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ON GENERALIZED GAMMA CONVOLUTION DISTRIBUTIONS

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Abstract

We present here characterizations of certain families of generalized gamma convolution distributions of L. Bondesson based on a simple relationship between two truncated moments. We also present a list of well-known random variables whose distributions or the distributions of certain functions of them belong to the class of generalized gamma convolutions.

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1. INTRODUCTION

It is widely known that the problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus, various characterizations have been established in many different directions. The present work deals with the characterizations of certain generalized gamma convolution (GGC) distributions which appeared in the excellent work of Bondesson [1]. We also present a list of well-known univariate continuous random variables whose distributions or the distributions of certain functions of them are members of the GGC class.

In what follows we copy some statements and results due to Bondesson [1, 2] and a definition due to Thorin [15]. The class of GGC was introduced by Thorin [15], as a useful tool for proving infinite divisibility of special distributions. Bondesson called this class of distributions, which are infinitely divisible , $\Im - class$ in honor of Thorin. It is the smallest class of distributions on $\mathbb{R}_+ = [0, \infty)$ that contains gamma distributions and is closed with respect to convolution and weak limits.

Definition 1.1 (Thorin 1977 as stated on p. 29 of [2]). A generalized gamma convolution (or $\Im - distribution$) is a probability distribution F on $\mathbb{R}_+$ with moment generating function of the form

$$\varphi(s) = \int e^{sx} F(dx) = \exp\left\{as + \int \log\left(\frac{t}{t-s}\right) U(dt)\right\},\ s \leq 0\ (or\ s \in \mathbb{C}\setminus\ (0, \infty)),$$

where $a \geq 0$ and $U(dt)$ is a nonnegative measure on $(0, \infty)$ satisfying

$$\int_{(0,1]} |\log t|\, U(dt) < \infty\ and\ \int_{(1,\infty)} t^{-1} U(dt) < \infty\,.$$

The following theorem and corollary are due to Bondesson appearing in his important contribution to the field of Probability Theory [1].

Theorem 1.2 (Bondesson 1979). All density functions on $(0, \infty)$ of the form

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$$f(x) = C\, x^{\beta-1} \Pi_{j=1}^{M} \left(1 + \sum_{k=1}^{N_j} c_{jk}\, x^{\alpha_{jk}}\right)^{-\gamma_j}, \quad x > 0\,, \qquad (1.1)$$

where all the parameters are strictly positive and the $\alpha_{jk}\,'s$ less than or equal to 1 and C is a normalizing constant, are generalized gamma convolutions; consequently all densities (distributions) which are weak limits of densities of the form (1.1) are generalized gamma convolutions (and thus infinitely divisible) as well. In the nonlimit case the total spectral mass $U(\infty)$ of the generalized gamma convolution is β.

Corollary 1.3 (Bondesson 1979). All densities of the form

$$f(x) = C\, x^{\beta-1} \exp\left\{-\sum_{k=1}^{N} c_k\, x^{\alpha_k}\right\}, \quad x > 0\,, \qquad (1.2)$$

where $c_k > 0$ and $|\alpha_k| \leq 1$, and their weak limits are generalized gamma convolutions and thus infinitely divisible.

Remark 1.4. (Bondesson 1979). Note that if a random variable X has a density of the form (1.2), so has X^q, $|q| \geq 1$.

Bondesson [1], states that he exposed some special cases of Theorem 1.2 and Corollary 1.3 which have been considered earlier by other authors using Thorin's (Grosswald's) method. To simplify the identification of these distributions, Bondesson employed the names used by Johnson and Kotz [13], which we also continue to use here. From Theorem 1.2 and Corollary 1.3, Bondesson showed that all the members of the following families

$$f(x) = C\, x^{\beta-1} (1 + c\, x^{\alpha})^{-\gamma}, \; x > 0\,, \; 0 < \alpha \leq 1\,, \qquad (1.3)$$

$$f(x) = C\, x^{\beta-1} \exp\{-c\, x^{\alpha}\}, \; x > 0\,, \; 0 < |\alpha| \leq 1\,, \qquad (1.4)$$

$$f(x) = C\, x^{\beta-1} \exp\left\{-\left(c_1 x + c_2 x^{-1}\right)\right\}, \; x > 0, -\infty < \beta < \infty, \; (1.5)$$

where the natural restrictions are put on the unspecified parameters, are densities of generalized gamma convolutions and thus infinitely divisible.

Setting in (1.4) $\beta = \sigma^{-2}\left(\mu + \alpha^{-1}\right)$ and $c = \sigma^{-2}\alpha^{-2}$, where μ and σ $(\sigma > 0)$ are constant, and letting α tend to zero, we obtain as a weak limit the density

$$f(x) = C\, x^{-1} \exp\left\{-(\log x - \mu)^2 / \left(2\sigma^2\right)\right\}, \; x > 0, \qquad (1.6)$$

which is lognormal density.

Remark 1.5. Bondesson ([2], Theorem 6.2.4) pointed out that multiplying densities (1.3) – (1.6) by $C_1(\delta + x)^{-\nu}$ for $\delta > 0$ and $\nu > 0$, will result in densities belonging to $\Im - class$.

2. Characterization Results

In this section we present characterizations of the families of distributions with densities of the form (1.3) – (1.6) in terms of a simple relationship between two truncated moments. We like to mention here the works of Galambos and Kotz [3], Kotz and Shanbhag [14], Glänzel [5 – 6], Glänzel et al. [7, 8], Glänzel and Hamedani [9] and Hamedani [10 – 12] in this direction. Our characterization results presented here will employ an interesting result due to Glänzel [6] (Theorem G below).

Theorem G. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ ($a = -\infty$, $b = \infty$might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}\left[g\left(X\right) \mid X \geq x\right] = \mathbf{E}\left[h\left(X\right) \mid X \geq x\right] \eta\left(x\right), \qquad x \in H ,$$

is defined with some real function η . Assume that g , $h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H . Finally, assume that the equation $h\eta = g$ has no real solution in the interior of H . Then F is uniquely determined by the functions g , h and η , particularly

$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)h(u) - g(u)} \right| \exp\left(-s(u)\right) du,$

where the function s is a solution of the differential equation $s' = \frac{\eta' h}{\eta h - g}$ and C is a constant, chosen to make $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions g_n, h_n and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem G and let $g_n \to g$, $h_n \to h$ for some continuously differentiable real functions g and h. Let, finally, X be a random variable with distribution F . Under the condition that $g_n(X)$ and $h_n(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges weakly to η , where

$\eta(x) = \frac{E[g(X) \mid X \geq x]}{E[h(X) \mid X \geq x]}$.

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions g , h and η , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in [9] .

A further consequence of the stability property of Theorem G is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions g , h and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

Remark 2.1.1. In Theorem G, the interval H need not be closed since the condition is only on the interior of H.

Proposition 2.1.2. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = x^{\alpha - \beta}(1 + cx^{\alpha})^{-1}$ and $g(x) = x^{\alpha - \beta}$ for $x \in (0, \infty)$. The density of X is (1.3) for $\gamma > 1$ if and only if the function η defined in Theorem G has the form

$$\eta(x) = \frac{\gamma}{\gamma-1}(1 + cx^{\alpha}) \ , \quad x > 0.$$

Proof. Let X have density (1.3) , then

$$(1 - F(x)) \ \mathbf{E}[h(X) \mid X \geq x] = \frac{C}{\alpha\gamma}(1 + cx^{\alpha})^{-\gamma} \ , \qquad x > 0 \ ,$$

and

$$(1 - F(x)) \ \mathbf{E}[g(X) \mid X \geq x] = \frac{C}{\alpha(\gamma-1)}(1 + cx^{\alpha})^{1-\gamma} \ , \qquad x > 0 \ ,$$

and finally

$$\eta(x) h(x) - g(x) = \frac{1}{\gamma-1} \ x^{\alpha-\beta} > 0, \quad for \ \ \gamma > 1 \ , \ x > 0 \ .$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) \ h(x)}{\eta(x) \ h(x) - g(x)} = \gamma\alpha c \ x^{\alpha-1}(1 + cx^{\alpha})^{-1} , \quad x > 0 \ ,$$

and hence

$$s(x) = \ln(1 + cx^{\alpha})^{\gamma} , \qquad x > 0.$$

Now, in view of Theorem G, X has density (1.3) .

Corollary 2.1.3. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 2.1.2 . The density of X , with $\gamma > 1$, is (1.3) if and only if there exist functions g and η defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = \gamma\alpha c \ x^{\alpha-1}(1 + cx^{\alpha})^{-1} , \qquad x > 0.$$

Remarks 2.1.4. (i) The general solution of the differential equation in Corollary 2.1.3 is

$$\eta(x) = (1 + cx^{\alpha})^{\gamma} \left[-\int g(x) \ \gamma\alpha c \ x^{\beta-1}(1 + cx^{\alpha})^{-\gamma} \, dx + D \right],$$

for $x > 0$, where D is a constant. One set of appropriate functions is given in Proposition 2.1.2 with $D = 0$.

(ii) Clearly there are other triplets of functions (h, g, η) satisfying the conditions of Theorem G. We presented one such triplet in Proposition 2.1.2.

We replace (1.4) with the following density to make it a little more general (see Remark 1.5) and call it $(1.4)^*$

$$f(x) = C \ x^{\beta-1}(\delta + x)^{-\nu} \exp\{-c \ x^{\alpha}\} \ , \quad x > 0 \ , \ \ 0 < |\alpha| \leq 1 \ , \ \nu > 0. \qquad (1.4)^*$$

The proofs of the following 3 propositions are similar to that of Proposition 2.1.2 and hence are omitted.

Proposition 2.1.5. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h(x) = x^{\alpha-\beta}(\delta+x)^{\nu}$ and $g(x) = (\delta+x)^{\nu}$ for $x \in (0,\infty)$. The density of X is $(1.4)^*$ if and only if the function η defined in Theorem G has the form

$$\begin{aligned}
\eta(x) &= c^{1-\frac{\beta}{\alpha}} e^{cx^{\alpha}} \gamma^* \left[cx^{\alpha}, \frac{\beta}{\alpha}\right], & 0<\alpha\le 1\,,\quad x>0,\\
&= c^{1-\frac{\beta}{\alpha}} e^{cx^{\alpha}} \gamma \left[cx^{\alpha}, \frac{\beta}{\alpha}\right] \left(e^{cx^{\alpha}}-1\right)^{-1}, & -1\le\alpha<0\,,\quad x>0,
\end{aligned}$$

where $\gamma[x,\delta] = \int_0^x u^{\delta-1}e^{-u}du$ and $\gamma^*[x,\delta] = \int_x^{\infty} u^{\delta-1}e^{-u}du$.

Proposition 2.1.6. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h(x) = x^{1-\beta}\, e^{c_2 x^{-1}}$ and $g(x) = x^{-\beta}\, e^{c_2 x^{-1}}$ for $x \in (0,\infty)$. The density of X is (1.5) if and only if the function η defined in Theorem G has the form

$\eta(x) = x + \frac{1}{c_1}$, $\quad x>0.$

Proposition 2.1.7. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h(x) \equiv 1$ and $g(x) = x$ for $x \in (0,\infty)$. The density of X is (1.6), with $\mu = 0$ and $\sigma = 1$, if and only if the function η defined in Theorem G has the form

$\eta(x) = e^{1/2}\left[\frac{1-\Phi(\log x - 1)}{1-\Phi(\log x)}\right]$, $\quad x>0,$

where Φ is the cumulative distribution function of the standard normal random variable.

Remark 2.1.8. A corollary and a remark similar to Corollary 2.1.3 and Remark 2.1.4 (part (i)) can be stated for $(1.4)^*$, (1.5) and (1.6) as well.

3 Well-known distributions belonging to $\Im -$ class

We will divide this section to four subsection for (1.3) – (1.6) respectively. In each subsection we list the densities of the well-known distributions belonging to $\Im - class$ as well as the well-known distributions which do not belong (as far as we know) to $\Im - class$, but certain functions of them belong to $\Im - class$. The following lists clearly are not complete, we just gathered what we could at this time.

3.1. Random variables and certain functions of them whose densities are of the form (1.3)

We list the densities of the distributions in alphabetical order rather than that of their importance. Some of these densities have already been listed in Bondesson [1, 2] and some have appeared in our previous works in different contexts.

1) Beta of second kind has density given by
$f(x) = \frac{1}{\delta^\beta B(\beta,\delta)} x^{\beta-1} \left(1 + \frac{1}{\delta} x\right)^{-(\beta+\delta)}, \quad x > 0,$
which is of the form (1.3) with $c = \frac{1}{\delta}$, $\alpha = 1$ and $\gamma = \beta + \delta$.

2) Beta Prim has density
$f(x) = \frac{1}{B(\beta,\delta)} x^{\beta-1} (1 + x)^{-(\beta+\delta)} , \quad x > 0,$
which is of the form (1.3) with $c = 1$, $\alpha = 1$ and $\gamma = \beta + \delta$.

3) Burr Type II has density
$f(x) = \beta\, e^{-x} (e^{-x} + 1)^{-(\beta+1)} , \quad \beta > 0 , \quad x \in \mathbb{R}.$
Letting $Y = e^X$, the density of Y , f_Y , will be
$f_Y(y) = \beta\, y^{\beta-1} (1 + y)^{-(\beta+1)} , \quad y > 0 ,$
which is of the form (1.3) with $c = 1$, $\alpha = 1$ and $\gamma = \beta + 1$. The density of Y is actually that of ratio of two independent gamma and exponential random variables with appropriate parameters.

4) Burr Type III has density

$$\begin{aligned} f(x) &= \tau\delta\, x^{-(\tau+1)} \left(1 + x^{-\tau}\right)^{-(\delta+1)} , \quad \tau > 0 , \ \delta > 0 , \quad x > 0 \\ &= \tau\delta\, x^{\tau\delta-1} (1 + x^{\tau})^{-(\delta+1)} , \end{aligned}$$

which is of the form (1.3) with $\beta = \tau\delta$, $0 < \alpha = \tau \le 1$ and $\gamma = \delta + 1$. We like to mention here that for $\delta = 1$, this density is also the density of the ratio of two independent Weibull random variables with appropriate parameters.

5) Burr Type IV has density
$f(x) = k\, x^{-2} \left(\frac{\delta-x}{x}\right)^{\frac{1}{\delta}-1} \left(1 + \left(\frac{\delta-x}{x}\right)^{\frac{1}{\delta}}\right)^{-(k+1)}, \quad k > 0 , \quad x \in (0, \delta) .$
Letting $Y = \frac{c-X}{X}$, the density of Y is given by

$$f_Y(y) = \frac{k}{\delta}\, y^{\frac{1}{\delta}-1} \left(1 + y^{\frac{1}{\delta}}\right)^{-(k+1)} , \quad y > 0,$$

which is of the form (1.3) with $\alpha = \beta = \frac{1}{\delta}$ $(\delta \ge 1)$, $c = 1$ and $\gamma = k + 1$.

6) Burr Type IX has density
$f(x) = 2k\delta\, e^x (1 + e^x)^{k-1} \left\{\delta \left[(1 + e^x)^k - 1\right] + 2\right\}^{-2}, \ k > 0 , \delta > 0 , \quad x \in \mathbb{R}.$
Taking $k = 1$ and letting $Y = e^X$, the density of Y is
$f_Y(y) = \frac{\delta}{2} \left(1 + \frac{\delta}{2} y\right)^{-2}, \quad y > 0,$
which is of the form (1.3) with $\alpha = \beta = 1$, $c = \frac{\delta}{2}$ and $\gamma = 2$.

7) Burr Type XII has density
$f(x) = (\gamma - 1)\, \beta\, x^{\beta-1} \left(1 + x^{\beta}\right)^{-\gamma} , \ \beta > 0 , \ \gamma > 1 , \quad x > 0,$
which is of the form (1.3) with $c = 1$, $\alpha = \beta \le 1$ and $\gamma > 1$.

8) Compound Gamma (also called Pearson Type VI) distribution has density
$f(x) = \frac{1}{\theta^\beta B(\beta,\delta)}\, x^{\beta-1} \left(1 + \frac{1}{\theta}\, x\right)^{-(\beta+\delta)}, \ \theta > 0 , \ \beta > 0 , \ \delta > 0 , \quad x > 0,$
which is of the form (1.3) with $c = \frac{1}{\theta}$, $\alpha = 1$ and $\gamma = \beta + \delta$.

9) Exponentiated Pareto (or Lomax) has density

$$f(x) = \theta\alpha\xi\, x^{\theta\alpha-1} (\xi + x^{\alpha})^{-(\theta+1)}$$

$$= \theta\alpha\xi^{-\theta}\ x^{\theta\alpha-1}\left(1+\frac{1}{\xi}x^{\alpha}\right)^{-(\theta+1)}\ ,\ \theta>0\ ,\ \alpha>0\ ,\ \xi>0\ ,\quad x>0,$$

which is of the form (1.3) with $c=\frac{1}{\xi}$, $\beta=\theta\alpha$ $(\alpha\leq 1)$ and $\gamma=\theta+1$.

10) F-distribution has density

$f(x)=\frac{\theta^{\theta/2}\xi^{-\theta/2}}{B\left(\frac{\theta}{2},\frac{\xi}{2}\right)}\ x^{\frac{\theta}{2}-1}\left(1+\frac{\theta}{\xi}\ x\right)^{-\left(\frac{\theta+\xi}{2}\right)}$, $\theta>0$, $\xi>0$, $x>0$,

which is of the form (1.3) with $c=\frac{\theta}{\xi}$, $\alpha=1$, $\beta=\frac{\theta}{2}$ and $\gamma=\frac{\theta+\xi}{2}$.

11) Generalized beta 2 has density

$f(x)=\frac{1}{\theta^{\alpha\xi}B(\xi,\delta)}\ x^{\alpha\xi-1}\left(1+\left(\frac{x}{\theta}\right)^{\alpha}\right)^{-(\xi+\delta)}$, $\theta>0$, $\alpha>0$, $\xi>0$, $\delta>0$, $x>0$,

which is of the form (1.3) with $c=\theta^{-\alpha}$, $\alpha\leq 1$, $\beta=\alpha\xi$ and $\gamma=\xi+\delta$.

12) Generalized Generalized Logistic has density

$$f(x)=\theta\delta\xi\ x^{\delta-1}e^{-\xi x^{\delta}}\left(1+e^{-\xi x^{\delta}}\right)^{-(\theta+1)},$$

$$\theta>0\ ,\ \xi>0\ ,\ \delta>0\ \textit{an odd}\ \text{integer}\ ,\quad x\in\mathbb{R}.$$

Taking $\delta=1$, it would reduce to the density of the Logistic distribution. Letting $Y=e^{-X}$, the density of Y is

$f(y)=\ \theta\xi\ y^{\xi-1}\left(1+y^{\xi}\right)^{-(\theta+1)}$, $\theta>0$, $y>0$,

which is of the form (1.3) with $c=1$, $\alpha=\beta=\xi\leq 1$ and $\gamma=\theta+1$.

13) Generalized Logistic (Dubey) has density

$f(x)=\frac{\delta}{\zeta}\ e^{\frac{1}{\xi}x}\left(1+\frac{\xi}{\zeta}e^{\frac{1}{\xi}x}\right)^{-(\delta+1)}$, $\delta>0$, $\xi>0$, $\zeta>0$, $x\in\mathbb{R}$.

Letting $Y=e^{X}$, the density of Y is

$f(y)=\frac{\delta}{\zeta}\ y^{\frac{1}{\xi}-1}\left(1+\frac{\xi}{\zeta}\ y^{\frac{1}{\xi}}\right)^{-(\delta+1)}$, $y>0$,

which is of the form (1.3) with $c=\frac{\xi}{\zeta}$, $\alpha=\beta=\frac{1}{\xi}\leq 1$ and $\gamma=\delta+1$.

14) Generalized Pareto has density

$f(x)=\frac{\theta+1}{\delta}\left(1+\frac{\theta}{\delta}\ x\right)^{-\left(\frac{1}{\theta}+2\right)}$, $\theta>0$, $\delta>0$, $x>0$,

which is of the form (1.3) with $c=\frac{\theta}{\delta}$, $\alpha=\beta=1$ and $\gamma=\frac{1}{\theta}+2$.

15) As observed by Bondesson ([1] , page 976), (1.3) is the general form for the density of a power with exponent $1/\alpha$ of the ratio of two independent gamma random variables. Thus, for $\alpha=1$, the distribution of the ratio of two independent gamma random variables belongs to $\Im-class$.

16) Ratio of two independent Rayleigh random variables has density

$f(x)=2\theta\delta\ x\ \left(\theta+\delta\ x^{2}\right)^{-2}$, $\theta>0$, $\delta>0$, $x>0$.

Letting $Y=X^{2}$, the density of Y is

$f_{Y}(y)=\frac{\delta}{\theta}\left(1+\frac{\delta}{\theta}\ y\right)^{-2}$, $y>0$,

which is a special case of (6) above.

17) Ratio of two independent Maxwell and Rayleigh random variables has density

$f(x)=3a^{3/2}\sigma^{3}x^{2}\left(1+a\sigma^{2}x^{2}\right)^{-5/2}$, $a>0$, $\sigma>0$, $x>0$.

Letting $Y=X^{2}$, the density of Y is

$f_{Y}(y)=\frac{3}{2}a^{3/2}\sigma^{3}y^{1/2}\left(1+a\sigma^{2}y\right)^{-5/2}$, $y>0$, which is of the form (1.3) with $c = a\sigma^2$, $\alpha = 1$, $\beta = \frac{3}{2}$ and $\gamma = \frac{5}{2}$.

18) Student t-distribution has density

$f(x) = \frac{1}{\theta^{1/2} B\left(\frac{1}{2}, \frac{\theta}{2}\right)} \left(1 + \frac{x^2}{\theta}\right)^{-(\theta+1)/2}, \ \theta = 1, 2, \ldots, \quad x \in \mathbb{R}.$

Letting $Y = X^2$, the density of Y is

$f_Y(y) = \frac{1}{\theta^{1/2} B\left(\frac{1}{2}, \frac{\theta}{2}\right)} y^{-1/2} \left(1 + \theta^{-1} y\right)^{-(\theta+1)/2}, \quad y > 0,$

which is of the form (1.3) with $c = \theta^{-1}$, $\alpha = 1$, $\beta = 1/2$ and $\gamma = (\theta + 1)/2$.

19) Symmetric distribution has density

$f(x) = \frac{1}{\pi} \left(e^{x/2} + e^{-x/2}\right)^{-1}, \quad x \in \mathbb{R}.$

Letting $Y = e^X$, the density of Y is

$f_Y(y) = \frac{1}{\pi} y^{-1/2} (1 + y)^{-1}, \quad y > 0,$

which is of the form (1.3) with $c = 1$, $\beta = \frac{1}{2}$ and $\alpha = \gamma = 1$.

3.2. Random variables and certain functions of them whose densities are of the form (1.4)

We list the densities of the distributions in alphabetical order rather than that of their importance. Some of these densities have already been listed in Bondesson [1, 2] and some have appeared in our previous works in different contexts.

1) Amoroso distribution has density

$f(x) = \frac{|\tau|}{\theta^{\tau k}\ \Gamma(k)} x^{\tau k - 1} e^{-\frac{1}{\theta^\tau} x^\tau}, \ \theta > 0, \ k > 0, \ \tau \in \mathbb{R}, \quad x > 0,$

which is of the form (1.4) with $c = \frac{1}{\theta^\tau}$, $0 < \alpha = \tau \leq 1$ and $\beta = \tau k$.

2) Extreme Value (Gumbel) distribution of Type 2 has density

$f(x) = \frac{k}{\theta} \left(\frac{x-\delta}{\theta}\right)^{-(k+1)} \exp\left\{-\left(\frac{x-\delta}{\theta}\right)^{-k}\right\}, \ k > 0, \ \theta > 0, \ \delta > 0, \quad x > \delta.$

Letting $Y = \left(\frac{X-\delta}{\theta}\right)^{-1}$, the density of Y is

$f_Y(y) = k\ y^{k-1} e^{-y^k}, \quad y > 0,$

which is of the form (1.4) with $c = 1$ and $\alpha = \beta = k \leq 1$.

3) Extreme Value (Gumbel) distribution of Type 3 has density

$f(x) = \frac{k}{\theta} \left(\frac{\delta - x}{\theta}\right)^{k-1} \exp\left\{-\left(\frac{\delta - x}{\theta}\right)^{k}\right\}, \ k > 0, \ \theta > 0, \ \delta > 0, \quad x < \delta.$

Letting $Y = \frac{\delta - X}{\theta}$, the density of Y is

$f_Y(y) = k\ y^{k-1} e^{-y^k}, \quad y > 0,$

which is of the form (1.4) with $c = 1$ and $\alpha = \beta = k \leq 1$.

4) Gamma distribution has density

$f(x) = \frac{1}{\theta^\beta \Gamma(\beta)} x^{\beta - 1} e^{-\frac{1}{\theta} x}, \ \theta > 0, \ \beta > 0, \quad x > 0,$

which is of the form (1.4) with $c = \frac{1}{\theta}$ and $\alpha = 1$.

5) Generalized Gamma (or Stacy) distribution has density

$f(x) = \frac{\theta}{\delta^{\theta\tau}\ \Gamma(\tau)} x^{\theta\tau - 1} e^{-\delta^{-\theta} x^\theta}, \ \theta > 0, \ \tau > 0, \ \delta > 0, \quad x > 0,$

which is of the form (1.4) with $c = \delta^{-\theta}$, $\alpha = \theta \leq 1$ and $\beta = \theta\tau$.

6) Log Modified Weibull distribution has density

$f(x) = \mu(\delta + \eta\ e^x) \exp\left\{\delta x + \eta e^x - \mu e^{\delta x + \eta e^x}\right\}, \quad x > 0,$

where $\mu > 0$, $\delta > 0$ and $\eta \geq 0$. For $\eta = 0$, the density will be

$f(x) = \mu\delta \exp\left\{\delta x - \mu e^{\delta x}\right\}, \quad x > 0.$

Letting $Y = e^X$, the density of Y is

$f_Y(y) = \mu\delta\ y^{\delta - 1}\ e^{-\mu\ y^\delta}, \quad y > 1,$

which is similar to (1.4) with $c = \mu$ and $\alpha = \beta = \delta \leq 1$, but the support of Y is $[1, \infty)$ rather than $\mathbb{R}_+$.

7) Kummer-gamma distribution has density

$f(x) = Cx^{\beta-1}(1+\delta x)^{-\nu} e^{-\xi x}, \quad x > 0,$

where $\beta > 0$, $\delta > 0$, $\xi > 0$ and $\nu > 0$ is of the form $(1.4)^*$ and thus belongs to $\Im - class$.

7) As pointed out by Bondesson ([1], page 976), (1.4) corresponds to a power with exponent $1/\alpha$ of a gamma random variable and hence belongs to $\Im - class$.

8) Again, as pointed out by Bondesson ([1], page 976), the density of the stable distribution of index $1/2$ has the form (1.4) and hence belongs to $\Im - class$.

9) Weibull distribution has density

$f(x) = c\beta\, x^{\beta-1}\, e^{-c\, x^{\beta}}, \; c > 0, \; \beta > 0, \quad x > 0,$

which is of the form (1.4) with $\alpha = \beta \leq 1$.

3.3. Random variables and certain functions of them whose densities are of the form (1.5)

We list here the densities of the distributions in alphabetical order rather than their importance. Some of these densities have already been listed in Bondesson [1, 2] and some have appeared in our previous work for different purposes.

1) Inverse Gamma distribution has density

$f(x) = \frac{\theta^{\delta}}{\Gamma(\delta)}\, x^{-\delta-1}\, e^{-\theta\, x^{-1}}, \; \theta > 0, \; \delta > 0, \quad x > 0,$

which is of the form (1.5) with $c_1 = 0$, $c_2 = \theta$ and $\beta = -\delta$.

2) As pointed out by Bondesson ([1], page 977), the distribution with density (1.5) has been named Generalized Inverse Gaussian. For $\beta = -1/2$, it reduces to Inverse Gaussian density.

3) Random Walk distribution has density

$f(x) = \delta^{1/2}\, e^{\delta/\mu}\, (2\pi\, x)^{-1/2} \exp\left\{-\frac{\delta}{2}\left(x + \frac{1}{\mu^2 x}\right)\right\}, \quad x > 0,$

where δ and μ are positive parameters. This density has the form (1.5) with $\beta = 1/2$, $c_1 = \delta/2$ and $c_2 = \frac{\delta}{2\mu^2}$.

4) As pointed out by Bondesson ([1], page 976), the density of Time-Homogeneous Diffusion Process has the form (1.5) and hence belongs to $\Im - class$.

3.4. A Random variable whose density is of the form (1.6)

This random variable is given in ([1], page 976) with lognormal density

$f(x) = C\, x^{-1} e^{-(\log\, x-\mu)^2/2\sigma^2}, \; \sigma > 0, \; \mu \in \mathbb{R}, \quad x > 0.$

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