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UNIFIED APPROACH TO THE IMPULSE RESPONSE AND GREEN FUNCTION IN THE CIRCUIT AND FIELD THEORY PART II : MULTI–DIMENSIONAL CASE

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In the circuit theory the concept of the impulse response of a linear system due to its excitation by the Dirac delta function $\delta(t)$ together with the convolution principle is widely used and accepted. The rigorous theory of symbolic functions, sometimes called distributions, where also the delta function belongs, is rather abstract and requires subtle mathematical tools [1–4]. Nevertheless, the most people intuitively well understand the delta function as a derivative of the (Heaviside) unit step function $\mathbf{1}(t)$ without too much mathematical rigor. In the previous part [5] the concept of the impulse response of linear systems was approached in a unified manner and generalized to the time-space phenomena in one dimension (transmission lines). Here the phenomena in more dimensions (static and dynamic electromagnetic fields) are treated. It is shown that many formulas in the field theory, which are often postulated in an inductive way as results of the experiments, and therefore appear as "deux ex machina" effects, can be mathematically deduced from a few starting equations.

K e y w o r d s: circuit theory, field theory, impulse response, Green function

1 INTRODUCTION

It has been shown in the previous paper [5] that the use of impulse response of a linear system as a reaction to the excitation by the Dirac impulse, or δ-function, is a very fruitful concept of electrical engineering education, making possible to calculate easily the response of the system to any excitation. Mainly in the circuit theory it is wide-spread and often used approach. We have shown that this approach can be easily and naturally transposed from time domain problems (circuit theory) to space domain problems as *eg* the one-dimensional harmonic steady state of wave propagation and excitation, as well as to one-dimensional time-space wave equation — representing *eg* the waves on the transmission line.

Here we shall further develop this approach to the treatment of more dimensional linear systems of electromagnetics, particularly the static electric and magnetic fields as well the dynamic electromagnetic fileds.

2 THE IMPULSE RESPONSE FOR THE LAPLACE OPERATOR

Let the solution of the Poisson equation

$$\nabla^2 V(\boldsymbol{r}) = -q(\boldsymbol{r}) , \qquad (1)$$

in the infinite space be considered, where the radius vector of the Cartesian coordinate system is $\boldsymbol{r} = x\boldsymbol{u}_x + y\boldsymbol{u}_y + z\boldsymbol{u}_z$, $\varphi(\boldsymbol{r}) = V(x,y,z)$, $q(\boldsymbol{r}) = q(x,y,z)$, and the Laplace operator in Cartesian coordinate system, reads

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \qquad (2)$$

The impulse response $g(\boldsymbol{r})$ for the Laplace operator is the solution of the equation

$$\nabla^2 g(\boldsymbol{r}) = -\delta(\boldsymbol{r}) . \qquad (3)$$

The three dimensional δ-function $\delta(\boldsymbol{r})$ is in the Cartesian co-ordinate system defined as

$$\delta(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z) . \qquad (4)$$

The impulse response $g(\boldsymbol{r})$ can be obtained after the three dimensional Fourier transform (63) of (3)

$$\left[(jk_x)^2 + (jk_y)^2 + (jk_z)^2\right]\mathcal{G}(j\boldsymbol{k}) = -1 . \qquad (5)$$

Since it possesses spherical symmetry one can simply write (see Appendix I.)

$$\mathcal{G}(j\boldsymbol{k}) = \mathcal{G}_3(k) = 1/k^2$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. Using the one dimensional inverse Fourier transform of (5) analogously as the formulae (12) and (14) in [5] we obtain

$$1/k^2 \overset{1D\text{–}FT}{\longleftrightarrow} (-r/2)[\mathbf{1}(r) - \mathbf{1}(-r)] , \qquad (6)$$

and the use of (73) yields

$$g(r) = -\frac{1}{2\pi r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(-\frac{r}{2}\Big) = \frac{1}{4\pi r} , \qquad (7)$$

or written vectorially

$$g(\boldsymbol{r}) = 1/4\pi\,|\boldsymbol{r}| . \qquad (8)$$

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The solution of the Poisson equation of electrostatics

$$\nabla^2 V(\boldsymbol{r}) = -\rho(\boldsymbol{r})/\varepsilon\,, \tag{9}$$

where $V(\boldsymbol{r})$ is the scalar potential of the intensity of electric field, $\rho(\boldsymbol{r})$ the volume charge density and ε_0 the permittivity of vacuum, is then in the whole infinite space merely the convolution

$$V(\boldsymbol{R}) = \iiint\limits_{\infty} \frac{\rho(\boldsymbol{r})}{4\pi\varepsilon_0\,|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,. \tag{10}$$

Similarly, the Poisson equation for the vector potential $\boldsymbol{A}(\boldsymbol{r})$ of a stationary magnetic field

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = -\mu_0 \boldsymbol{J}(\boldsymbol{r})\,, \tag{11}$$

where $\boldsymbol{J}(\boldsymbol{r})$ is the density of volume current and μ_0 the permeability of vacuum, possesses the analogous convolutional solution

$$\boldsymbol{A}(\boldsymbol{R}) = \iiint\limits_{\infty} \frac{\mu_0 \boldsymbol{J}(\boldsymbol{r})}{4\pi\,|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,, \tag{12}$$

provided the Coulomb gauge is used for calibration of the vector potential.

Notice the difference in Laplace operator applied to the scalar or vector field. While $\nabla^2 V(\boldsymbol{r})$ has the menaing $\nabla^2 V(\boldsymbol{r}) = \operatorname{div}\operatorname{grad} V(\boldsymbol{r})$, the vectorial expression $\nabla^2 \boldsymbol{A}(\boldsymbol{r})$ means

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = \operatorname{grad}\operatorname{div}\boldsymbol{A}(\boldsymbol{r}) - \operatorname{curl}\operatorname{curl}\boldsymbol{A}(\boldsymbol{r})\,.$$

Herewith we have obtained the alternate expressions for the three-dimensional delta function $\delta(\boldsymbol{R}-\boldsymbol{r})$ in the form

$$\delta(\boldsymbol{R}-\boldsymbol{r}) = -\nabla_r^2 \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|} = -\operatorname{div}_r \frac{\boldsymbol{R}-\boldsymbol{r}}{4\pi|\boldsymbol{R}-\boldsymbol{r}|^3}\,, \tag{13}$$

or

$$\delta(\boldsymbol{R}-\boldsymbol{r}) = -\nabla_R^2 \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|} = \operatorname{div}_R \frac{\boldsymbol{R}-\boldsymbol{r}}{4\pi|\boldsymbol{R}-\boldsymbol{r}|^3}\,. \tag{14}$$

In (13) and (14) $\boldsymbol{R}$ has the same meaning of a radius vector as $\boldsymbol{r}$, one needs just to distinguish with respect to which variable the differential operators are applied (indexes r, or R).

3 FUNDAMENTAL THEOREM BY HERMANN HELMHOLTZ FOR VECTOR FIELDS

Let us consider the vector field $\boldsymbol{F}(\boldsymbol{r})$ and the sifting property of the δ-function together with (14) over the infinite space

$$\boldsymbol{F}(\boldsymbol{R}) = \iiint\limits_{\infty} \boldsymbol{F}(\boldsymbol{r})\delta(\boldsymbol{R}-\boldsymbol{r})\,\mathrm{d}v\,. \tag{15}$$

Substituting to (15) from (14) gives

$$\boldsymbol{F}(\boldsymbol{R}) = -\iiint\limits_{\infty} \boldsymbol{F}(\boldsymbol{r})\nabla_R^2 \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,. \tag{16}$$

where ∇_R^2 is understood as $\nabla_R^2 = \operatorname{div}_R \operatorname{grad}_R$. Since the Laplace operator operates on the variable $\boldsymbol{R}$, it can be taken outside the integral

$$\boldsymbol{F}(\boldsymbol{R}) = -\nabla_R^2 \iiint\limits_{\infty} \boldsymbol{F}(\boldsymbol{r}) \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,, \tag{17}$$

but now it operates on the vectorial expression, therefore ∇_R^2 must be understood as
$\nabla_R^2 = \operatorname{grad}_R \operatorname{div}_R - \operatorname{curl}_R \operatorname{curl}_R$. Since

$$\begin{aligned}\operatorname{div}_R \iiint\limits_{\infty} \frac{\boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v &= \iiint\limits_{\infty} \boldsymbol{F}(\boldsymbol{r})\cdot\operatorname{grad}_R \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v \\ = -\iiint\limits_{\infty} \boldsymbol{F}(\boldsymbol{r})\cdot\operatorname{grad}_r \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v &= \iiint\limits_{\infty} \frac{\operatorname{div}_r \boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v \\ &- \iiint\limits_{\infty} \operatorname{div}_r \frac{\boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,, \end{aligned} \tag{18}$$

and similarly

$$\begin{aligned}&\operatorname{curl}_R \iiint\limits_{\infty} \frac{\boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v \\ &\quad = -\iiint\limits_{\infty} \boldsymbol{F}(\boldsymbol{r}) \times \operatorname{grad}_R \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v \\ &\quad = \iiint\limits_{\infty} \boldsymbol{F}(\boldsymbol{r}) \times \operatorname{grad}_r \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v \\ &= \iiint\limits_{\infty} \frac{\operatorname{curl}_r \boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v - \iiint\limits_{\infty} \operatorname{curl}_r \frac{\boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,, \end{aligned} \tag{19}$$

and taking into account that the last integrals in (18) and (19) must be in case of infinite domain and continuous non-singular vector field $\boldsymbol{F}(\boldsymbol{r})$ equal to zero, we arrive to the expression

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{R}) = &-\operatorname{grad}_R \iiint\limits_{\infty} \frac{\operatorname{div}\boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v \\ &+ \operatorname{curl}_R \iiint\limits_{\infty} \frac{\operatorname{curl}\boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,. \end{aligned} \tag{20}$$

Formula (20) can be written also in the form

$$\boldsymbol{F}(\boldsymbol{R}) = -\operatorname{grad} V(\boldsymbol{R}) + \operatorname{curl}\boldsymbol{A}(\boldsymbol{R})\,. \tag{21}$$

where $V(\boldsymbol{R})$ and $\boldsymbol{A}(\boldsymbol{R})$ are the scalar and vector potential of the vector field $\boldsymbol{F}(\boldsymbol{R})$.

$$V(\boldsymbol{R}) = \iiint\limits_{\infty} \frac{\operatorname{div} \boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,, \tag{22}$$

$$\boldsymbol{A}(\boldsymbol{R}) = \iiint\limits_{\infty} \frac{\operatorname{curl} \boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,. \tag{23}$$

This is the fundamental theorem saying that each vector field can have only two constituents, the lamellar, or sometimes called conservative constituent $-\operatorname{grad} V(\boldsymbol{R})$ with its origin in the volume density of scalar (spring-type) sources $\operatorname{div} \boldsymbol{F}(\boldsymbol{R})$, and the solenoidal constituent $\operatorname{curl} \boldsymbol{A}(\boldsymbol{R})$ originated by the volume density of vectorial (vortex- type) sources $\operatorname{curl} \boldsymbol{F}(\boldsymbol{R})$.

Explicitly written (20) gives

$$\boldsymbol{F}(\boldsymbol{R}) = \iiint\limits_{\infty} \frac{\operatorname{div} \boldsymbol{F}(\boldsymbol{r})(\boldsymbol{R}-\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v + \iiint\limits_{\infty} \frac{\operatorname{curl} \boldsymbol{F}(\boldsymbol{r}) \times (\boldsymbol{R}-\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,, \tag{24}$$

where one easily recognizes the form of the first term as the well-known Coulomb law of electrostatics and the form of the second one as the Biot-Savart law of stationary magnetic fields.

Let us divide the whole infinite space into two nonoverlapping simply connected domains Ω_1 and Ω_2 with a common boundary — the closed surface Σ. Let us define the unit-step function $\mathbf{1}_{\Omega 1}(\boldsymbol{r})$ over Ω_1, and analogously $\mathbf{1}_{\Omega 2}(\boldsymbol{r})$ over Ω_2. Then the relation (79)

$$\begin{aligned} \operatorname{grad} \mathbf{1}_{\Omega 1}(\boldsymbol{r}) &= -\boldsymbol{n}_{12}\delta_\Sigma(\boldsymbol{r})\,, \\ \operatorname{grad} \mathbf{1}_{\Omega 2}(\boldsymbol{r}) &= \boldsymbol{n}_{12}\delta_\Sigma(\boldsymbol{r})\,, \end{aligned} \tag{25}$$

where $\boldsymbol{n}_{12}$ is the normal unit vector of the boundary surface directed from Ω_1 to Ω_2, holds.

Now, the vector field $\boldsymbol{F}(\boldsymbol{r})$ in the whole infinite space will be written in the form

$$\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{F}_1(\boldsymbol{r})\mathbf{1}_{\Omega 1}(\boldsymbol{r}) + \boldsymbol{F}_2(\boldsymbol{r})\mathbf{1}_{\Omega 2}(\boldsymbol{r}) \tag{26}$$

denoting thus explicitly the values of $\boldsymbol{F}(\boldsymbol{r})$ in Ω_1 as $\boldsymbol{F}_1(\boldsymbol{r})$ and in Ω_2 as $\boldsymbol{F}_2(\boldsymbol{r})$. This leads to the formula

$$\begin{aligned} \operatorname{div} \boldsymbol{F}(\boldsymbol{r}) &= \operatorname{div} \boldsymbol{F}_1(\boldsymbol{r})\mathbf{1}_{\Omega 1}(\boldsymbol{r}) + \boldsymbol{F}_1(\boldsymbol{r}) \cdot \operatorname{grad} \mathbf{1}_{\Omega 1}(\boldsymbol{r}) \\ &\quad + \operatorname{div} \boldsymbol{F}_2(\boldsymbol{r})\mathbf{1}_{\Omega 2}(\boldsymbol{r}) + \boldsymbol{F}_2(\boldsymbol{r}) \cdot \operatorname{grad} \mathbf{1}_{\Omega 2}(\boldsymbol{r})\,, \end{aligned} \tag{27}$$

or

$$\begin{aligned} \operatorname{div} \boldsymbol{F}(\boldsymbol{r}) &= \operatorname{div} \boldsymbol{F}_1(\boldsymbol{r})\mathbf{1}_{\Omega 1}(\boldsymbol{r}) - \boldsymbol{F}_1(\boldsymbol{r}) \cdot \boldsymbol{n}_{12}\delta_\Sigma(\boldsymbol{r}) \\ &\quad + \operatorname{div} \boldsymbol{F}_2(\boldsymbol{r})\mathbf{1}_{\Omega 2}(\boldsymbol{r}) + \boldsymbol{F}_2(\boldsymbol{r}) \cdot \boldsymbol{n}_{12}\delta_\Sigma(\boldsymbol{r})\,. \end{aligned} \tag{28}$$

The substitution into (22) yields

$$V(\boldsymbol{R}) = \iiint\limits_{\infty} \frac{\operatorname{div} \boldsymbol{F}_1(\boldsymbol{r})\mathbf{1}_{\Omega 1}(\boldsymbol{r}) + \operatorname{div} \boldsymbol{F}_2(\boldsymbol{r})\mathbf{1}_{\Omega 2}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v + \iiint\limits_{\infty} \frac{[\boldsymbol{F}_2(\boldsymbol{r}) - \boldsymbol{F}_1(\boldsymbol{r})] \cdot \boldsymbol{n}_{12}\delta_\Sigma(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,, \tag{29}$$

or

$$V(\boldsymbol{R}) = \iiint\limits_{\infty} \frac{\operatorname{div} \boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v + \oiint\limits_{\Sigma} \frac{\boldsymbol{n}_{12} \cdot [\boldsymbol{F}_2(\boldsymbol{r}) - \boldsymbol{F}_1(\boldsymbol{r})]}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}S\,, \tag{30}$$

and similarly for the vector potential

$$\boldsymbol{A}(\boldsymbol{R}) = \iiint\limits_{\infty} \frac{\operatorname{curl} \boldsymbol{F}(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v + \oiint\limits_{\Sigma} \frac{\boldsymbol{n}_{12} \times [\boldsymbol{F}_2(\boldsymbol{r}) - \boldsymbol{F}_1(\boldsymbol{r})]}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}S\,. \tag{31}$$

The second terms in (30) and (31) — the discontinuities in the normal and tangential values of the vector field — are called surface divergence and surface curl and represent the origins of the field in form of surface densities of the scalar (spring-type) and vectorial (vortex type) sources.

4 GREEN FORMULAE FOR THE BOUNDED PROBLEMS OF THE ELECTROSTATICS AND STATIONARY MAGNETIC FIELDS

Using (79) one can easily obtain the well known Green formula in the differential form. Let us apply the Laplace operator to the function $V(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})$, *ie* to $V(\boldsymbol{r})$ having been cut to zero outside Ω,

$$\nabla^2\{V(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\} = \operatorname{div}\operatorname{grad}\{V(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\} = \operatorname{div}\big\{\mathbf{1}_\Omega(\boldsymbol{r})\operatorname{grad} V(\boldsymbol{r}) - \boldsymbol{n}V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r})\big\}. \tag{32}$$

Further applying the divergence operator yields

$$\begin{aligned} \nabla^2\big\{V(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\big\} &= \mathbf{1}_\Omega(\boldsymbol{r})\nabla^2 V(\boldsymbol{r}) - \boldsymbol{n} \cdot \operatorname{grad} V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) \\ &\quad - \boldsymbol{n} \cdot V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\operatorname{grad}\delta_\Sigma(\boldsymbol{r})\,. \end{aligned} \tag{33}$$

Substitution from (9) gives the generalized Poisson equation of electrostatics

$$\begin{aligned} \nabla^2\big\{V(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\big\} &= -\rho(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})/\varepsilon_0 \\ &\quad - \boldsymbol{n} \cdot \operatorname{grad} V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) - \boldsymbol{n} \cdot V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\operatorname{grad}\delta_\Sigma(\boldsymbol{r})\,. \end{aligned} \tag{34}$$

Its solution is easily obtained by the convolution with the impulse response of the free infinite space

$$V(\boldsymbol{R})\mathbf{1}_\Omega(\boldsymbol{R}) = \iiint\limits_\Omega \frac{\rho(\boldsymbol{r})}{4\pi\varepsilon_0|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v + \oiint\limits_\Sigma \frac{\operatorname{grad} V(\boldsymbol{r})}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\cdot \mathrm{d}\boldsymbol{S}' - \oiint\limits_\Sigma V(\boldsymbol{r})\operatorname{grad}_r \frac{1}{4\pi|\boldsymbol{R}-\boldsymbol{r}|}\cdot \mathrm{d}\boldsymbol{S}\,. \tag{35}$$

Observe that dividing the whole infinite space into two parts — the domain Ω and the complement $\infty-\Omega$ (10) may be written as

$$V(\boldsymbol{R}) = \iiint\limits_\Omega \frac{\rho_{\text{inside}}(\boldsymbol{r})}{4\pi\varepsilon_0|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v + \iiint\limits_{\infty-\Omega} \frac{\rho_{\text{outside}}(\boldsymbol{r})}{4\pi\varepsilon_0|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v, \tag{36}$$

ie to the scalar potential $V(\boldsymbol{r})$ in any point contribute the charge densities inside Ω as well as outside Ω.

On the other hand the formula (35) says that the effect of two surface integrals on the potential inside Ω is the same as the effect of outside charges. In this sense they represent outside volume charge densities transformed into the surface charge density $-\varepsilon_0\boldsymbol{n}\cdot\operatorname{grad} V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}$ and the surface dipole moment density $-\varepsilon_0\boldsymbol{n}V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}$.

Observe also that accordingly (35) the potential outside Ω is zero. The two surface terms have the twofold effect of adding the contribution of the outside charge densities to the potential inside Ω, as well as compensating the contribution of inside charge densities to the zero potential outside Ω.

Similar considerations can be performed also for the vector potential of the magnetic field. Using the rules of the ∇ operator algebra one obtains

$$\operatorname{div}\{\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\} = \operatorname{div}\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r}) - \boldsymbol{n}\cdot\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) \tag{37}$$

and

$$\operatorname{grad}\operatorname{div}\{\boldsymbol{A}(\boldsymbol{r})\} = \operatorname{grad}\operatorname{div}\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r}) - \boldsymbol{n}\operatorname{div}\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) - \boldsymbol{n}\cdot\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\operatorname{grad}\delta_\Sigma(\boldsymbol{r})\,, \tag{38}$$

together with

$$\operatorname{curl}\{\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\} = \operatorname{curl}\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r}) - \boldsymbol{n}\times\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) \tag{39}$$

and

$$\operatorname{curl}\operatorname{curl}\{\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\} = \operatorname{curl}\operatorname{curl}\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r}) - \boldsymbol{n}\times\operatorname{curl}\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) + \boldsymbol{n}\times\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\times\operatorname{grad}\delta_\Sigma(\boldsymbol{r})\,. \tag{40}$$

Hence

$$\nabla^2\{\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\} = \mathbf{1}_\Omega(\boldsymbol{r})\nabla^2\boldsymbol{A}(\boldsymbol{r}) - \boldsymbol{n}\operatorname{div}\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) - \boldsymbol{n}\cdot\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\operatorname{grad}\delta_\Sigma(\boldsymbol{r}) - \boldsymbol{n}\times\operatorname{curl}\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) + +\boldsymbol{n}\times\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\times\operatorname{grad}\delta_\Sigma(\boldsymbol{r})\,. \tag{41}$$

With the Coulomb gauge $\operatorname{div}\boldsymbol{A}=0$, the generalized Poisson equation of stationary magnetic field reads

$$\nabla^2\{\boldsymbol{A}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\} = -\mu_0\boldsymbol{J}(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r}) - \boldsymbol{n}\times\operatorname{curl}\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\delta_\Sigma(\boldsymbol{r}) + \boldsymbol{n}\times\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\times\operatorname{grad}\delta_\Sigma(\boldsymbol{r}) - \boldsymbol{n}\cdot\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}\operatorname{grad}\delta_\Sigma(\boldsymbol{r})\,. \tag{42}$$

Comparing (42) to (33) one sees that in analogy to electrostatic case, where the two surface terms represent the field sources of scalar type — the surface charge density $-\varepsilon_0\boldsymbol{n}\cdot\operatorname{grad}V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}$ and the surface electric dipole moment density $-\varepsilon_0\boldsymbol{n}V(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}$, in (42) there are two surface terms representing the field sources of vector type — the surface current density $\boldsymbol{n}\times\operatorname{curl}\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}/\mu_0$ and the surface magnetic dipole moment density $\boldsymbol{n}\times\boldsymbol{A}(\boldsymbol{r})\big|_{\boldsymbol{r}\in\Sigma}/\mu_0$. Moreover, the last term in (42), though mathematically sound misses any physical meaning.

However, the form of this integral is formally identical with the electric field intensity due to the surface charge density. Therefore, since the curl of such a field is zero, it does not contribute to the magnetic induction vector $\boldsymbol{B}=\operatorname{curl}\boldsymbol{A}$, but is needed to guarantee the value $\boldsymbol{A}$ being zero outside closed domain Ω. The differential formulation (42) can be easily transformed into the integral expression similarly as (34) was recast into (35). To obtain this integral expression in a classical way is a much more tedious operation as shown in [6, pp. 250-253].

5 GREEN FUNCTION FOR THE INFINITE HALFSPACE

As it was already discussed, only if the surface charge density and the surface dipole moment density in (35) truly correspond to some physically realizable distribution of outside charges, then (35) holds correctly. Therefore they cannot be given arbitrarily. In this sense (35) is again over determined.

For the Dirichlet, Neumann or Cauchy problem, instead of using the impulse response for the homogeneous infinite space as in (35), the Green function must be constructed which fulfils the pertaining zero boundary conditions.

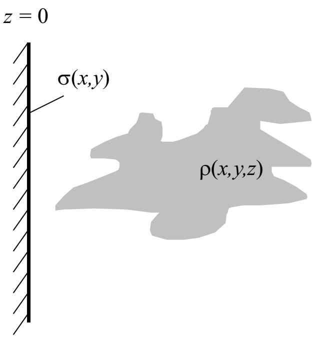

Fig. 1. Volume and surface charge density distribution representing zero boundary conditions

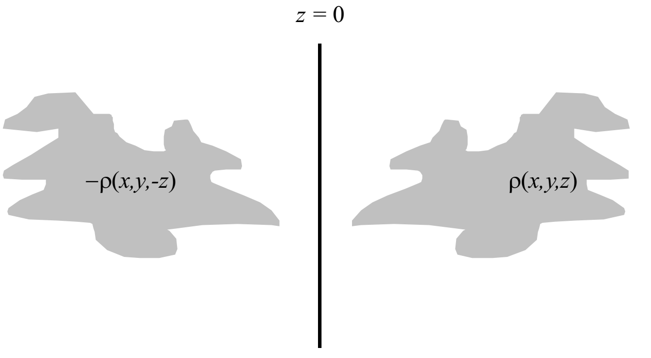

Fig. 2. Zero boundary conditions represented through volume charge density distribution and its mirror image

For the infinite half-space $z > 0$ and its boundary Σ equal to the plane $z = 0$ the Green function for the Dirichlet problem is the solution of

$$\nabla^2 G(\boldsymbol{R},\boldsymbol{r}) = -\delta(\boldsymbol{R}-\boldsymbol{r}), \quad Z>0, z>0, \tag{43}$$

together with the requirement $G(\boldsymbol{R},\boldsymbol{r})|_{z=0} = 0$. It can be for $Z > 0$ obtained as an impulse response of the free space (8) to the excitation in form of two δ -functions with opposite sign placed symmetrically with respect to z plane, $\delta(x,y,z) - \delta(x,y,-z)$ in the form

$$G(\boldsymbol{R},\boldsymbol{r}) = \frac{1}{4\pi\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} - \frac{1}{4\pi\sqrt{(X-x)^2+(Y-y)^2+(Z+z)^2}}\,. \tag{44}$$

Here $G(\boldsymbol{R},\boldsymbol{r})\big|_{z=0} = 0$ and

$$\operatorname{grad}_r G(\boldsymbol{R},\boldsymbol{r})\big|_{z=0} = \frac{(X-x)\boldsymbol{u}_x + (Y-y)\boldsymbol{u}_y + Z\boldsymbol{u}_z}{2\pi[(X-x)^2+(Y-y)^2+Z^2]^{3/2}}. \tag{45}$$

Thus, instead of (35) one has the formula

$$V(\boldsymbol{R}) = \iiint\limits_{\Omega} \frac{\rho(\boldsymbol{r})G(\boldsymbol{R},\boldsymbol{r})}{\varepsilon_0}\,\mathrm{d}v - \oiint\limits_{\Sigma} V(\boldsymbol{r})\operatorname{grad}_r G(\boldsymbol{R},\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{S}\,, \tag{46}$$

where now the boundary values $V(\boldsymbol{r})\big|_{\mathbf{r}\in\Sigma}$ can be chosen arbitrarily.

On the other hand if one takes for the Green function the impulse response of the free space (8) to the excitation $\delta(x,y,z)+\delta(x,y,-z)$ *ie*

$$G(\boldsymbol{R},\boldsymbol{r}) = \frac{1}{4\pi\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} + \frac{1}{4\pi\sqrt{(X-x)^2+(Y-y)^2+(Z+z)^2}}\,, \tag{47}$$

then one gets

$$G(\boldsymbol{R},\boldsymbol{r})\big|_{z=0} = \frac{1}{2\pi[(X-x)^2+(Y-y)^2+Z^2]^{3/2}} \tag{48}$$

and $\operatorname{grad}_r G(\boldsymbol{R},\boldsymbol{r})\big|_{z=0} = 0$. Instead of (46) for the Dirichlet problem one has the formula for the Neumann problem

$$V(\boldsymbol{R}) = \iiint\limits_{\Omega} \frac{\rho(\boldsymbol{r})G(\boldsymbol{R},\boldsymbol{r})}{\varepsilon_0}\,\mathrm{d}v + \oiint\limits_{\Sigma} \operatorname{grad} V(\boldsymbol{r})G(\boldsymbol{R},\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{S}, \tag{49}$$

where the values $\operatorname{grad} V(\boldsymbol{r})\big|_{\mathbf{r}\in\Sigma}$ can be chosen arbitrarily.

Let us take as an example the well known problem of the planar metalic surface $z = 0$ and the volume density of charge $\rho(x,y,z)$ in the half space $z > 0$ as illustrated in Fig. 1. The field intensity $\boldsymbol{E}(\boldsymbol{r}) = -\operatorname{grad} V(\boldsymbol{r})$ is due to the volume charge distribution $\rho(x,y,z)$ and the not-a-priori-known surface density $\sigma(x,y)$ of the induced charge on the metallic surface. The problem is standardly solved by the method of images sketched in Fig. 2. The scalar potential due to both the physically existing charge distribution $\rho(x,y,z)$ in $z > 0$ and the virtual image distribution $\rho_M(x,y,z) = -\rho(x,y,-z)$ in $z < 0$ leads using (10) to

$$V(X,Y,Z) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_{z>0} \frac{\rho(x,y,z)}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}\,\mathrm{d}v + \frac{1}{4\pi\varepsilon_0}\iiint\limits_{z<0} \frac{\rho_M(x,y,z)}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}\,\mathrm{d}v\,. \tag{50}$$

This is the same result as if we used the first term in (46) with the Green function (44), *ie*

$$V(X,Y,Z) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_{z>0}\frac{\rho(x,y,z)}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}}\,\mathrm{d}v - \frac{1}{4\pi\varepsilon_0}\iiint\limits_{z>0}\frac{\rho(x,y,z)}{\sqrt{(X-x)^2+(Y-y)^2+(Z+z)^2}}\,\mathrm{d}v, \quad Z>0\,. \tag{51}$$

Since $V(\boldsymbol{r})|_{z=0} = 0$ the second term in (46) does not contribute to $V(\boldsymbol{R})$. The density of the induced surface charge $\sigma(x,y)$ can be determined as

$$\sigma(x,y) = \varepsilon_0 E_z(x,y) = -\varepsilon_0 \mathbf{u}_z \cdot \operatorname{grad} V(\boldsymbol{r})|_{z=0}$$

The scalar potential can be then determined also by

$$V(X,Y,Z) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_{z>0}\frac{\rho(x,y,z)\,\mathrm{d}v}{\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2}} + \frac{1}{4\pi\varepsilon_0}\iint\limits_{z=0}\frac{\sigma(x,y)\,\mathrm{d}S}{\sqrt{(X-x)^2+(Y-y)^2+Z^2}},\ Z>0. \tag{52}$$

that corresponds to the first two terms of general formula (35).

6 WAVE EQUATION IN THREE DIMENSIONS

The impulse response function for the three-dimensional wave equation in the case of the lossless homogeneous isotropic infinitely extended medium is solution of the equation

$$\nabla^2 g(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2 g(\boldsymbol{r},t)}{\partial t^2} = -\delta(\boldsymbol{r})\delta(t)\,, \tag{53}$$

After the three dimensional Fourier transform in the space domain we arrive at the equation formally identical with (81) in [5]

$$\frac{\partial^2 \mathcal{G}(jk,t)}{\partial t^2} + c^2k^2\mathcal{G}(jk,t) = c^2\delta(t)\,. \tag{54}$$

After the one-dimensional inverse Fourier transform giving the same result as (83) in [5]

$$g_1(r,t) = \frac{c}{2}\mathbf{1}(t)\{\mathbf{1}(r+ct) - \mathbf{1}(r-ct)\}\,. \tag{55}$$

The three-dimensional impulse response accordingly (73) equals

$$g(r,t) = \frac{c}{4\pi r}\mathbf{1}(t)\{\delta(r+ct) - \delta(r-ct)\}\,, \tag{56}$$

or after the transformation of arguments of δ-functions

$$g(r,t) = \frac{\mathbf{1}(t)}{4\pi r}\{\delta(t+r/c) + \delta(t-r/c)\}\,. \tag{57}$$

Since for the three-dimensional space as well as for causal functions, $r > 0$ and $t > 0$ always hold, the first δ-function (representing the "advanced" solution) is meaningless, since the argument of the δ-function cannot be equal to zero for any value of r and t. Therefore only the "retarded" result remains

$$g(\boldsymbol{r},t) = \frac{\delta(t-|\boldsymbol{r}|/c)}{4\pi|\boldsymbol{r}|}\,. \tag{58}$$

This leads to the well-known solution of the wave equation for the dynamic potentials

$$\nabla^2\varphi(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2\varphi(\boldsymbol{r},t)}{\partial t^2} = -\frac{\rho(\boldsymbol{r},t)}{\varepsilon_0} \tag{59}$$

in form of the retarded potential

$$\varphi(\boldsymbol{R},t) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_{\infty}\frac{\rho\big(\boldsymbol{r},t-|\boldsymbol{R}-\boldsymbol{r}|/c\big)}{|\boldsymbol{R}-\boldsymbol{r}|}\,\mathrm{d}v\,, \tag{60}$$

simply obtained as the convolution of the source function $\rho(\boldsymbol{r},t)/\varepsilon_0$ with the impulse response (58).

7 CONCLUSIONS

It has been shown that the formalism of symbolic functions together with their proper application and utilization enables one to easily obtain in a unified manner complicated formulas from the circuit theory and field theory and to give them rather simple interpretation that facilitates deeper insight into the meaning of these formulas. In many textbooks they are merely presented without deeper mathematical insight how they have been obtained.

Appendix I

Three-Dimensional Fourier Transform of the Centro Symmetric Functions

The definition of one dimensional direct and inverse Fourier transform is given in previous paper [5] formula (96) in the form

$$\mathcal{F}(jk) = \int_{-\infty}^{\infty} f(x)\exp(-jkx)\,\mathrm{d}x\,, \tag{61}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(jk)\exp(jkx)\,\mathrm{d}k\,, \tag{62}$$

The three dimensional direct Fourier transform of the function in the Cartesian co-ordinate system $f(\boldsymbol{r}) = f(x,y,z)$ is defined analogously by

$$\mathcal{F}(j\boldsymbol{k}) = \mathcal{F}(jk_x, jk_y, jk_z) = \iiint\limits_{\infty} f(\boldsymbol{r}) \exp\{-j\boldsymbol{k}\cdot\boldsymbol{r}\}\,\mathrm{d}v = \\ = \iiint\limits_{-\infty}^{\infty} f(x,y,z)\exp\{-j(xk_x + yk_y + zk_z)\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \tag{63}$$

and the inverse Fourier transform similarly by

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\iiint\limits_{\infty} \mathcal{F}(j\boldsymbol{k})\exp\{j\boldsymbol{k}\cdot\boldsymbol{r}\}\mathrm{d}v_k\,, \tag{64}$$

where $\mathrm{d}v_k = \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}k_z$.

If the function $f(\boldsymbol{r})$ possesses the spherical symmetry, *ie* $f(x,y,z) = f_3(r)$, where $r = \sqrt{x^2+y^2+z^2}$, then in spherical co-ordinate system the three dimensional Fourier transform is also rotationally symmetrical in spectral domain, *ie* $\mathcal{F}(jk_x, jk_y, jk_z) = \mathcal{F}_3(k)$, where $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. The direct three dimensional Fourier transform, in spherical coordinates with ϑ and θ, the polar and azimuthal angles, yields

$$\mathcal{F}_3(k) = \int\limits_0^{\infty}\int\limits_{-\pi}^{\pi}\int\limits_0^{2\pi} f_3(r) r^2 \sin\vartheta \exp\{-jrk(\cos\vartheta\cos\theta + \\ \cos\vartheta\sin\theta + \sin\vartheta)\}\,\mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\theta\,. \tag{65}$$

After performing the two angle integrations one obtains

$$\mathcal{F}_3(k) = \frac{4\pi}{k}\int\limits_0^{\infty} f_3(r)\sin(kr) r\,\mathrm{d}r\,. \tag{66}$$

Analogous formula holds for the inverse transform too

$$f_3(r) = \frac{1}{2\pi^2 r}\int\limits_0^{\infty} \mathcal{F}_3(k)\sin(kr) k\,\mathrm{d}k\,. \tag{67}$$

Comparing with the standard one-dimensional Fourier transform (61), (62) re-casted for the even functions into the form

$$\mathcal{F}_1(k) = 2\int\limits_0^{\infty} f_1(r)\cos\{kr\}\,\mathrm{d}r\,, \tag{68}$$

$$f_1(r) = \frac{1}{\pi}\int\limits_0^{\infty} \mathcal{F}_1(k)\cos\{kr\}\,\mathrm{d}k\,, \tag{69}$$

one can easily recognise that

$$\mathcal{F}_3(k) = \frac{4\pi}{k}\int\limits_0^{\infty} f(r)\sin(kr) r\,\mathrm{d}r \\ = -\frac{4\pi}{k}\frac{\mathrm{d}}{\mathrm{d}k}\int\limits_0^{\infty} f(r)\cos(kr)\,\mathrm{d}r = -\frac{2\pi}{k}\frac{\mathrm{d}\mathcal{F}_1(k)}{\mathrm{d}k}\,, \tag{70}$$

as well as

$$f_3(r) = \frac{1}{2\pi^2 r}\int\limits_0^{\infty} \mathcal{F}_3(k)\sin(kr) k\,\mathrm{d}k \\ = -\frac{1}{2\pi^2 r}\frac{\mathrm{d}}{\mathrm{d}r}\int\limits_0^{\infty} \mathcal{F}_3(k)\cos(kr)\,\mathrm{d}k = -\frac{1}{2k\pi r}\frac{\mathrm{d}f_1(r)}{\mathrm{d}r}\,. \tag{71}$$

Shortly, if $f_3(r) = f_1(r)$ then

$$\mathcal{F}_3(k) = -\frac{2\pi}{k}\frac{\mathrm{d}\mathcal{F}_1(k)}{\mathrm{d}k} \tag{72}$$

and, on the other hand, if $\mathcal{F}_3(k) = \mathcal{F}_1(k)$ then

$$f_3(r) = -\frac{1}{2\pi r}\frac{\mathrm{d}f_1(r)}{\mathrm{d}r}\,. \tag{73}$$

Appendix II

Three-Dimensional Delta Function and Surface Delta Function

The three dimensional δ-function is in the Cartesian coordinate system defined as a product of three one dimensional δ-functions

$$\delta(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)\,, \tag{74}$$

where $\boldsymbol{r} = x\boldsymbol{u}_x + y\boldsymbol{u}_y + z\boldsymbol{u}_z$ is the radius vector in the Cartesian coordinate system with the unit vectors $\{\boldsymbol{u}_x, \boldsymbol{u}_y, \boldsymbol{u}_z\}$. The fundamental property of the δ-function

$$\iiint\limits_{\infty} f(\boldsymbol{r})\delta(\boldsymbol{r})\,\mathrm{d}v = f(0)\,, \tag{75}$$

is preserved also in three dimensions. Written in Cartesian coordinate system it simply means

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x,y,z)\delta(x)\delta(y)\delta(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = f(0,0,0)\,. \tag{76}$$

The sifting property of the δ-function, *ie* the three dimensional convolution integral reads

$$\iiint\limits_{\infty} f(\boldsymbol{r})\delta(\boldsymbol{R}-\boldsymbol{r})dv = f(\boldsymbol{R})\,. \tag{77}$$

From (75) we maintain

$$\iiint\limits_{\infty} \delta(\boldsymbol{r})\,\mathrm{d}v = 1\,. \tag{78}$$

The same formulas hold also for vector functions, *ie*

$$\iiint\limits_{\infty} \boldsymbol{F}(\boldsymbol{r})\delta(\boldsymbol{R}-\boldsymbol{r})\,\mathrm{d}v = \boldsymbol{F}(\boldsymbol{R})\,. \tag{79}$$

Let us consider the simply connected three dimensional domain Ω with its closure — the closed boundary surface Σ. The unit-step function $\mathbf{1}_\Omega(\boldsymbol{r})$ equal to unity inside Ω, zero outside, and $1/2$ on the boundary Σ has the property

$$\iiint\limits_{\infty} f(\boldsymbol{r})\mathbf{1}_\Omega(\boldsymbol{r})\,\mathrm{d}v = \iiint\limits_{\Omega} f(\boldsymbol{r})\,\mathrm{d}v\,, \tag{80}$$

and, let us define also the surface delta function $\delta_\Sigma(\boldsymbol{r})$ with the following property

$$\iiint\limits_{\infty} f(\boldsymbol{r})\delta_\Sigma(\boldsymbol{r})\,\mathrm{d}v = \oiint\limits_{\Sigma} f(\boldsymbol{r})\,\mathrm{d}S\,. \tag{81}$$

We maintain that

$$\operatorname{grad}\mathbf{1}_\Omega(\boldsymbol{r}) = -\boldsymbol{n}\delta_\Sigma(\boldsymbol{r})\,, \tag{82}$$

where $\boldsymbol{n}$ is the normal unit vector of the boundary surface directed outwards Ω. This is the three dimensional analogy to the one dimensional relation (119) $\mathrm{d}\mathbf{1}(x)/\,\mathrm{d}x = \delta(x)$ shown in [5]. In a similar way also the $\operatorname{grad}\delta_\Sigma(\boldsymbol{r})$ can be defined by the relation

$$\iiint\limits_{\infty} f(\boldsymbol{r})\operatorname{grad}\delta_\Sigma(\boldsymbol{r})\,\mathrm{d}v = -\oiint\limits_{\Sigma} \operatorname{grad} f(\boldsymbol{r})\,\mathrm{d}S \tag{83}$$

as an analogy to

$$\int\limits_{-\infty}^{\infty} f(t)\delta'(t)\,\mathrm{d}t = -f'(0)\,. \tag{84}$$

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