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UNIFIED APPROACH TO THE IMPULSE RESPONSE AND GREEN FUNCTION IN THE CIRCUIT AND FIELD THEORY, PART I: ONE–DIMENSIONAL CASE

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In the circuit theory the concept of the impulse response of a linear system due to its excitation by the Dirac delta function $\delta(t)$ together with the convolution principle is widely used and accepted. The rigorous theory of symbolic functions, sometimes called distributions, where also the delta function belongs, is rather abstract and requires subtle mathematical tools [1], [2], [3], [4]. Nevertheless, the most people intuitively well understand the delta function as a derivative of the (Heaviside) unit step function $\mathbf{1}(t)$ without too much mathematical rigor. The concept of the impulse response of linear systems is here approached in a unified manner and generalized to the time-space phenomena in one dimension (transmission lines), as well as in a subsequent paper [5] to the phenomena in more dimensions (static and dynamic electromagnetic fields).

K e y w o r d s: circuit theory, field theory, impulse response, Green function

1 INTRODUCTION

The response of a linear system to the excitation by the Dirac impulse, or δ-function, is well-known as the impulse response $h(t)$ and is widely used in the circuit theory. The linear system is characterized by its system operator $\mathfrak{L}\{\dots\}$, where $\mathfrak{L}\{h(t)\} = \delta(t)$ holds. Knowing the impulse response of the system, the response $f(t)$ to any excitation $y(t)$, $\mathfrak{L}\{f(t)\} = y(t)$, can be easily determined in terms of the convolution

$$f(t) = \int_{-\infty}^{\infty} y(\tau) h(t-\tau) \mathrm{d}\tau \,. \tag{1}$$

The idea of impulse excitation and impulse response in time domain can be naturally extended to space domain and to the treatment of more dimensional linear systems of electromagnetics as well, particularly of the:

a) time and one spatial dimension – the case of transmission lines,

b) more spatial dimensions without time-dependence – the case of static fields,

c) time and more spatial dimensions – the case of dynamic fields.

One of the simple examples in electrostatics is the representation of the point charge Q placed in the origin in terms of the charge density equal to $\rho(\boldsymbol{r}) = Q\delta(\boldsymbol{r})$. Solution of the Poisson equation for the scalar potential $\varphi(\boldsymbol{r})$ of the point charge in the origin

$$\nabla^2 \varphi(\boldsymbol{r}) = -Q\delta(\boldsymbol{r})/\varepsilon_0 \,, \tag{2}$$

where ε_0 is the permittivity, is given by the well-known formula

$$\varphi(\boldsymbol{r}) = Q/4\pi\varepsilon_0|\boldsymbol{r}| \,. \tag{3}$$

Thus, the impulse response for the Laplace operator $g(\boldsymbol{r})$ is $g(\boldsymbol{r}) = 1/4\pi|\boldsymbol{r}|$ The well-known result of electrostatics for any volume charge density distribution $\rho(\boldsymbol{r})$

$$\varphi(\boldsymbol{r}) = \iiint\limits_{\infty} \frac{\rho(\mathbf{r}')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|} \mathrm{d}v' \tag{4}$$

is in fact merely a convolution of the impulse response with $\rho(\boldsymbol{r})/\varepsilon_0$.

In what follows we try to pursue and generalize these ideas further and show that some theoretically tricky results can be obtained by relatively simple mathematical tools starting from the plain second order differential operator.

2 SECOND ORDER OPERATOR IN TIME DOMAIN AND IN ONE–DIMENSIONAL SPACE DOMAIN

Let us consider the simplest second order operator $\mathfrak{L} = \partial^2/\partial t^2$ in the time domain, or $\mathfrak{L} = \partial^2/\partial x^2$ in the space domain, and pertaining equations

$$\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = y(t) \,, \qquad \frac{\mathrm{d}^2 \varphi(x)}{\mathrm{d}x^2} = q(x) \,, \tag{5), (6}$$

where $f(t)$ in (5) can be interpreted as *eg* a one-dimensional position function of the mass-point moving in the force field proportional to $y(t)$, and $\varphi(x)$ in (6) physically represents the electric scalar potential due to the charge density source $\rho(x)$, $q(x) = -\rho(x)/\varepsilon_0$, *ie* the one-dimensional Poisson equation of electrostatics.

The equations

$$\frac{\mathrm{d}^2 h_c(t)}{\mathrm{d}t^2} = \delta(t) \,, \qquad \frac{\mathrm{d}^2 g_s(x)}{\mathrm{d}x^2} = \delta(x) \tag{7), (8}$$

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define the impulse responses in time and space domains. Due to the causality principle valid in the time domain $h_c(t)$ is given as a causal function

$$h_c(t) = h(t)\mathbf{1}(t)\,, \tag{9}$$

and due to the symmetry of the homogeneous space the impulse response $g_s(t)$ as a centro-symmetric function (see (110) in Appendix I)

$$g_s(t) = \frac{1}{2}\left\{h_c(x) + h_c(-x)\right\}. \tag{10}$$

The Laplace transform of (7) for causal problems and the Fourier transform of (8) for centro-symmetric problems lead to

$$H_c(p) = 1/p^2, \qquad \mathcal{G}_s(j\omega) = 1/(j\omega)^2, \quad (11,12)$$

yielding after the inverse transforms

$$h_c(t) = t\mathbf{1}(t), g_s(x) = \tfrac{1}{2}\left\{x\mathbf{1}(x) - x\mathbf{1}(-x)\right\}. \quad (13,14)$$

The response to the arbitrary excitation $y(t)$, or $q(x)$ is expressed as the respective convolution integrals

$$f(t) = \int_{-\infty}^{\infty} y(\tau)h_c(t-\tau)\mathrm{d}\tau\,, \tag{15}$$

$$\varphi(x) = \int_{-\infty}^{\infty} q(\xi)g_s(x-\xi)\mathrm{d}\xi\,. \tag{16}$$

Other form of (15) due to the causal character of $h_c(t) = h(t)\mathbf{1}(t)$ in (15) reads

$$f(t) = \int_{-\infty}^{t} y(\tau)h(t-\tau)\mathrm{d}\tau\,, \tag{17}$$

clearly illustrating the causality principle – to the result $f(t)$ at the time instant t contribute only the "previous" values of the excitation function $y(\tau)$, $\tau \in (-\infty, t)$.

For $t > 0$ can be (17) recast into the form

$$f(t) = f_0(t) + \int_0^t y(\tau)h(t-\tau)\mathrm{d}\tau\,,\ t > 0\,,$$
$$f_0(t) = \int_{-\infty}^{0} y(\tau)h(t-\tau)\mathrm{d}\tau\,,\ t > 0\,. \quad (18,19)$$

Reformulating (5) in terms of causal functions $f_c(t) = f(t)\mathbf{1}(t)$, $y_c(t) = y(t)\mathbf{1}(t)$ leads, due to (122) to the equation with the embodied *initial conditions* $f(0)$ and $f'(0)$

$$\frac{\mathrm{d}^2 f_c(t)}{\mathrm{d}t^2} = y_c(t) + f'(0)\delta(t) + f(0)\delta'(t) \tag{20}$$

with the solution

$$f_c(t) = f(t)\mathbf{1}(t) = \mathbf{1}(t)\int_0^t y(\tau)h(t-\tau)\mathrm{d}\tau + f'(0)h_c(t) + f(0)h'_c(t)\,. \tag{21}$$

Notice the difference between the formulas (17) and (21) as well as between (5) and (20). Both are defined for the whole time axis $t \in (-\infty, \infty)$, but the latter formulate the solution in terms of causal functions $f_c(t)$, *ie* the solution $f_c(t)$ equal to zero for $t < 0$, and to the true values $f(t)$ for $t > 0$, while the former gives $f(t)$ for any $t \in (-\infty, \infty)$.

Comparison of (19) with the first two terms of (21) yields

$$f_0(t)\mathbf{1}(t) = f'(0)h_c(t) + f(0)h'_c(t) \tag{22}$$

ie in the positive time instances $t > 0$ the history due to the past excitation (at the negative time values $t < 0$) is encoded into the *initial conditions* $f(0)$ and $f'(0)$.

Taking into account the centro-symmetric character (10) of $g_s(x)$ the convolution integral (16) can be written as

$$\varphi(x) = \frac{1}{2}\int_{-\infty}^{x} q(\xi)h_c(x-\xi)\mathrm{d}\xi + \frac{1}{2}\int_{x}^{\infty} q(\xi)h_c(x-\xi)\mathrm{d}\xi\,. \tag{23}$$

For $x \in (0, a)$ one can write

$$\varphi(x) = \varphi_0(x) + \int_0^a q(\xi)g_s(x-\xi)\mathrm{d}\xi + \varphi_a(x)\,,\ x \in (0,a) \tag{24}$$

$$\varphi_0(x) = \frac{1}{2}\int_{-\infty}^{0} q(\xi)h_c(x-\xi)\mathrm{d}\xi\,,\ x \in (0,a)\,, \tag{25}$$

$$\varphi_a(x) = \frac{1}{2}\int_{a}^{\infty} q(\xi)h_c(x-\xi)\mathrm{d}\xi\,,\ x \in (0,a)\,, \tag{26}$$

Similar to the previous development one can force the solution to be zero outside the bounded interval $x \in (0, a)$ by introducing the function $\varphi_\Omega(x)$

$$\varphi_\Omega(x) = \varphi(x)\left[\mathbf{1}(x) - \mathbf{1}(x-a)\right] = \varphi(x)\mathbf{1}_\Omega(x)\,, \tag{27}$$

where $\mathbf{1}_\Omega(x) = \big[\mathbf{1}(x) - \mathbf{1}(x-a)\big]$, *ie* $\varphi_\Omega(x)$ is equal to zero outside the simply connected domain Ω with finite support, which, in the one-dimensional case, is simply the interval $x \in (0, a)$, Ω: $\{x \in (0, a)\}$. Then (6) is modified into

$$\frac{\mathrm{d}^2\varphi_\Omega(x)}{\mathrm{d}x^2} = q(x)\mathbf{1}_\Omega(x) + \varphi'(x)\big|_{x\in\Sigma}\delta_\Sigma(x) + \varphi(x)\big|_{x\in\Sigma}\delta'_\Sigma(x)\,, \tag{28}$$

where Σ is the closure of Ω, in our one dimensional case it means simply the two points $x = 0$ and $x = a$, $\Sigma : \{x = 0, x = a\}$, and the meaning of symbols in (28) is

$$\mathrm{d}\mathbf{1}_\Omega(x)/\mathrm{d}x = \delta_\Sigma(x) = \left[\delta(x) - \delta(x-a)\right], \tag{29}$$

$$\varphi'(x)\big|_{x\in\Sigma}\delta_\Sigma(x) = \varphi'(0)\delta(x) - \varphi'(a)\delta(x-a)\,, \tag{30}$$

$$\varphi(x)\big|_{x\in\Sigma}\delta'_\Sigma(x) = \varphi(0)\delta'(x) - \varphi(a)\delta'(x-a)\,. \tag{31}$$

The solution of (28) in terms of convolution reads

$$\varphi_\Omega(x) = \int_{-\infty}^{\infty} \Big\{ q(\xi)\mathbf{1}_\Omega(\xi) + \varphi'(\xi)\big|_{\xi\in\Sigma}\delta_\Sigma(\xi) + \varphi(\xi)\big|_{\xi\in\Sigma}\delta'_\Sigma(\xi)\Big\} g_s(x-\xi)\mathrm{d}\xi\,, \quad (32)$$

or explicitly

$$\begin{aligned}\varphi_\Omega(x) = \mathbf{1}_\Omega(x)\int_0^a q(\xi)g_s(x-\xi)\mathrm{d}\xi \\ + \big\{\varphi'(0)g_s(x) + \varphi(0)g'_s(x)\big\} \\ - \big\{\varphi'(a)g_s(x-a) + \varphi(a)g'_s(x-a)\big\}.\end{aligned} \quad (33)$$

Evaluation of (33) leads to the true values $\varphi(x)$ within the interval $x \in (0,a)$, and to zero outside this interval. This means that the influence of the excitation $q(x)$ outside $x \in (0,a)$ is transformed into the *boundary values*, $\varphi(0)$, $\varphi'(0)$, $\varphi(a)$ and $\varphi'(a)$, *ie* as far as the correct boundary values are known, (33) provides the correct result within $x \in (0,a)$ and zero outside.

The classical way to obtain (33) is as follows: Consider the integral

$$\int_0^a \big\{\varphi'(\xi)g_s(x-\xi) - \varphi(\xi)g'_s(x-\xi)\big\}'\mathrm{d}\xi\,. \quad (34)$$

Performing both, the derivative in the curly brackets of (34), as well as the integration of (34), one arrives to the one-dimensional variant of the well known Green identity

$$\begin{aligned}\int_0^a \big\{\varphi''(\xi)g_s(x-\xi) - \varphi(\xi)g''_s(x-\xi)\big\}\mathrm{d}\xi \\ = \big\{\varphi'(a)g_s(x-a) + \varphi(a)g'_s(x-a)\big\} \\ - \big\{\varphi'(0)g_s(x+a) + \varphi(0)g'_s(x+a)\big\}\,.\end{aligned} \quad (35)$$

It should be noticed that the derivatives in (34) are with respect to ξ, while in the RHS of (35) they are meant with respect to x (therefore the change of the sign). Substitution from (6) and (8) yields

$$\begin{aligned}\varphi(x) = \int_0^a q(\xi)g_s(x-\xi)\mathrm{d}\xi \\ + \big\{\varphi'(0)g_s(x) + \varphi(0)g'_s(x)\big\} \\ - \big\{\varphi'(a)g_s(x-a) + \varphi(a)g'_s(x-a)\big\}\,.\end{aligned} \quad (36)$$

This is seemingly the same result as in (33) However, from (33) it is apparent that for $x \notin (0,a)$ the result is zero and that cannot be easily seen from (36). In fact (28) is the differential formulation of the classical result (36).

3 THE GREEN FUNCTION

With the four boundary values $\varphi(0)$, $\varphi'(0)$, $\varphi(a)$ and $\varphi'(a)$ in (33), or (36), the problem is clearly over determined, *ie* they cannot be chosen arbitrarily. For the operators of second order only two boundary conditions can be freely chosen, *ie* either the pair $\varphi(0)$, $\varphi(a)$ (Dirichlet problem), or the pair $\varphi'(0)$, $\varphi'(a)$ (Neumann problem), or the combination pairs either $\varphi(0)$, $\varphi'(a)$, or $\varphi'(0)$, $\varphi(a)$ (Cauchy problem), respectively.

To achieve this goal the special type of impulse response called the Green function $G(x,\xi)$, must be constructed, which fulfils the equation

$$\frac{\mathrm{d}^2 G(x,\xi)}{\mathrm{d}x^2} = \delta(x-\xi) \quad (37)$$

and the pertaining zero boundary conditions on the domain $x,\xi \in \Omega$, *ie* instead of (32) one obtains

$$\varphi(x) = \int_\Omega q(\xi)G(x,\xi)\mathrm{d}\xi + \int_\Omega \big\{\varphi'(\xi)\big|_{\xi\in\Sigma}\delta_\Sigma(\xi) + \varphi(\xi)\big|_{\xi\in\Sigma}\delta'_\Sigma(\xi)\big\}G(x,\xi)\mathrm{d}\xi\,, \quad (38)$$

leading specifically to

$$\begin{aligned}\varphi(x) = \int_0^a q(\xi)G(x,\xi)\mathrm{d}\xi + \\ \big\{\varphi'(0)G(x,0) - \varphi(0)G'(x,0)\big\} \\ - \big\{\varphi'(a)G(x,a) - \varphi(a)G'(x,a)\big\}\end{aligned} \quad (39)$$

where derivatives of $G(x,\xi)$ are with respect to ξ.

As mentioned above $G(x,\xi)$ has to fulfil pertaining boundary conditions, *ie* either $G(x,0) = 0$, $G(x,a) = 0$ (the Dirichlet problem), or $G'(x,0) = 0$, $G'(x,a) = 0$ (the Neumann problem) and analogously for two possible variants of the Cauchy problem. Then only two boundary values remain in (39). In what follows we shall focus only on the Dirichlet problem.

The Green function $G(x,\xi)$ can be constructed by either of the following three ways [4].

The first way: Since we are interested only in solution on the interval $x,\xi \in (0,a)$ one can construct the odd periodic series of δ-impulses $\delta(x-\xi) - \delta(x+\xi)$ for x on the interval $x \in (-a,a)$, and for ξ on the interval $\xi \in (0,a)$, *ie*

$$\sum_{k=-\infty}^{\infty} \delta(x-\xi+2ka) - \delta(x+\xi+2ka)\,, \quad (40)$$

leading formally to the solution

$$G(x,\xi) = \sum_{k=-\infty}^{\infty} g_s(x-\xi+2ka) - g_s(x+\xi+2ka)\,, \quad (41)$$

provided (41) converges.

The second way: Both (40) and (41) can be expressed in form of the Fourier series. For (40) one obtains

$$\delta(x-\xi) - \delta(x+\xi) = \frac{2}{a}\sum_{n=1}^{\infty}\sin(n\pi\xi/a)\sin(n\pi x/a)\,, \quad (42)$$

and (41) stems from (12) and (42) in the form

$$G(x,\xi)=\sum_{n=1}^{\infty}\frac{-2a}{(n\pi)^2}\sin(n\pi\xi/a)\sin(n\pi x/a)\,. \tag{43}$$

The third way: Direct integration of (37) with respect to x on the interval $x\in(0,a)$ on assumption of zero mean value of $G'(x,\xi)$ leads to

$$G'(x,\xi)=\frac{1}{a}\begin{cases}(\xi-a)\,, & 0<x<\xi<a\,,\\ \xi\,, & 0<\xi<x<a\,.\end{cases} \tag{44}$$

Second integration yields the Green function for the Dirichlet problem in the closed form

$$G(x,\xi)=\frac{1}{a}\begin{cases}(\xi-a)x\,, & 0<x<\xi<a\,,\\ \xi(x-a)\,, & 0<\xi<x<a\,.\end{cases} \tag{45}$$

The equivalence of (43) and (45) can be proved also by *eg* the direct calculation of the Fourier-series-expansion of (45).

Thus instead of (39) one obtains for $x\in(0,a)$ the formula

$$\varphi(x)=\int_0^a q(\xi)G(x,\xi)\mathrm{d}\xi-\varphi(0)G'(x,0)+\varphi(a)G'(x,a) \tag{46}$$

where now $\varphi(0)$ and $\varphi(a)$ can be chosen arbitrarily.

For (6) and the pure boundary-value-problem, $q(x)=0$, the solution of the Laplace equation reads

$$\varphi(x)=\varphi(a)G'(x,a)-\varphi(0)G'(x,0)\,. \tag{47}$$

With (45) it represents the linear dependence of the electric field potential between the two infinitely extended plate electrodes

$$\varphi(x)=\varphi(a)[x/a]-\varphi(0)[(x-a)/a]\,. \tag{48}$$

Note in (45) that if $\xi=0$, or $\xi=a$ the values $G(x,0)$ and $G(x,a)$ are identically equal to zero, therefore $\varphi'(0)$ and $\varphi'(a)$ disappear from (39).

Taking into account (14), the Green function (45), can be written in the form

$$G(x,\xi)=g_s(x-\xi)+\frac{a-2\xi}{2a}g_s\Big(x-\frac{a\xi}{2\xi-a}\Big). \tag{49}$$

This can be understood as a kind of *mirror image* of the point $x=\xi$ inside the interval $(0,a)$ to the point $x_m=a\xi/(2\xi-a)$ outside the interval $(0,a)$, particularly the points $x\in(0,a/2)$ are mirrored to the points $x_m\in(-\infty,0)$, and the points $x\in(a/2,a)$, to the points $x_m\in(a,\infty)$. The integral in (46) can be interpreted as a contribution of the virtual $q_v(x)$ outside the interval $x\in(0,a)$ to the result inside $x\in(0,a)$, *ie*

$$\varphi(x)=\int_{-\infty}^{0}q_v(\xi)g_s(x-\xi)\mathrm{d}\xi+\int_0^a q(\xi)g_s(x-\xi)\mathrm{d}\xi \\ +\int_a^{\infty}q_v(\xi)g_s(x-\xi)\mathrm{d}\xi\,,\quad x\in(0,a)\,, \tag{50}$$

$$q_v(x)=\frac{a}{2a-4x}q\Big(\frac{ax}{2x-a}\Big),\quad x\notin(0,a)\,. \tag{51}$$

Thus, using this *mirror-image-source*, the problem on the bounded domain Ω can be transformed into the problem on the infinite space and the simple impulse response (14) used instead of the Green function (45).

4 TRANSIENTS IN THE RLC–CIRCUIT

Let us take a simple example in time domain — the current $i(t)$ in the serial RLC-circuit driven by the voltage source $u(t)$ is governed by the equation

$$LCi''(t)+RCi'(t)+i(t)=Cu'(t)\,. \tag{52}$$

By using the transformation $i(t)=f(t)\exp(-tR/2L)$ one can transform (52) into the equation of the type

$$\mathfrak{L}\{f(t)\}=\frac{\mathrm{d}^2f(t)}{\mathrm{d}t^2}+\omega_0^2f(t)=y(t)\,, \tag{53}$$

$$\omega_0^2=\frac{1}{LC}-\frac{R^2}{4L^2}\,,\quad y(t)=\frac{1}{L}u'(t)\exp\frac{tR}{2L}$$

Equation for the impulse response $h_c(t)$ reads

$$\frac{\mathrm{d}^2h_c(t)}{\mathrm{d}t^2}+\omega_0^2h_c(t)=\delta(t)\,, \tag{54}$$

or in the Laplace transform domain

$$(p^2+\omega_0^2)H_c(p)=1\,. \tag{55}$$

The inverse transform yields the causal impulse response

$$h_c(t)=\mathbf{1}(t)\sin(\omega_0t)/\omega_0\,. \tag{56}$$

The causal solution of (53) is

$$f(t)=\int_{-\infty}^{t}y(\tau)\frac{\sin\omega_0(t-\tau)}{\omega_0}\mathrm{d}\tau\,, \tag{57}$$

and for the current in (52) one obtains

$$i(t)=\frac{1}{L}e^{-tR/2L}\int_{-\infty}^{t}u'(\tau)e^{\tau R/2L}\frac{\sin\omega_0(t-\tau)}{\omega_0}\mathrm{d}\tau\,. \tag{58}$$

For the initial problem (53) yields the equation

$$\frac{\mathrm{d}^2f_c(t)}{\mathrm{d}t^2}+\omega_0^2f_c(t)=y_c(t)+f'(0)\delta(t)+f(0)\delta'(t) \tag{59}$$

with the solution

$$f_c(t)=f(t)\mathbf{1}(t)=\mathbf{1}(t)\int_0^t y(\tau)\frac{\sin\omega_0(t-\tau)}{\omega_0}\mathrm{d}\tau \\ +f(0)\mathbf{1}(t)\cos(\omega_0t)+f'(0)\mathbf{1}(t)\sin(\omega_0t)/\omega_0\,. \tag{60}$$

For the causal current response one obtains

$$i(t)\mathbf{1}(t)=\frac{1}{L}e^{-tR/2L}\frac{1}{t}\Big\{\int_0^t u'(\tau)e^{\tau R/2L}\frac{\sin\omega_0(t-\tau)}{\omega_0}\mathrm{d}\tau \\ +i_0\Big[\cos(\omega_0t)-\frac{R}{2L}\frac{\sin\omega_0t}{\omega_0}\Big]+i_0'\frac{\sin\omega_0t}{\omega_0}\Big\}. \tag{61}$$

The initial conditions $i(0)$ and $i'(0)$ represent the initial energy state of the circuit, as well as the initial value of the excitation $u(0)$ that is lost in the derivative $u'(t)$ in the RHS of (52). The energy of the magnetic field of an inductor $Li^2(0)/2$ is encoded in $i(0)$ directly, and the energy content of the electric field of a capacitor $Cu_C^2(0)/2$ is encoded in $i'(0)$, since $i'(0)=[u(0)-u_C(0)-Ri(0)]/L$.

5 HARMONIC STEADY–STATE OF ONE–DIMENSIONAL WAVE PROPAGATION

The same type of operator in space domain (Helmholtz equation)

$$\mathfrak{L}\{\varphi(x)\} = \frac{\mathrm{d}^2\varphi(x)}{\mathrm{d}x^2} + \beta_0^2\varphi(x) = q(x) \qquad (62)$$

describes (in complex representation) the one-dimensional case of the steady-state harmonic wave propagation with the amplitude $f(x,t) = \varphi(x)\exp(j\omega t)$, *eg* wave along the lossless transmission line, characterised by the complex amplitude $\varphi(x)$ and the complex excitation source $q(x)$. Equation for the centro-symmetric response $g_s(x)$ is

$$\frac{\mathrm{d}^2 g_s(x)}{\mathrm{d}x^2} + \beta_0^2 g_s(x) = \delta(x)\,. \qquad (63)$$

In the Fourier transform domain it reads

$$\big[(jk)^2 + \beta_0^2\big]\mathcal{G}_s(jk) = 1\,. \qquad (64)$$

Inverse transform yields in analogy to (56)

$$g_s(x) = \{\mathbf{1}(x) - \mathbf{1}(-x)\}\sin(\beta_0 x)/2\beta_0\,. \qquad (65)$$

One should consider that, for the time dependance $\exp(j\omega t)$, since $\sin(\beta_0 x) = \frac{1}{2j}[\exp(j\beta x) - \exp(-j\beta x)]$, both types of waves — outbound waves propagating “away” from the source $\delta(x)$, *ie* $\exp(-j\beta x)$ for $x > 0$ and $\exp(j\beta x)$ for $x < 0$, as well as inbound waves propagating towards the source, *ie* $\exp(j\beta x)$ for $x > 0$ and $\exp(-j\beta x)$ for $x < 0$, are encompassed in (65). For physical reasons there may exist only outbound waves on the infinite domain $x \in (-\infty, \infty)$. The remedy is to add to (65) the solution of the homogeneous equation in the form $g_0(x) = j\cos(\beta_0 x)/2\beta_0$ leading thus to

$$g_s(x) + g_0(x) = j\,\frac{\mathbf{1}(x)\exp(-j\beta_0 x) - \mathbf{1}(-x)\exp(j\beta_0 x)}{2\beta_0}\,. \qquad (66)$$

The response $\varphi(x)$ due to $q(x)$ on the infinite domain $x \in (-\infty, \infty)$, can be obtained by the convolutory integral of type (16)

$$\varphi(x) = \frac{j\exp(-j\beta_0 x)}{2\beta_0}\int_{-\infty}^{x} q(\xi)\exp(j\beta_0\xi\mathrm{d}\xi + \frac{j\exp(j\beta_0 x)}{2\beta_0}\int_{x}^{\infty} q(\xi)\exp(-j\beta_0\xi\mathrm{d}\xi\,. \qquad (67)$$

The first integral are contributions to $\varphi(x)$, in accordance with the Huyghen’s principle, of the forward propagating waves emanating from the source distribution $q(\xi)$ in the points $\xi < x$ and the second of the backwards propagating waves from the source distribution $q(\xi)$ in the points $\xi > x$.

The Green function for the Helmholtz equation (62) and the Dirichlet problem on the interval $x \in (0, a)$ can be constructed following the development in the preceding paragraph 3 in the form of Fourier series

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{-2/a}{(n\pi/a)^2 + \beta_0^2}\sin(n\pi\xi/a)\sin(n\pi x/a)\,, \qquad (68)$$

as well as in the closed form

$$G(x,\xi) = \frac{1}{\beta_0\sin(\beta_0 a)} \times \begin{cases} \sin\{\beta_0(\xi - a)\}\sin(\beta_0 x)\,, & 0 \le x \le \xi \le a\,, \\ \sin(\beta_0\xi)\sin\{\beta_0(x - a)\}\,, & 0 \le \xi \le x \le a\,. \end{cases} \qquad (69)$$

In case of the transmission lines and plate waveguides the pure boundary-value-problem is typical. The feeding of the line by a harmonic voltage source at $x = 0$ is given by the (complex amplitude) boundary value $\varphi(0) = \varphi_0$, and short-circuited line at $x = a$, $\varphi(a) = 0$, leads, using (47), to the standing wave

$$\varphi(x) = -\varphi_0 G'(x,0) = \varphi_0\frac{\sin\{\beta_0(a - x)\}}{\sin(\beta_0 a)}\,. \qquad (70)$$

thus comprising the interference of both, the forward (direct) and backward (reflected) wave.

6 ONE DIMENSIONAL WAVE EQUATION — THE TRANSMISSION LINE

Waves on the transmission line are described by the homogeneous equation

$$\frac{\partial^2 f(x,t)}{\partial x^2} - af(x,t) - 2b\frac{\partial f(x,t)}{\partial t} - \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2} = 0 \qquad (71)$$

with $a = R_0G_0$, $b = \frac{1}{2}\big(R_0C_0 + G_0L_0\big)$, $1/c^2 = L_0C_0$, where the constants R_0, G_0, L_0 and C_0 are the distributed resistance, conductance, inductance and capacitance of the homogeneous line respectively.

Taking the initial condition $f(x,0) = \exp(-jkx)$, *ie* in form of the harmonic (monochromatic) wave amplitude with the wavenumber k and the wavelength $\lambda = 2\pi/k$ one easily arrives to the solution [2]

$$f(x,t) = \exp(-bc^2 t)\exp\{-jk(x \pm vt)\}\,, \qquad (72)$$

representing thus the direct and reverse wave attenuated in time and propagating along x with the phase velocity

$$v = c\sqrt{1 - (b^2c^2 - a)/k^2}\,. \qquad (73)$$

The pertaining angular frequency of this harmonic wave is

$$\omega = kv = c\sqrt{k^2 - (b^2c^2 - a)}\,, \qquad (74)$$

leading to the frequency dependence (dispersion) of the wavenumber

$$k(\omega) = \sqrt{(\omega/c)^2 + (b^2c^2 - a)}\,, \tag{75}$$

and of the phase velocity

$$v(\omega) = \frac{c}{\sqrt{1 + c^2(b^2c^2 - a)/\omega^2}}\,. \tag{76}$$

For $b^2c^2 - a > 0$, *ie* the phase velocity increasing with the frequency, we have the anomalous dispersion case, while for $b^2c^2 - a < 0$, *ie* the phase velocity decreasing with the frequency, we have the normal dispersion case. Since for the transmission line

$$b^2c^2 - a = \tfrac{1}{4}\big(R_0\sqrt{C_0/L_0} - G_0\sqrt{L_0/C_0}\big)^2 \tag{77}$$

holds, the transmission line is always an anomalous-dispersion-system, provided the distributed parameters of the line are constants independent from ω.

Using the substitution $f(x,t) = \psi(x,t)\exp(-bc^2t)$ in (71) yields

$$\frac{\partial^2\psi(x,t)}{\partial x^2} - \alpha\psi(x,t) - \frac{1}{c^2}\frac{\partial^2\psi(x,t)}{\partial t^2} = 0\,, \tag{78}$$

where $\alpha = a - b^2c^2$. If $a = b^2c^2$, *ie* $R_0C_0 = G_0L_0$, the transmission line is distortionless and (78) yields

$$\mathfrak{L}\{\psi(x,t)\} = \frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi(x,t)}{\partial t^2} = 0\,. \tag{79}$$

Only this dispersionfree case will be treated further.

The impulse response $g(x,t)$ for the infinite free space, $x \in (-\infty,\infty)$, is the solution of the equation

$$\frac{\partial^2 g(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 g(x,t)}{\partial t^2} = -\delta(x)\delta(t)\,, \tag{80}$$

where the two δ-functions on the right side represent in space domain the radiating point source, and in the time domain an infinitely short flash.

The space-domain Fourier transform $g(x,t) \stackrel{FT}{\longleftrightarrow} \mathcal{G}(jk,t)$ of (80) yields

$$\frac{\partial^2\mathcal{G}(jk,t)}{\partial t^2} + c^2k^2\mathcal{G}(jk,t) = c^2\delta(t) \tag{81}$$

analogous to (54). The solution analogous to (56) reads

$$\mathcal{G}(jk,t) = \mathbf{1}(t)c\sin(ckt)/k\,, \tag{82}$$

The inverse Fourier transform of (82) yields

$$g(x,t) = \frac{c}{2}\mathbf{1}(t)\{\mathbf{1}(x+ct) - \mathbf{1}(x-ct)\}\,. \tag{83}$$

On the other hand, performing first the Laplace transform $g(x,t) \stackrel{LT}{\longleftrightarrow} \mathcal{G}(x,p)$ of (80) yields the equation

$$\frac{\partial^2 G(x,p)}{\partial x^2} - \frac{p^2}{c^2}\mathcal{G}(x,p) = -\delta(x) \tag{84}$$

with the solution analogous to (66) that yields

$$g(x,t) = \frac{c}{2}\{\mathbf{1}(-x)\mathbf{1}(t+x/c) + \mathbf{1}(x)\mathbf{1}(t-x/c)\}\,. \tag{85}$$

an expression equivalent to (83).

The wave character of the response is clearly manifested — either from (83) on the positive portion of the x-axis the trailing edge of the spatial rectangular step running with the velocity c to $+\infty$ (forward wave) and on the negative portion of the x-axis the leading edge of the spatial rectangular step running to $-\infty$ (backward wave), or from (85) shoving the time delay, equal to x/c, needed for unit step to arrive to the point x.

The response to any excitation $q(x,t)$, *ie* the solution of the equation $\mathfrak{L}\{\psi(x,t)\} = -q(x,t)$, reads

$$\begin{aligned}\psi(x,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(\xi,\tau)g(x-\xi,t-\tau)\mathrm{d}\tau\mathrm{d}\xi \\ &= \frac{c}{2}\int_{-\infty}^{t}\int_{x-c(t-\tau}^{x+c(t-\tau)} g(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau\,.\end{aligned} \tag{86}$$

For the causal and bounded response $\psi_{c\Omega}(x,t) = \psi_{c\Omega}(x,t)\mathbf{1}(t)\mathbf{1}_\Omega(x)$ one obtains the equation

$$\begin{aligned}\mathfrak{L}\{\psi_{c\Omega}(x,t)\} = -q_{c\Omega}(x,t) - c^{-2}[\psi'(x,0)\delta(t) + \psi(x,0)\delta'(t)] \\ + \partial\psi(x,t)/\partial x\big|_{x\in\Sigma}\delta_\Sigma(x) + \psi(x)\big|_{x\in\Sigma}\delta'_\Sigma(x)\,.\end{aligned} \tag{87}$$

The Green function for zero boundary conditions (Dirichlet problem) on the interval $x \in (0,a)$ is obtained as a solution of (81) having replaced $\delta(x)$ in the RHS with infinite series (40). The solution in terms of infinite series of mirror images represents the waves reflected on the boundaries $x = 0$ and $x = a$

$$G(x,\xi,t) = \begin{cases} \frac{c}{2}\big\{\sum_{k=0}^{\infty}\mathbf{1}\big(t - \frac{x-\xi+2ka}{c}\big) - \mathbf{1}\big(t + \frac{x+\xi-2ka}{c}\big) \\ \sum_{k=0}^{\infty}\mathbf{1}\big(t - \frac{\xi-x+2ka}{c}\big) - \mathbf{1}\big(t - \frac{\xi+x+2ka}{c}\big)\big\}, \\ \qquad\qquad 0 \le \xi \le x \le a\,, \\ \frac{c}{2}\big\{\sum_{k=0}^{\infty}\mathbf{1}\big(t - \frac{x-\xi+2ka}{c}\big) - \mathbf{1}\big(t - \frac{x+\xi+2ka}{c}\big) \\ \sum_{k=0}^{\infty}\mathbf{1}\big(t - \frac{x-\xi+2ka}{c}\big) - \mathbf{1}\big(t - \frac{\xi+x-2ka}{c}\big)\big\}, \\ \qquad\qquad 0 \le x \le \xi \le a\,. \end{cases} \tag{88}$$

For the pure boundary problem, *ie* the transmission line fed by a voltage source $\psi(0,t) = \psi_0(t)$ at $x = 0$, and short-circuited at $x = a$, $\psi(a,t) = 0$ the wave distribution along the line is obtained as

$$\psi(x,t) = \int_{-\infty}^{\infty}\psi_0(\tau)G'(x,0,t-\tau)\mathrm{d}\tau\,,\ x \in (0,a)\,, \tag{89}$$

where $G'(x,0,t) = \partial G(x,\xi,t)/\partial\xi\big|_{\xi=0}$ for $0 \le \xi \le x \le a$. Since

$$G'(x,0,t) = \delta(t-x/c) + \sum_{k=1}^{\infty}\left\{\delta\left(t-\tfrac{2ka+x}{c}\right) - \delta\left(t-\tfrac{2ka-x}{c}\right)\right\}, \quad (90)$$

the result

$$\psi(x,t) = \int_{-\infty}^{\infty}\psi_0(\tau)G'(x,0,t-\tau)\mathrm{d}\tau = \psi_0(t-x/c) + \sum_{k=1}^{\infty}\left\{\psi_0\left(t-\tfrac{x+2ka}{c}\right) - \psi_0\left(t+\tfrac{x-2ka}{c}\right)\right\} \quad (91)$$

holds, representing thus the time signal $\psi_0(t)$ projected on the interval $x \in (0,a)$ of the x-axis with subsequent negative reflections in the boundary points.

The impulse response for distortion-free transmission line with losses is

$$g(x,t) = \tfrac{c}{2}\exp(-bc^2t)\left\{\mathbf{1}(-x)\mathbf{1}\left(t+x/c\right) + \mathbf{1}(x)\mathbf{1}\left(t-x/c\right)\right\}, \quad (92)$$

that leads to analogous formula for the boundary problem as (93) now in the form

$$G'(x,0,t) = \exp(-bcx)\delta(t-x/c) + \sum_{k=1}^{\infty}\left\{\exp(-bc[2ka+x])\delta(t-[2ka+x]/c) - \exp(-bc[2ka-x])\delta(t-[2ka-x]/c)\right\} \quad (93)$$

with the result

$$\psi(x,t) = \exp(-bcx)\psi_0(t-x/c) + \sum_{k=1}^{\infty}\left\{\exp(-bc[2ka+x])\psi_0(t-[x+2ka]/c) - \exp(-bc[2ka-x])\psi_0(t+[x-2ka]/c)\right\} \quad (94)$$

The impulse response for the full wave equation operator with dispersion (71) has been solved in [2] with the result

$$g(x,t) = \frac{c}{2}\exp(-bc^2t)J_0\left(\sqrt{[b^2c^2-a][c^2t^2-x^2]}\right) \times \left\{\mathbf{1}(-x)\mathbf{1}(t+x/c) + \mathbf{1}(x)\mathbf{1}(t-x/c)\right\} \quad (95)$$

where J_0 is the Bessel function. Thus, in the case of anomalous dispersion the impulse response possesses an oscillatory character.

7 CONCLUSIONS

It was shown that the formalism of symbolic functions together with their proper application and utilization of the convolution principle enables one to easily obtain in a unified manner complicated formulas for the linear systems - from the circuit theory to the theory of transmission lines - and to give them rather simple interpretation that facilitates deeper insight into the meaning of these formulas.

Appendix I. The Fourier and Laplace transform

The Fourier transform pair in the time domain, $f(t) \xleftrightarrow{FT} \mathcal{F}(j\omega)$, where $\mathcal{F}(j\omega)$ denotes the spectral density of the function $f(t)$, is defined as the direct and inverse transform in the form

$$\mathcal{F}(j\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\mathrm{d}t, \quad (96)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(j\omega)\exp(j\omega t)\mathrm{d}\omega, \quad (97)$$

where ω is termed the angular frequency. The Laplace transform $F(p)$, $f_c(t) \xleftrightarrow{LT} F(p)$, is a special modification of the Fourier transform for "*causal*" functions $f_c(t) = f(t)\mathbf{1}(t)$, where $\mathbf{1}(t)$ is the Heaviside unit-step function. Since $f_c(t) = f(t)\mathbf{1}(t)$ is different from zero only on the positive part of the axis t, it can be additionally damped by an exponential factor $\exp(-\sigma t)$. Let $f_c(t,\sigma) = f(t)\exp(-\sigma t)\mathbf{1}(t)$. Then the Fourier transform

$$\mathcal{F}(j\omega,\sigma) = \int_{-\infty}^{\infty} f(t)\mathbf{1}(t)\exp(-\sigma t)\exp(-j\omega t)\mathrm{d}t, \quad (98)$$

is in fact the standard definition of the Laplace transform

$$F(p) = \mathcal{F}(j\omega,\sigma) = \int_0^{\infty} f(t)\exp(-pt)\mathrm{d}t, \quad (99)$$

where $p = \sigma + j\omega$ is the complex frequency.

The inverse Laplace transform can be easily obtained from the inverse Fourier transform (102) in the well-known form

$$f_c(t) = f(t)\mathbf{1}(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(p)\exp(pt)\mathrm{d}p. \quad (100)$$

For the Fourier transform in the space domain the following notation will be used

$$\mathcal{F}(jk) = \int_{-\infty}^{\infty} f(x)\exp(-jkx)\mathrm{d}x, \quad (101)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(jk)\exp(jkx)\mathrm{d}k, \quad (102)$$

where k is commonly termed the wavenumber.

The Fourier and Laplace transforms of the Dirac function $\delta(t)$, the Heaviside unit-step function $\mathbf{1}(t)$, and $t\mathbf{1}(t)$ read

$$1 \xleftrightarrow{LT} \delta(t) \xleftrightarrow{FT} 1, \quad (103)$$

$$1/p \xleftrightarrow{LT} \mathbf{1}(t) \xleftrightarrow{FT} 1/j\omega + \pi\delta(\omega), \quad (104)$$

$$1/p^2 \xleftrightarrow{LT} t\mathbf{1}(t) \xleftrightarrow{FT} 1](j\omega)^2 + j\pi\delta')\omega). \quad (105)$$

Here the complication with the functions not converging to zero for $t \to \infty$, in form of δ-functions in their Fourier spectra is clearly manifested.

The formally identical expressions in the Laplace and Fourier transform domains lead to different originals, *eg* on the contrary to (104) for the inverse space-domain Fourier transform the symmetry in space holds

$$1/(jk) \overset{FT}{\longleftrightarrow} \frac{1}{2}\operatorname{sgn}(x) = \frac{1}{2}\big(\mathbf{1}(x) - \mathbf{1}(-x)\big), \quad (106)$$

$$1/(jk)^2 \overset{FT}{\longleftrightarrow} \frac{1}{2}\big(x\mathbf{1}(x) - x\mathbf{1}(-x)\big), \quad (107)$$

or as the further examples show

$$\frac{1}{p^2+\Omega^2} \overset{LT}{\longleftrightarrow} \frac{\sin(\Omega t)}{\Omega}\mathbf{1}(t), \quad (108)$$

$$\frac{1}{(jk)^2+\beta^2} \overset{FT}{\longleftrightarrow} \frac{1}{2\beta}[\sin(\beta x)\mathbf{1}(x) - \sin(\beta x)\mathbf{1}(-x)]. \quad (109)$$

Hence, while the inverse Laplace transform is suitable for solving the time domain problems, where the principle of causality holds, the inverse Fourier transform is suitable to solve the problems of the infinite homogeneous and isotropic space where the principle of symmetry holds. Generally, if the impulse responses $h_c(t)$, $g_s(x)$ are solutions of the formally identical equations $\mathfrak{L}\{h_c(t)\} = \delta(t)$ and $\mathfrak{L}\{g_s(x)\} = \delta(x)$, *ie* if for $p = jk$ $H(p)$ and $\mathcal{G}(jk)$ are formally of the same form, as *eg* in (108) and (109), then, depending on whether the operator $\mathfrak{L}$ is of even or odd order, the relation

$$g_s(x) = \tfrac{1}{2}[h_c(x) \pm h_c(-x)] \quad (110)$$

between the causal impulse response $h_c(t)$ in time domain and the (even or odd) centrosymmetric impulse response $g_s(x)$ in space domain holds.

Appendix II. Some properties of the Dirac δ-function and the unit step function

The fundamental property of the Dirac δ-function is the well-known relation

$$\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{d}t = f(0). \quad (111)$$

Since no regular continuous or discontinuous function possesses this property $\delta(t)$ is often denoted as the symbolic function, distribution, or defined only in the sense of a functional. From (111) it stems straightforwardly

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1, \quad (112)$$

ie the unit δ-impulse has always unit area. The convolution integral

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\mathrm{d}\tau = f(t) \quad (113)$$

represents the so called *sifting* property of the δ-function.

From the above the following properties can be easily inferred

$$f(t)\delta(t) = f(0)\delta(t), \ t\delta(t) = 0, \ \delta(-t) = \delta(t), \quad (114)$$

For the derivatives of δ-function the following properties hold

$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t)\mathrm{d}t = (-1)^n f^{(n)}(0), \quad (115)$$

$$\int_{-\infty}^{\infty} f(t-\tau)\delta^{(n)}(\tau)\mathrm{d}\tau = f^{(n)}(t), \quad (116)$$

$$\int_{-\infty}^{\infty} f(t-\tau)\tau^n\delta^{(n)}(\tau)\mathrm{d}\tau = n!f(t), \quad (117)$$

$$\delta^{(n)}(t) = (-1)^n\delta^{(n)}(-t), \ t^n\delta^{(n)}(t) = (-1)^n n!\delta(t). \quad (118)$$

The derivatives of the unit step function $\mathbf{1}(t)$ lead to the δ-function and its derivatives

$$\frac{\mathrm{d}\mathbf{1}(t)}{\mathrm{d}t} = \delta(t), \ \frac{\mathrm{d}^2[t\mathbf{1}(t)]}{\mathrm{d}t^2} = \delta(t), \quad (119)$$

$$\frac{\mathrm{d}^n[t^n\mathbf{1}(t)]}{\mathrm{d}t^n} = n!\delta(t). \quad (120)$$

Hence, the derivatives of the causal function $f_c(t) = f(t)\mathbf{1}(t)$ are

$$f_c'(t) = f'(t)\mathbf{1}(t) + f(0)\delta(t), \quad (121)$$

$$f_c''(t) = f''(t)\mathbf{1}(t) + f'(0)\delta(t) + f(0)\delta'(t), \quad (122)$$

$$f_c^{(n)}(t) = f^{(n)}(t)\mathbf{1}(t) + \sum_{i=1}^{n} f^{(n-i)}(0)\delta^{(i-1)}(t). \quad (123)$$

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