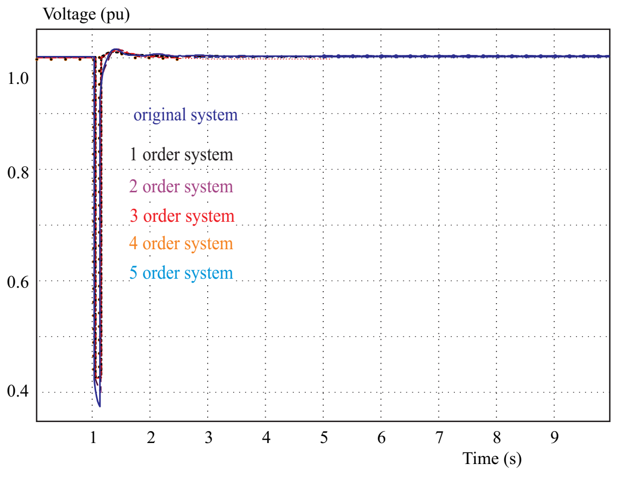

Fig. 4. Voltages of reduced systems

Table 1. Parameters of single generator-to-infinite bus system parameters of single generator-to-infinite bus system

Parameter	Data	Parameter	Data
x_d	2.07	H	6
x_q	1.99	T'_{d0}	4.1
x'_d	0.28	T'_{q0}	0.215
$x'{}_q$	0.49	T''_{d0}	0.033
x''_d	0.215	T''_{d0}	0.56
x''_q	0.49	x_Σ	0.1

Using the method introduced in Section 2.2 and 3.1, the diagonal singular matrix $\boldsymbol{\Sigma}$ and matrices $\overline{\boldsymbol{W}}_c$ and $\overline{\boldsymbol{W}}_0$ can be calculated as

$$\begin{aligned}\boldsymbol{\Sigma} = \overline{\boldsymbol{W}}_c = \overline{\boldsymbol{W}}_0 = \\ = \operatorname{diag}\{0.40311,\ 0.07592, \\ 0.05340,\ 0.00080, \\ 0.00018,\ 0.00001\}\end{aligned}$$

As we can see that $\lambda_3 \gg \lambda_4$, or from the view of the energy, the first 3 singular values carry about 99.8 % of the overall energy, therefore, the 6-order nonlinear system can be reduced to a 3-order system.

Figures 2–4 give the outputs of the reduced system and the original system under three-phase short circuit failure, *ie* the simulation curves of power angle, angular velocity, and generator terminal voltage. It can be clearly seen, in Figs. 2–4, that the dynamics of 5-order, 4-order, and 3-order reduced system are almost exactly the same as that of the original system, with very small errors. When the system is further reduced to 2-order system, even 1-order system, the error along the dynamics is relatively big because too much energy is lost. In another word, the dynamic behavior between system input and output and system states can be enough preserved when using 3-order reduced system to approximate the original 6-order system. Although there is 3-order utility model of the synchronous machine in the power system generator model, the 3-order reduction model has the similar precision of 6-order model of the synchronous machine and the approximative calculation complexity of 3-order utility model of the synchronous machine.

5.2 Multi-generator power system model analysis

A 15-generator power system model is taken as an example for model reduction analysis, and its topology is given in Fig. 5.

In Fig. 5, a synchronous generator model is described by its six-order utility dynamic model, and the load models can be chosen to be the constant impedance, the constant current and the constant power. The parameters for synchronous generators, transmission lines and loads in Fig. 5 are from [22].

Because the generator model is described under dq coordinates, and the network model is about xy coordinates, the xy–dq coordinate transformation is needed for the interconnection of generators and network. Meanwhile, the constant current loads and the constant power

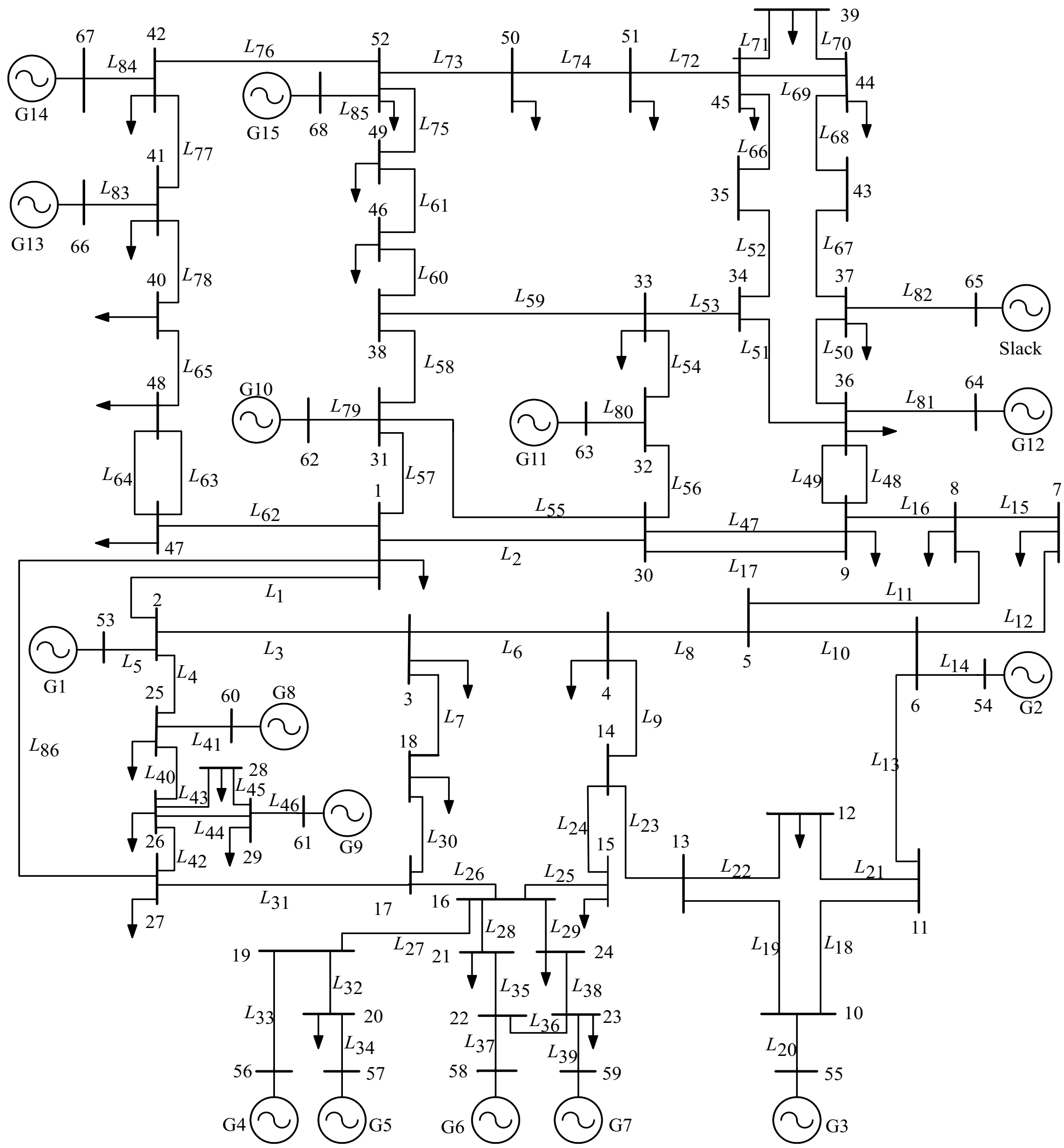

Fig. 5. 15-generator test system

loads are the non-conforming loads, and they are connected to the network via the current injections which are functions of the load buses voltages. Therefore, the equations the transmission network with generators and loads are

$$\begin{bmatrix} \boldsymbol{I}_{xyG} \\ \boldsymbol{I}_{xyL} \end{bmatrix} = -\boldsymbol{Y} \begin{bmatrix} \boldsymbol{V}_{xyG} \\ \boldsymbol{V}_{xyL} \end{bmatrix}. \tag{24}$$

Finally, a 90-order nonlinear power system dynamic model in the form of (13) is obtained. This 90-order nonlinear model is a dynamic model of the whole power system, and should not be seen as just the generator model, only due to not considering dynamic model of loads in the test system.

By using the previous model reduction algorithm (BRNPS), the balanced transformation matrix $\boldsymbol{T}$ and Galerkin projection $\boldsymbol{P}$ can be obtained. The distribution of the Hankel singular values of the balanced system is given in Fig. 6.

When we choose the threshold ε to be 0.99, the original 90-order nonlinear power system can be projected onto a 35-order subspace; when ε is 0.95, the order of the reduced system is 19. The smaller the ε, the lower the order of the reduced system model, and also the bigger the distortion of the dynamic behavior of the reduced system. Careful consideration is needed when choosing ε. The comparison of various ε and the simulation of the corresponding simulation result are given in Figs. 7–9.

In the simulation, governor control and excitation control for each generator are considered, and the parameters of these controllers are kept the same in the reduce system model and in the original system model. Reduced model

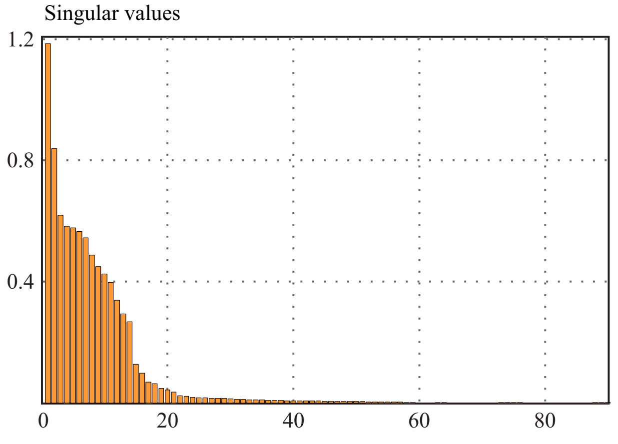

Fig. 6. Hankel singular values

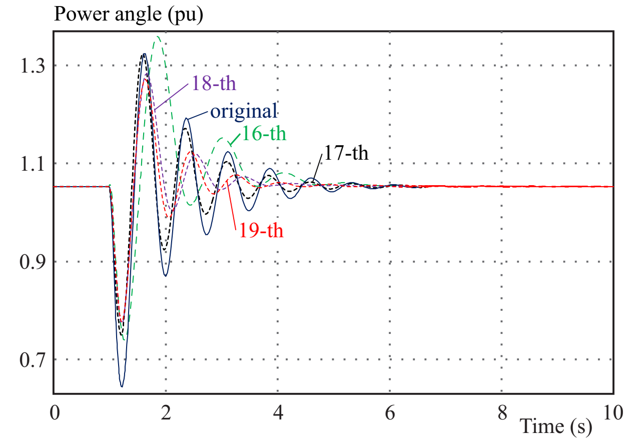

Fig. 7. Power angle curves of G14

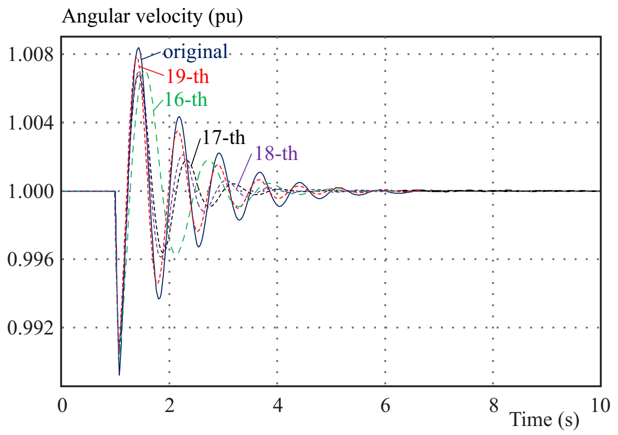

Fig. 8. Angle velocity curves of G14

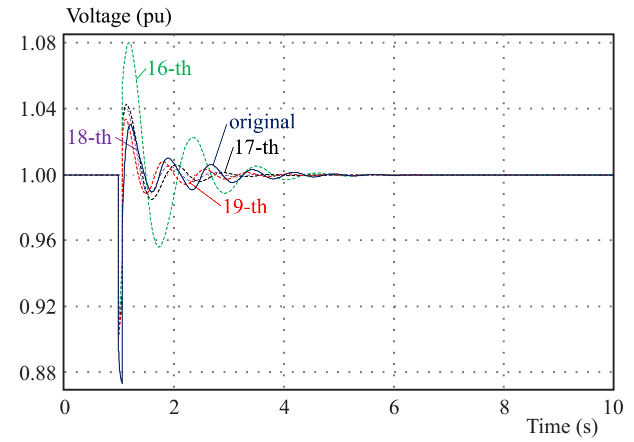

Fig. 9. Angle velocity curves of G14

with different orders are simulated under a three-phase short circuit failure.

Without loss generality, any generator or bus in the test system could be chosen for detail analysis. Figure 5 gives the simulation curves of the generator G14 with a three-phase short circuit failure on the line L_{11}. Since the dynamics (such as the frequency and the amplitude of the output) in 20-order and above reduced systems are almost the same as those of the original systems, only power angle, angular velocity and generators terminal voltage of the original system, the 19-order reduced system to the 16-order reduced system are given in Figs. 7–9.

It can be seen from Figs. 7–9 that, for the 19-order reduced system, the errors of angular velocity and the amplitude of the voltage are very small, while that of the power angle are a little bigger, especially, the frequency response of the three variables (that is, the transients) are almost the same, compared to those of the original system. When the order of the model is reduced to 18, the amplitudes of the three outputs change very little, while the frequencies of those outputs have bigger changes, and there are distortions in the transients. When the system model is further reduced to 16, even bigger distortions happen in the amplitude and frequency of the outputs. By analyzing Hankel singular values, the energy projected onto from the 16-order to the 19-order subspaces is about 93.22 %, 94.04 %, 94.8 % and 95.36 % of the original system respectively.

By analyzing the simulation result of all nodes in the 15-generatro test system, it is found when $\varepsilon > 0.94$, the error introduced by model reduction is small, and the dynamics of the original system can be maintained very well. Therefore, $\varepsilon = 0.95$ is chosen for the test system, and the order of the reduced system is chosen to be 19.

Meantime, it is can be seen from the simulation results that, for the reduced system with various orders, the outputs steady states of the reduced system and those of the original system are equal. Especially, empirical Gramians balanced reduction approach can keep the systems steady state the same for lower order (even 1-order) reduced system.

6 Conclusion

In this paper, an empirical Gramians balanced reduction approach for multi-generator nonlinear power system model is proposed, and simulation tests are performed on a single generator-to-infinite bus power system and 15-generator nonlinear power system. Simulation results show that the reduced model can preserve the dynamics of inputs and outputs and the steady state of the original nonlinear power system. Especially, for the 15-generator nonlinear power system, the order of the 90-order nonlinear dynamic model can be reduced to 19, being 1/5 of that of the original system model. Simulation results verify the effectiveness of the proposed approach.

Meantime, the research of the paper shows that the order of the reduced model can be chosen according to the accuracy requirements of the power system application. When focusing on the overall dynamic behaviors of the power system (for example, the higher level dynamic model in hierarchical control), a subspace with a lower order could be chosen, and a rough dynamic model maintaining the overall dynamic behavior can be obtained; while when focusing on the transient behavior of the system, a subspace with a higher order could be chosen to obtain a reduced model with smaller error to the original system.

Our further research work will be on studying the error analysis method of the balanced reduced nonlinear power system model.

References

[1] A. M. Miah, "Study of a Coherency-based Simple Dynamic Equivalent for Transient Stability Assessment", *IET Gener. Transm. Distrib.*, 2011, vol 5, no. 4, pp. 405–416.

[2] A. Akhavein, M. Billinton, R. Fotuhi-firuzabad, *et al* "Adequacy Equivalent Development Composite Generation and Transmission Systems Using Network Screening", *IET Gener. Transm. Distrib.*, 2011, vol. 5, no. 11, pp. 1141–1148.

[3] A. Vahidnia, G. Ledwich, E. Palmer and A. Ghosh, "Generator Coherency and Area Detection Large Power Systems", *IET Gener. Transm. Distrib.*, 2012, vol. 6, no. 9, pp. 874–883.

[4] S. Kim and T. J. Overbye, "Mixed Power Flow Analysis using AC and DC Models", *IET Gener. Transm. Distrib.*, 2012, vol. 6, no. 10, pp. 1053–1059.

[5] Y. Mishra, Z. Y. Dong, J. Ma and D. J. Hill, "Induction Motor Load Impact on Power System Eigenvalue Sensitivity Analysis", *IET Gener. Transm. Distrib.*, 2009, vol. 3, no. 7, pp. 690–700.

[6] R. Agrawal and D. Thukaram, "Support Vector Clustering-based Direct Coherency Identification of Generators a Multi-machine Power System", *IET Gener. Transm. Distrib.*, 2013, vol. 7, no. 12, pp. 1357–1366.

[7] E. Pajuelo, R. Gokaraju and M. S. Sachdev, "Identification of Generator Loss-of-excitation From Power-swing Conditions using a Fast Pattern Classification Method", *IET Gener. Transm. Distrib.*, 2013, vol. 7, no. 1, pp. 24-36.

[8] G. Y. Cao and D. J. Hill, "Power system voltage small-disturbance stability studies based on the power flow equation", *IET Gener. Transm. Distrib.*, 2010, vol. 4, no. 7, pp. 873-882.

[9] A. Ishchenko, J. M. A. Myrzik and W. L. Kling, "Dynamic Equivalencing of Distribution Networks with Dispersed Generation using Hankel Norm Approximation", *IET Gener. Transm. Distrib.*, 2007, vol. 1, no. 5, pp. 818–825.

[10] J. Thambirajah, E. Barocio and N. F. Thornhill, "Comparative Review of Methods for Stability Monitoring Electrical Power Systems and Vibrating Structures", *IET Gener. Transm. Distrib.*, 2010, vol. 4, no. 10, pp. 1086–1103.

[11] D. Chaniotis and M. A. Pai, "Model Reduction Power Systems Using Krylov Subspace Methods", *IEEE Trans.Power Syst.*, 2005, vol. 3, no. 20, pp. 888–894.

[12] J. J. Sanchez-Gasca and J. H. Chow, "Power System Reduction to Simplify the Design of Damping Controllers for Interarea Oscillations", *IEEE Trans.Power Syst.*, 1996, vol. 3, no. 11, pp. 1342–1349.

[13] N. Martins and P. E. Quintao, "Computing Dominant Poles of Power System Multivariable Transfer Functions", *IEEE Trans.Power Syst.*, 2003, vol. 18, no.1, pp. 152-159.

[14] A. C. Antoulas and D. C. Sorensen, "Approximation of Large -scale Dynamical Systems: an Overview", *Int. J. Appl. Math. Comput. Sci.*, 2001, vol. 5, no. 11, pp. 1093–1121.

[15] L. Sirovich, "Turbulence and the Dynamic of Coherent Structures part III: Dynamics and Scaling", *Quarterly of Applied Mathematics*, 1987, vol. 45, pp. 583–590.

[16] S. Lall, J. E. Marsden and S. Glavaski, "A Subspace Approach to Balanced Truncation for Model Reduction of Nonlinear Control System", *Int.Journal of Robust and Nonlinear Control*, 2002, vol. 12, pp. 519–535.

[17] J. Hahn and F. Thomas, "Balancing Approach to Minimal Realization and Model Reduction of Stable Nonlinear System. System", *Industrial and Engineering Chemistry Research*, 2002, vol. 9, no. 41, pp. 2204–2212.

[18] C. Sun and J. Hahn, "Reduction of Stable Differential-algebraic Equation Systems via Projection and System Identification", *Journal of process control*, 2005, vol. 15, pp. 639–650.

[19] C. Sun and J. Hahn, "Model Reduction the Presence of Uncertainty Model Parameters", *Journal of Process Control*, 2006, vol. 16, pp. 645–649.

[20] P. M. Anderson and A. A. Fouad, *Power System Control and Stability*, New Jersey, 1994, IEEE Press.

[21] A. C. Antoulas, "A new result on passivity preserving model reduction", *System and Control Letters*, 2005, vol. 54, pp. 361-374.

[22] J. H. Chow and G. J. Rogers, "Power System Toolbox", [Online], http://www.eps.ee.kth.se/personal/vanfretti/pst/Power_System_ToolboxWebpage/PST.html.

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