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AXIOMS FOR (α, β, γ)-ENTROPY OF A GENERALIZED PROBABILITY SCHEME

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Abstract

In this communication, we characterize a measure of information of type (α, β, γ) by taking certain axioms parallel to those considered earlier by Harvda and Charvat along with the recursive relation (1.7). Some properties of this measure are also studied. This measure includes Shannon information measure as a special case.

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1. INTRODUCTION

Shannon's measure of entropy for a discrete probability distribution

$$P = (p_1, ..., p_n), \ \ p_i \geq 0, \ \ \sum_{i=1}^{n} p_i = 1,$$

given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i$$

has been characterized in several ways (see Aczel[1]). Out of the many ways of characterization the two elegant approaches are to found in the work of

(i) Fadeev [5], who uses branching property viz.,

$$H_n(p_1, ..., p_n) = H_{n-1}(p_1 + p_2, p_3, ..., p_n) + (p_1 + p_2) H_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right), \tag{1.1}$$

n =3,4,... for the above distribution P, as the basic postulate, and

(ii) Chaundy and McLeod [3], who studied the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) \quad for \ p_i \geq 0, \, q_j \geq 0. \tag{1.2}$$

Both the above mentioned approaches have been extensively exploited and generalized. The most general form of (1.2) has been studied by Sharma and Taneja ,

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who considered the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i)g(q_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i)f(q_j) \quad ,$$

$$\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1, \qquad p_i \geq 0, \quad q_j \geq 0. \tag{1.3}$$

We define the information measure as:

$$H_n(p_1, ..., p_n; \alpha, \beta, \gamma) = (2^{\frac{\gamma-\alpha}{\gamma}} - 2^{\frac{\gamma-\beta}{\gamma}})^{-1} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}),$$

$$\alpha \neq \gamma \neq \beta, \qquad \alpha, \beta, \gamma > 0, \tag{1.4}$$

for a complete probability distribution $P = (p_1, ..., p_n)$, $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$. Measure(1.4) reduces to entropy of type β(or α) when $\alpha = \gamma = 1$(or $\beta = \gamma = 1$) given by

$$H_n(p_1, ..., p_n; \beta) = (2^{1-\beta} - 1)^{-1} \left[\sum_{i=1}^{n} p_i^{\beta} - 1 \right], \quad \beta \neq 1, \ \beta > 0 \tag{1.5}$$

When $\beta \to 1$, measure (1.5) reduces to Shannon's entropy[7], viz.

$$H_n(p_1, ..., p_2) = -\sum_{i=1}^{n} p_i \log_2 p_i. \tag{1.6}$$

The measure (1.5) was characterized by many authors by different approaches. Harvda and Charvat[6] characterized (1.5) by an axiomatic approach. Darcozy [4] studied (1.5) by a functional equation. A joint characterization of the measure (1.5) and (1.6) has been done by author in two different ways. Firstly by a generalized functional equation and secondly by an axiomatic approach. Later on Tsallis[8] gave the applications of (1.5) in Physics.

In this communication, we characterized the measure (1.4) by taking certain axioms parallel to those considered earlier by Harvda and Charvat[6] along with the recursive relation (1.7). Some properties of this measure are also studied.

The measure (1.4) satisfies a recursive relation as follows:

$$H_n(p_1, ..., p_n; \alpha, \beta, \gamma) - H_{n-1}(p_1 + p_2, p_3, ..., p_n; \alpha, \beta, \gamma)$$

$$= \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (p_1 + p_2)^{\alpha/\gamma} H_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \alpha, \gamma \right)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} (p_1 + p_2)^{\beta/\gamma} H_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \gamma, \beta \right),$$

$$\alpha \neq \gamma \neq \beta, \qquad \alpha, \beta, \gamma > 0 \tag{1.7}$$

where $p_1+p_2>0, \quad A_{(\alpha,\gamma)}=(2^{\frac{\gamma-\alpha}{\gamma}}-1) \quad and \;\; A_{(\beta,\gamma)}=(2^{\frac{\gamma-\beta}{\gamma}}-1)\,.$

$$H(p_1,p_2,...,p_n;\alpha,\gamma)=A^{-1}_{(\alpha,\gamma)}\left[\sum_{i=1}^{n}p_i^{\alpha/\gamma}-1\right]\;;\;\alpha\neq\gamma;\,\alpha,\gamma>0\neq 1$$

$$H(p_1,p_2,...,p_n;\gamma,\beta)=A^{-1}_{(\beta,\gamma)}\left[1-\sum_{i=1}^{n}p_i^{\beta/\gamma}\right]\;;\;\beta\neq\gamma;\,\beta,\gamma>0\neq 1.$$

Proof:

$H_n(p_1,...,p_n;\alpha,\beta,\gamma)-H_{n-1}(p_1+p_2,p_3,...,p_n;\alpha,\beta,\gamma)$

$=(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{(p_1^{\alpha/\gamma}-p_1^{\beta/\gamma})+(p_2^{\alpha/\gamma}-p_2^{\beta/\gamma})+....+(p_n^{\alpha/\gamma}-p_n^{\beta/\gamma})\}$

$-(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{(p_1+p_2)^{\alpha/\gamma}-(p_1+p_2)^{\beta/\gamma}+(p_3^{\alpha/\gamma}-p_3^{\beta/\gamma})+....+(p_n^{\alpha/\gamma}-p_n^{\beta/\gamma})\}$

$=(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{p_1^{\alpha/\gamma}-p_1^{\beta/\gamma}+p_2^{\alpha/\gamma}-p_2^{\beta/\gamma}-(p_1+p_2)^{\alpha/\gamma}+(p_1+p_2)^{\beta/\gamma}\}$

$=(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{p_1^{\alpha/\gamma}+p_2^{\alpha/\gamma}-(p_1+p_2)^{\alpha/\gamma}\}$

$$+(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}\{(p_1+p_2)^{\beta/\gamma}-p_1^{\beta/\gamma}-p_2^{\beta/\gamma}\}$$

$=(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}(p_1+p_2)^{\alpha/\gamma}\left[\frac{p_1^{\alpha/\gamma}}{(p_1+p_2)^{\alpha/\gamma}}+\frac{p_2^{\alpha/\gamma}}{(p_1+p_2)^{\alpha/\gamma}}-1\right]$

$$+(2^{\frac{\gamma-\alpha}{\gamma}}-2^{\frac{\gamma-\beta}{\gamma}})^{-1}(p_1+p_2)^{\beta/\gamma}\left[1-\frac{p_1^{\beta/\gamma}}{(p_1+p_2)^{\alpha/\gamma}}-\frac{p_2^{\beta/\gamma}}{(p_1+p_2)^{\alpha/\gamma}}\right]$$

$=\frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}}(p_1+p_2)^{\alpha/\gamma}H_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\,;\,\alpha\,,\,\gamma\right)$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)}-A_{(\alpha,\gamma)}}(p_1+p_2)^{\beta/\gamma}H_2\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\,;\;\gamma,\,\beta\right),$$

which proves (1.7).

2. SET OF AXIOMS

For characterizing a measure of information of type (α,β,γ) associated with a probability distribution $P=(p_1,...,p_n)\,,\ p_i\geq 0,\ \sum_{i=1}^{n}p_i=1\,,$ we introduce the following axioms:

(1) $H_n(p_1,...,p_n;\,\alpha,\,\beta,\gamma)$ is continuous in the region
$p_i\geq 0,\ \ \sum_{i=1}^{n}p_i=1, \quad \alpha,\,\beta,\gamma>0;$

(2) $H_2(1,\,0\,;\,\alpha,\,\beta,\gamma)\,=0$

(3) $H_2(\frac{1}{2},\,\frac{1}{2}\,;\alpha,\,\beta,\gamma)\ \ =\ \ 1,\ \alpha,\,\beta,\gamma>0$

(4) $H_n(p_1,...,p_{i-1},0,\ p_{i+1},...,p_n;\,\alpha,\,\beta,\gamma)$

$$=H_{n-1}(p_1,...,p_{i-1},\ p_{i+1},...,p_n;\,\alpha,\,\beta,\gamma)$$

for every i = 1,2,. . . ,n;

(5) $H_{n+1}(p_1, ..., p_{i-1}, v_{i_1}, v_{i_2}, p_{i+1}, ..., p_n; \alpha, \beta, \gamma)$

$$-H_n(p_1, ..., p_{i-1}, p_i, p_{i+1}, ..., p_n; \alpha, \beta, \gamma)$$

$$= \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} p_i^{\alpha/\gamma} H_2(v_{i_1}/p_i,\, v_{i_2}/p_i\,;\, \alpha,\, \gamma)$$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} p_i^{\beta/\gamma} H_2(v_{i_1}/p_i,\, v_{i_2}/p_i\,;\, \gamma,\, \beta),$$

$$\alpha \neq \gamma \neq \beta\,, \quad \alpha\,,\, \beta, \gamma > 0$$

for every $v_{i_1} + v_{i_2} = p_i > 0, \;\; i = 1,\, 2..., n,$
where, $A_{(\alpha,\gamma)} = (2^{\frac{\gamma-\alpha}{\gamma}} - 1) \;\; and \;\; A_{(\beta,\gamma)} = (2^{\frac{\gamma-\beta}{\gamma}} - 1)\,, \quad \alpha \neq \gamma \neq \beta$

Theorem 2.1. If $\alpha \neq \beta \neq \gamma$; $\alpha, \beta, \gamma > 0$, then the axioms $(1) - (5)$ determine a measure given by

$$H_n(p_1, ..., p_n; \alpha, \beta, \gamma) = (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma})\,,$$

$$\alpha \neq \gamma \neq \beta\,, \quad \alpha\,,\, \beta, \gamma > 0 \tag{2.1}$$

where $A_{(\alpha,\gamma)} = (2^{\frac{\gamma-\alpha}{\gamma}} - 1) \;\; and \;\; A_{(\beta,\gamma)} = (2^{\frac{\gamma-\beta}{\gamma}} - 1)\,.$
Before proving the theorem we prove some intermediate results based on the above axioms:

Lemma 1. If $v_k \geq 0$, $k = 1, 2, ..., m$; $\sum_{k=1}^{m} v_k = p_i > 0$,then

$H_{n+m-1}(p_1, ..., p_{i-1,} v_1, ..., v_m,\; p_{i+1}, ..., p_n;\; \alpha,\; \beta, \gamma)$

$$= H_n(p_1, ..., p_n;\; \alpha,\; \beta, \gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} p_i^{\alpha/\gamma} H_m(v_1/p_i, ..., v_m/p_i\,;\, \alpha,\, \gamma)$$

.

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} p_i^{\beta/\gamma} H_m(v_1/p_i, ..., v_m/p_i\,;\; \gamma,\, \beta) \tag{2.2}$$

Proof :- To prove the lemma, we proceed by induction.For m = 2, the desired statement holds(cf. Axiom(IV)). Let us suppose that the result is true for numbers less than or equal to m, we shall prove it for m+1. We have

$H_{n+m}(p_1, ..., p_{i-1,} v_1, ..., v_{m+1},\; p_{i+1}, ..., p_n;\; \alpha,\; \beta, \gamma)$

$$= H_{n+1}(p_1, ..., p_{i-1,} v_1, L,\; p_{i+1}, ..., p_n;\; \alpha,\; \beta, \gamma)$$

$$+\frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} L^{\alpha/\gamma} H_m(v_2/L, ..., v_{m+1}/L;\; \alpha,\; \gamma)$$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} L^{\beta/\gamma} H_m(v_2/L, ..., v_{m+1}/L;\; \gamma, \beta)$$

(where $L = v_2 + ... + v_{m+1}$)

$$= H_n(p_1, ..., p_n;\, \alpha,\, \beta, \gamma) + \tfrac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} p_i^{\alpha/\gamma} H_2(v_1/p_i, L/p_i;\, \alpha,\, \gamma)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} p_i^{\beta/\gamma} H_2(v_1/p_i, L/p_i;\; \gamma, \beta)$$

$$+ \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} L^{\alpha/\gamma} H_m(v_2/L, ..., v_{m+1}/L;\, \alpha,\, \gamma)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} L^{\beta/\gamma} H_m(v_2/L, ..., v_{m+1}/L;\; \gamma, \beta)$$

$$= H_n(p_1, ..., p_n;\, \alpha,\, \beta, \gamma) + \tfrac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} \{p_i^{\alpha/\gamma} H_2(v_1/p_i, L/p_i;\, \alpha,\, \gamma)$$

$$+ L^{\alpha/\gamma} H_m(v_2/L, ..., v_{m+1}/L;\, \alpha,\, \gamma)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} \{p_i^{\beta/\gamma} H_2(v_1/p_i, L/p_i;\; \gamma, \beta)$$

$$+ L^{\beta/\gamma} H_m(v_2/L, ..., v_{m+1}/L;\; \gamma, \beta)\} \tag{2.3}$$

where $p_i = v_1 + L > 0$

One more application of induction premise yields

$$H_{m+1}(v_1/p_i, ..., v_{m+1}/p_i;\, \alpha,\, \beta, \gamma) = H_2(v_1/p_i, L/p_i; \alpha, \beta, \gamma)$$

$$+ \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (L/p_i)^{\alpha/\gamma} H_m(v_2/L,\, ..., v_{m+1}/L\,;\, \alpha,\, \gamma)$$

$$+ \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (L/p_i)^{\beta/\gamma} H_m(v_2/L,\, ..., v_{m+1}/L\,;\, \gamma,\, \beta) \tag{2.4}$$

For $\beta = \gamma$, (2.4) reduces to

$$H_{m+1}(v_1/p_i, ..., v_{m+1}/p_i;\, \alpha,\, \gamma) =$$

$$= H_2(v_1/p_i,\, L/p_i\,;\, \alpha,\, \gamma) + (L/p_i)^{\alpha/\gamma} H_m(v_2/L,\, ..., v_{m+1}/L\,;\, \alpha,\, \gamma) \tag{2.5}$$

Similarly for $\alpha = \gamma$, (2.4) reduces to

$$H_{m+1}(v_1/p_i, ..., v_{m+1}/p_i;\, \gamma,\, \beta) =$$

$$= H_2(v_1/p_i,\, L/p_i\,;\, \gamma,\, \beta) + (L/p_i)^{\beta/\gamma} H_m(v_2/L,\, ..., v_{m+1}/L\,;\, \gamma,\, \beta) \tag{2.6}$$

Expression (2.3) together with (2.5) and (2.6) gives the desired result.
Lemma 2 If $v_{i\,j} \geq 0,\ j = 1, 2, ..., m_i,\ \sum_{j=1}^{m_i} v_{i\,j} = p_i > 0,\ i = 1, 2, ..., n,$ $\sum_{i=1}^{n} p_i = 1$, then

$$H_{m_1+...+m_n}(v_{1\,1}, v_{1\,2}, ..., v_{1\,m_1} : ... : v_{n\,1}, v_{n\,2}, ..., v_{n\,m_n}; \alpha,\ \beta, \gamma)$$

$$= H_n(p_1, p_2, ..., p_n; \alpha,\ \beta, \gamma) +$$

$$+ \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} \sum_{i=1}^{n} p_i^{\alpha/\gamma} H_{m_i}(v_{i\,1}/p_i, ..., v_{i\,m_i}/p_i; \alpha,\ \gamma) +$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} \sum_{i=1}^{n} p_i^{\beta/\gamma} H_{m_i}(v_{i\,1}/p_i, ..., v_{i\,m_i}/p_i;\ \gamma,\ \beta) \tag{2.7}$$

Proof: Proof of this lemma is directly follow from lemma 1.
Lemma 3. If $F(n; \alpha, \beta, \gamma) = H_n(1/n, ..., 1/n;\ \alpha,\ \beta, \gamma)$, then

$$F(n; \alpha, \beta, \gamma) = \tfrac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} F(n; \alpha, \gamma) + \tfrac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} F(n; \gamma, \beta) \tag{2.8}$$

Where $F(n; \alpha,\ \gamma) =\ A_{(\alpha,\gamma)}^{-1}(n^{\frac{\gamma-\alpha}{\gamma}} - 1),\ \ \alpha \neq \gamma$ and
$F(n;\ \gamma,\ \beta) =\ A_{(\beta,\gamma)}^{-1}(n^{\frac{\gamma-\beta}{\gamma}} - 1),\ \ \beta \neq \gamma$ (2.9)
Proof. Replacing in Lemma 2 m_i by m and putting $v_{i\,j} = 1/mn$, i=1,2,. . . ,n; j=1,2,. . . ,m, where m and n are positive integer, we have

$$F(mn;\ \alpha,\ \beta, \gamma) = F(m;\ \alpha,\ \beta, \gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (1/m)^{\frac{\alpha-\gamma}{\gamma}} F(n;\ \alpha, \gamma)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} (1/m)^{\frac{\beta-\gamma}{\gamma}} F(n;\ \gamma,\ \beta) \tag{2.10}$$

$$F(mn;\ \alpha,\ \beta, \gamma) = F(n;\ \alpha,\ \beta, \gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (1/n)^{\frac{\alpha-\gamma}{\gamma}} F(m;\ \alpha, \gamma)$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} (1/n)^{\beta/\gamma - 1} F(m;\ \gamma,\ \beta), \tag{2.11}$$

Putting m = 1 in (2.10) and using $F(1; \alpha, \beta, \gamma) = 0$(by axiom 2), we get

$$F(n;\ \alpha,\ \beta, \gamma) = \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\lambda)} - A_{(\beta,\gamma)}} F(n;\ \alpha, \gamma)\ + \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} F(n;\ \gamma,\ \beta),$$

which is (2.8).
Comparing the right hand sides of (2.10) and (2.11), we get.

$$F(m;\ \alpha,\ \beta, \gamma) + \tfrac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} (1/m)^{\frac{\alpha}{\alpha-\gamma}} F(n;\ \alpha, \gamma) +$$

$$+ \frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}} (1/m)^{\frac{\beta}{\beta-\gamma}} F(n;\ \gamma,\ \beta)$$

$$= F(n;\,\alpha,\,\beta,\gamma) + \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}}\,(1/n)^{\frac{\alpha}{\alpha-\gamma}}\,F(m;\,\alpha,\gamma)+$$

$$+\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}}\,(1/n)^{\frac{\beta}{\beta-\gamma}}\,F(m;\,\gamma,\,\beta) \qquad (2.12)$$

Equation (2.12) together with (2.8) gives

$$A_{(\alpha,\gamma)}\{[1-(1/n)^{\alpha/\gamma-1}]\,F(m;\,\alpha,\,\gamma)\,+\,[(1/m)^{\alpha/\gamma-1}-1]\,F(n;\,\alpha,\,\gamma)\}$$
$$= A_{(\beta,\gamma)}\{[1-(1/n)^{\beta/\gamma-1}]\,F(m;\,\gamma,\,\beta)\,+\,[(1/m)^{\beta/\gamma-1}-1]\,F(n;\,\gamma,\,\beta)\}. \qquad (2.13)$$

Putting n = 2 in (2.13) and use $F(2,\alpha,\beta,\gamma) = H_2(\frac{1}{2},\frac{1}{2};\,\alpha,\,\beta,\gamma) = 1$, we get

$$A_{(\alpha,\gamma)}\{(1-2^{1-\alpha/\gamma})\,F(m;\,\alpha,\,\gamma) - (1-(1/m)^{\alpha/\gamma-1})\}$$

$$= A_{(\beta,\gamma)}\{(1-2^{1-\beta/\gamma})\,F(m;\,\gamma,\,\beta) - (1-(1/m)^{\beta/\gamma-1})\} = C\ \ (say),$$

i.e., $A_{(\alpha,\gamma)}\{(1-2^{1-\alpha/\gamma})\,F(m;\,\alpha,\,\gamma) - (1-(1/m)^{\alpha/\gamma-1})\}$ =C ,
where C is an arbitrary constant.
For m = 1 , we get C=0
Thus, we have

$$F(m;\,\alpha,\,\gamma) == \frac{1-m^{1-\alpha/\gamma}}{1-2^{1-\alpha/\gamma}} = A^{-1}_{(\alpha,\gamma)}(m^{1-\alpha/\gamma}-1), \quad \alpha\neq\gamma.$$

Similarly,

$$F(m;\,\gamma,\,\beta) = \frac{1-m^{1-\beta/\gamma}}{1-2^{1-\beta/\gamma}} = A^{-1}_{(\beta,\gamma)}(m^{1-\beta/\gamma}-1), \quad \beta\neq\gamma,$$

which is (2.9).
Now (2.8) together with (2.9) gives

$$F(n;\,\alpha,\,\beta,\gamma) = \frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}}F(n;\,\alpha,\gamma)\,+\,\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}}F(n;\,\gamma,\,\beta),$$

$$= (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}(n^{1-\alpha/\gamma} - n^{1-\beta/\gamma}). \qquad (2.14)$$

Proof of the theorem:
We prove the theorem for rationals and then the continuity axiom (1) extends the result for reals. For this let m and $r_i's$ be positive integers such that $\sum_{i=1}^{n} r_i = m$ and if we put $p_i = r_i/m$, i=1,2,. . . .,n then an application of lemma 2 gives

$$H_m(\underbrace{1/m,...,1/m}_{r_1},...,\underbrace{1/m,...,1/m}_{r_n};\alpha,\beta,\gamma) = H_n(p_1,p_2,...,p_n;\alpha,\,\beta,\gamma)+$$

$$+\,\frac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}}\sum_{i=1}^{n} p_i^{\alpha/\gamma} H_{r_i}(1/r_i,...,1/r_i;\alpha,\,\gamma)$$

$$+\,\frac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)} - A_{(\alpha,\gamma)}}\sum_{i=1}^{n} p_i^{\beta/\gamma} H_{r_i}(1/r_i,...,1/r_i;\,\gamma,\,\beta)$$

i.e., $H_n(p_1, ..., p_n;\, \alpha,\, \beta, \gamma) = F(m;\, \alpha,\, \beta, \gamma)$

$$-\tfrac{A_{(\alpha,\gamma)}}{A_{(\alpha,\gamma)}-A_{(\beta,\gamma)}} \textstyle\sum_{i=1}^{n} p_i^{\alpha/\gamma} F(r_i;\, \alpha, \gamma)$$

$$-\tfrac{A_{(\beta,\gamma)}}{A_{(\beta,\gamma)}-A_{(\alpha,\gamma)}} \textstyle\sum_{i=1}^{n} p_i^{\beta/\gamma} F(r_i;\, \gamma, \beta) \tag{2.15}$$

Equation (2.15) together with (2.9) and (2.14) gives

$$H_n(p_1, ..., p_n;\, \alpha,\, \beta, \gamma) = \frac{1}{A_{(\alpha,\gamma)} - A_{(\beta,\gamma)}} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}),$$

$$\alpha \neq \gamma \neq \beta\,, \quad \alpha\,, \beta, \gamma > 0$$

which is (2.1).
This completes the proof of the theorem.

3. PROPERTIES OF ENTROPY OF TYPE (α, β, γ)

The measure $H_n(P;\, \alpha,\, \beta, \gamma)$, where $P = (p_1, ..., p_n)\,,\ p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1$ is a probability distribution, as characterized in the preceding section, satisfies certain properties, which are given in the following theorems:

Theorem 3.1. The measure $H_n(P;\, \alpha,\, \beta, \gamma)$ is non-negative for

$$\alpha \neq \gamma \neq \beta\,, \quad \alpha\,, \beta, \gamma > 0$$

Proof:-

Case 1. $\alpha > \gamma\,;\ \ \beta < \gamma \Rightarrow \frac{\alpha}{\gamma} > 1\, and\, \frac{\beta}{\lambda} < 1;$

$$\Rightarrow \sum_{i=1}^{n} p_i^{\alpha/\gamma} < 1\, and \sum_{i=1}^{n} p_i^{\beta/\gamma} > 1$$

$$\Rightarrow \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}) < 0$$

Since , $\alpha > \gamma\, and\, \beta < \gamma$;we get

$$(2^{1-\alpha/\gamma} - 2^{1-\beta/\gamma})^{-1} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma}) > 0$$

Case 2.. Similarly for $\alpha < \gamma\, and\, \beta > \gamma$, we get

$$(2^{1-\alpha/\gamma} - 2^{1-\beta/\gamma})^{-1} \sum_{i=1}^{n} p_i^{\alpha/\gamma} - p_i^{\beta/\gamma} > 0$$

Therefore from case 1,case 2 and axiom 2, we get

$$H_n(P;\, \alpha,\, \beta, \gamma) \geq 0$$

This completes the proof of theorem.

Definition. We shall use the following definition of a convex function.

A function $f(.)$ over the points in a convex set R is convex $\cap$ if for all $r_1, r_2 \in R\ \ and\ \mu \in (0,1)$

$$\mu f(r_1) + (1-\mu) f(r_2) \leq f(\mu r_1 + (1-\mu) r_2). \tag{3.1}$$

The function $f(.)$ is convex $\cup$ if (3.1) holds with $\geq$ in place of $\leq$.
Theorem 3.2. The measure$H_n(P;\,\alpha,\ \beta,\gamma)$ is convex $\cap$ function of the probability distribution $P = (p_{1,\ldots\ldots,}p_n), p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1$,when either $\alpha > \gamma;\ \beta \leq \gamma\ or\ \beta > \gamma;\ \alpha \leq \gamma.$
Proof:- Let there be r distributions

$$P_k(X) = \{p_k(x_1), ..., p_k(x_n)\}, \quad \sum_{i=1}^{n} p_k(x_i) = 1,\ \ k = 1,2,...,r, \tag{3.2}$$

associated with the random variable $X = (x_1, ..., x_n)$
consider r numbers $(a_1, ..., a_r)$ such that $a_k \geq 0$ and $\sum_{k=1}^{r} a_k = 1$ and define

$$P_o(X) = \{p_o(x_1), ..., p_o(x_n)\},$$

where

$$p_o(x_i) = \sum_{k=1}^{r} a_k p_k(x_i),\ \ i = 1,2,...,n. \tag{3.3}$$

Obviously, $\sum_{i=1}^{n} p_o(x_i) = 1$ and thus P$_o$(x) is a bonafide distribution of X.
Let $\alpha > \gamma, 0 < \beta \leq \gamma$, then we have

$$\sum_{k=1}^{r} a_k H_n(p_k;\,\alpha,\,\beta,\gamma) - H_n(P_0(\alpha,\beta,\gamma) = \sum_{k=1}^{r} a_k H_n(p_k;\,\alpha,\,\beta,\gamma)$$

$$-(A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}\left\{\left[\sum_{j=1}^{r} a_j p_j\right]^{\alpha/\gamma} - \left[\sum_{j=1}^{r} a_j p_j\right]^{\beta/\gamma}\right\} \tag{3.4}$$

$$\leq \sum_{k=1}^{r} a_k H_n(p_k;\alpha,\beta,\gamma) - (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}(\sum_{j=1}^{r} a_j p_j^{\alpha/\gamma} - \sum_{j=1}^{r} a_j p_j^{\beta/\gamma}) = 0$$

(by Jensen inequality)

$\Rightarrow \sum_{k=1}^{r} a_k H_n(p_k;\alpha,\beta,\gamma) - H_n(P_o;\,\alpha,\ \beta,\gamma) \leq 0$
i.e., $\sum_{k=1}^{r} a_k H_n(p_k;\,\alpha,\,\beta,\gamma) \leq H_n(P_o;\,\alpha,\ \beta,\gamma)$
for $\alpha > \gamma, 0 < \beta \leq \gamma$,
By symmetry in α ,β and γ the above result is true for $\beta > \gamma, 0 < \alpha \leq \gamma$.

Theorem3.3. The measure$H_n(p;\,\alpha,\ \beta,\gamma)$ satisfies the following relations:
(i) Generalized –Additive:

$$H_{nm}(P*Q;\,\alpha,\,\beta,\gamma) = G_n(P;\alpha,\ \beta,\gamma) H_m(Q;\alpha,\ \beta,\gamma) +$$

$$+ G_m(Q;\alpha,\ \beta,\gamma) H_n(P;\alpha,\ \beta,\gamma),$$

$$\alpha,\ \beta,\gamma > 0, \tag{3.5}$$

where $G_n(P;\alpha,\ \beta,\gamma)=\frac{1}{2}\sum_{i=1}^{n}(p_i^{\alpha/\gamma}+p_i^{\beta/\gamma}),\ \alpha,\ \beta,\gamma>0.$ (3.6)
(ii) Sub-Additive: For $\alpha,\beta>\gamma$, the measure$H_n(p;\ \alpha,\ \beta,\gamma)$ is sub-additive
i.e., $H_{nm}(P*Q;\alpha,\beta,\gamma)\ \leq\ H_n(P;\alpha,\ \beta,\gamma)+H_m(Q;\alpha,\ \beta,\gamma)$, (3.7)
where$P=(p_1,...,p_n),\ \ Q=(q_1,...,q_m)$ and

$$P*Q=(p_1q_1,...,p_1q_m,...,p_nq_1,...,p_nq_m)$$

are complete probability distributions.
Proof.(i) We have

$$H_{nm}(P*Q;\alpha,\beta,\gamma)=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}[(p_iq_j)^{\alpha/\gamma}-(p_iq_j)^{\beta/\gamma}]$$

$$=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}[(p_iq_j)^{\alpha/\gamma}-(p_iq_j)^{\beta/\gamma}+p_i^{\alpha/\gamma}q_j^{\beta/\gamma}-\ p_i^{\alpha/\gamma}q_j^{\beta/\gamma}]$$

$$=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}[p_i^{\alpha/\gamma}q_j^{\alpha/\gamma}-p_i^{\beta/\gamma}q_j^{\beta/\gamma}+p_i^{\alpha/\lambda}q_j^{\beta/\gamma}-\ p_i^{\alpha/\gamma}q_j^{\beta/\gamma}]$$

$$=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}[p_i^{\alpha/\gamma}(q_j^{\alpha/\gamma}+q_j^{\beta/\gamma})-q_j^{\beta/\gamma}(p_i^{\alpha/\gamma}+p_i^{\beta/\gamma})]$$

$$=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}[\sum_{i=1}^{n}p_i^{\alpha/\gamma}\sum_{j=1}^{m}(q_j^{\alpha/\gamma}+q_j^{\beta/\gamma})-\sum_{j=1}^{m}\ q_j^{\beta/\gamma}\sum_{i=1}^{n}(p_i^{\alpha/\gamma}+p_i^{\beta/\gamma})] \tag{3.8}$$

Also

$$H_{nm}(P*Q;\alpha,\beta,\gamma)=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}[(p_iq_j)^{\alpha/\gamma}-(p_iq_j)^{\beta/\gamma}]$$

$$=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}[(p_iq_j)^{\alpha/\gamma}-(p_iq_j)^{\beta/\gamma}+p_i^{\beta/\gamma}q_j^{\alpha/\gamma}-\ p_i^{\beta/\gamma}q_j^{\alpha/\gamma}]$$

$$=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}[p_i^{\alpha/\gamma}q_j^{\alpha/\gamma}-p_i^{\beta/\gamma}q_j^{\beta/\gamma}+p_i^{\beta/\gamma}q_j^{\alpha/\gamma}-\ p_i^{\beta/\gamma}q_j^{\alpha/\gamma}]$$

$$=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}[q_j^{\alpha/\gamma}(p_i^{\alpha/\gamma}+p_i^{\beta/\gamma})-p_i^{\beta/\gamma}(q_j^{\alpha/\gamma}+q_j^{\beta/\gamma})]$$

$$=(A_{(\alpha,\gamma)}-A_{(\beta,\gamma)})^{-1}[\sum_{j=1}^{m}q_j^{\alpha/\gamma}\sum_{i=1}^{n}(p_i^{\alpha/\gamma}+p_i^{\beta/\gamma})-\sum_{i=1}^{n}\ p_i^{\beta/\gamma}\sum_{i=1}^{n}(q_j^{\alpha/\gamma}+q_j^{\beta/\gamma})] \tag{3.9}$$

Adding (3.8) and (3.9), we get

$$2\,H_{nm}(P*Q;\alpha,\beta,\gamma) = (A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}[\sum_{i=1}^{n} p_i^{\alpha/\gamma} \sum_{j=1}^{m} (q_j^{\alpha/\gamma} + q_j^{\beta/\gamma})$$

$$-\sum_{j=1}^{m} q_j^{\beta/\gamma} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} + p_i^{\beta/\gamma})]$$

$$+(A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1}[\sum_{j=1}^{m} q_j^{\alpha/\gamma} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} + p_i^{\beta/\gamma}) - \sum_{i=1}^{n} p_i^{\beta/\gamma} \sum_{i=1}^{n} (q_j^{\alpha/\gamma} + q_j^{\beta/\gamma})]$$

$$= \sum_{i=1}^{n} (p_i^{\alpha/\gamma} + p_i^{\beta/\gamma})\,(A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{j=1}^{m} (q_j^{\alpha/\gamma} - q_j^{\beta/\gamma})$$

$$+\sum_{j=1}^{m} (q_j^{\alpha/\gamma} + q_j^{\beta/\gamma})\,(A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma})$$

$$H_{nm}(P*Q;\alpha,\beta,\gamma) = \frac{1}{2} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} + p_i^{\beta/\gamma})\,(A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{j=1}^{m} (q_j^{\alpha/\gamma} - q_j^{\beta/\gamma})$$

$$+\frac{1}{2} \sum_{j=1}^{m} (q_j^{\alpha/\gamma} + q_j^{\beta/\gamma})\,(A_{(\alpha,\gamma)} - A_{(\beta,\gamma)})^{-1} \sum_{i=1}^{n} (p_i^{\alpha/\gamma} - p_i^{\beta/\gamma})$$

Using (3.6)

$$H_{nm}(P*Q;\,\alpha,\,\beta,s) = G_n(P;\alpha,\;\beta,\gamma)H_m(Q;\alpha,\;\beta,\gamma) +$$

$$+G_m(Q;\alpha,\;\beta,\gamma)H_n(P;\alpha,\;\beta,\gamma)$$

which is (3.5). This completes the proof of part (i).

Proof (ii):- From part (i), we have

$$H_{nm}(P*Q;\,\alpha,\,\beta,\gamma) = G_n(P;\alpha,\;\beta,\gamma)H_m(Q;\alpha,\;\beta,\gamma) +$$

$$+G_m(Q;\alpha,\;\beta,\gamma)H_n(P;\alpha,\;\beta,\gamma)$$

As $G_n(P;\alpha,\;\beta,\gamma) = \frac{1}{2}\sum_{i=1}^{n}(p_i^{\alpha/\gamma} + p_i^{\beta/\gamma})\;\leq 1\;for\;\alpha,\,\beta\,\geq\gamma$

Therefore,

$$H_{nm}(P*Q;\,\alpha,\,\beta,\gamma) \leq H_m(Q;\alpha,\;\beta,\gamma) + H_n(P;\alpha,\;\beta,\gamma)$$

This proves the sub-additivity.

Conclusion: In addition to well known information measure of Shannon, Renyi's, Harvda-Charvat, Vajda, Darcozy, we have characterize a measure which we call (α, β, γ) information measure. We have given some basic axioms and properties with recursive relation . The Shannon measure included in the (α, β, γ) information measure for the limiting case $\alpha = \gamma = 1$ and $\beta \to 1$ or $\beta = \gamma = 1$ and $\alpha \to 1$. This measure is generalization of Harvda Charvat entropy.

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