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APPLICATIONS OF SOME HYPERGEOMETRIC SUMMATION THEOREMS INVOLVING DOUBLE SERIES

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Abstract

The main object of this paper is to derive a number of general double series identities and to apply each of these identities in order to deduce several hypergeometric reduction formulas for the Srivastava-Daoust double hypergeometric function. The results presented in this paper are based essentially upon some known hypergeometric summation theorems in conjunction with series rearrangement techniques.

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General Terms: Hypergeometric Summation Theorems; Reduction Formulas.
Additional Key Words and Phrases: Series identities; Hypergeometric summation theorems; Series rearrangement techniques; Pochhammer's symbol; Gamma function; Hypergeometric reduction formulas; Srivastava-Daoust double and multiple hypergeometric functions; Gauss-Legendre multiplication formula.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

In terms of Pochhammer's symbol (or the *shifted* factorial) $(\lambda)_n$ defined by

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) \qquad (n \in \mathbf{N} := \{1,2,3,\cdots\}), \tag{1.1}$$

a generalized Gaussian hypergeometric function of one variable is defined by [14, p. 19, Equation (23)]

$$ {}_AF_B\left[\begin{array}{c}(a_A);\\ \\ (b_B);\end{array} z\right] = {}_AF_B\left[\begin{array}{c}a_1,a_2,\cdots,a_A;\\ \\ b_1,b_2,\cdots,b_B;\end{array} z\right] = {}_AF_B\left[\begin{array}{c}(a_j)_{j=1}^A;\\ \\ (b_j)_{j=1}^B;\end{array} z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_A)_n}{(b_1)_n(b_2)_n\cdots(b_B)_n}\,\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{[(a_A)]_n}{[(b_B)]_n}\,\frac{z^n}{n!}, \tag{1.2}$$

where, and in what follows, we find it to be convenient to abbreviate the array of A parameters $a_1, a_2, \cdots, a_A$ by (a_A), with similar interpretation for (b_B), *et cetera.*

In the year 1969, Srivastava and Daoust [14] introduced and studied the following general hypergeometric function of n variables, which is popularly known in the literature as the *Srivastava-Daoust multivariable hypergeometric function* (see, for details, [10, pp. 454–456], [11, p. 199], [13, pp. 153–158], [14, p. 37] and [15, pp.

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64–65]; see also [5] and [6]):

$$F^{A:B^{(1)};\cdots;B^{(n)}}_{D:E^{(1)};\cdots;E^{(n)}}\left(\begin{array}{l}\left[(a_A):\theta^{(1)},\cdots,\theta^{(n)}\right]\ :\left[(b^{(1)}_{B^{(1)}}):\phi^{(1)}\right];\cdots;\left[(b^{(n)}_{B^{(n)}}):\phi^{(n)}\right];\\ \\ \left[(d_D):\psi^{(1)},\cdots,\psi^{(n)}\right]:\left[(e^{(1)}_{E^{(1)}}):\delta^{(1)}\right];\cdots;\left[(e^{(n)}_{E^{(n)}}):\delta^{(n)}\right];\end{array} z_1,\cdots,z_n\right)$$

$$:= \sum_{m_1,\cdots,m_n=0}^{\infty} \Xi(m_1,\cdots,m_n)\,\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!} \tag{1.3}$$

where

$$\Xi(m_1,\cdots,m_n)=\frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}}\prod\limits_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}\cdots\prod\limits_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod\limits_{j=1}^{D}(d_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}}\prod\limits_{j=1}^{E^{(1)}}(e_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod\limits_{j=1}^{E^{(n)}}(e_j^{(n)})_{m_n\delta_j^{(n)}}}, \tag{1.4}$$

the coefficients

$$\theta_j^{(k)}\ \ (j=1,2,\cdots,A;\ k=1,2,\cdots,n),\quad \phi_j^{(k)}\ \ \left(j=1,2,\cdots,B^{(k)};\ k=1,\cdots,n\right),$$

$$\psi_j^{(k)}\ \ (j=1,2,\cdots,D;\ k=1,2,\cdots,n)$$

and

$$\delta_j^{(k)}\ \ \left(j=1,2,\cdots,E^{(k)};\ k=1,\cdots,n\right)$$

are real constants (positive, negative or zero) (see, for example, [14, pp. 37 and 270–272]) and (for $k\in\{1,2,\cdots,n\}$) we abbreviate by $(b^{(k)}_{B^{(k)}})$ the array of $B^{(k)}$ parameters $b_j^{(k)}$ $\left(j=1,2,\cdots,B^{(k)}\right)$, with similar interpretations for $(e^{(k)}_{E^{(k)}})$, *et cetera*.

We begin by recalling the definitions and properties of various Pochhammer symbols which are relevant to our present investigation. First of all, for the Pochhammer symbol $(\lambda)_n$, it is easily seen from the Gauss-Legendre multiplication formula (see [9, p. 26, Theorem 10] and [14, p. 23, Equation (27)]:

$$\Gamma(mz)=(2\pi)^{\frac{1}{2}(1-m)}\,m^{mz-\frac{1}{2}}\prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right) \tag{1.5}$$

$$\left(z\neq 0,-\frac{1}{m},-\frac{2}{m},\cdots;\ m\in\mathbf{N}\right)$$

that

$$(\lambda)_{mn}=m^{mn}\left(\frac{\lambda}{m}\right)_n\left(\frac{\lambda+1}{m}\right)_n\cdots\left(\frac{\lambda+m-1}{m}\right)_n \tag{1.6}$$

$$(m\in\mathbf{N};\ n\in\mathbf{N}_0:=\mathbf{N}\cup\{0\}).$$

We denote by $\Delta(N;\lambda)$ the set of N parameters

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \cdots, \frac{\lambda+N-1}{N} \qquad (N \in \mathbf{N}),$$

the set $\Delta(N;\lambda)$ being empty when $N=0$. Thus, for example, $\Delta[N;(a_A)]$ represents the set of NA parameters given by

$$\Delta(N;a_1), \Delta(N;a_2), \Delta(N;a_3), \cdots, \Delta(N;a_A).$$

The following known hypergeometric summation theorems will also be required in our investigation (see, for example, [7, p. 547, Entries (7.4.5.8), (7.4.5.9) and (7.4.5.10); p. 555, Entries (7.5.3.11) and (7.5.3.12)]):

$$ {}_3F_2\left[\begin{array}{r} a, a+\frac{1}{3}, a+\frac{2}{3}; \\ \frac{1}{3}, \frac{2}{3}; \end{array} -1\right] = \frac{2}{3}\left[2^{-1-3a} + \cos(a\pi)\right] \qquad \left(\Re(a) < \frac{1}{3}\right), \tag{1.7}$$

$$ {}_3F_2\left[\begin{array}{r} a, a+\frac{1}{3}, a+\frac{2}{3}; \\ \frac{2}{3}, \frac{4}{3}; \end{array} -1\right] = \frac{2}{3(1-3a)}\left[2^{-3a} + \cos\left(\frac{(1+3a)\pi}{3}\right)\right] \tag{1.8}$$

$$(\Re(a) < 0),$$

$$ {}_3F_2\left[\begin{array}{r} a, a+\frac{1}{3}, a+\frac{2}{3}; \\ \frac{4}{3}, \frac{5}{3}; \end{array} -1\right] = \frac{4}{3(1-3a)(2-3a)}$$

$$\cdot \left[2^{1-3a} + \cos\left(\frac{(2+3a)\pi}{3}\right)\right] \qquad \left(\Re(a) < -\frac{1}{3}\right), \tag{1.9}$$

$$ {}_4F_3\left[\begin{array}{r} a, a+\frac{1}{2}, b, b+\frac{1}{2}; \\ \frac{1}{2}, a-b+\frac{1}{2}, a-b+1; \end{array} 1\right] = 2^{-2a-1}\,\frac{\Gamma(2a-2b+1)}{\Gamma(a-2b+1)}$$

$$\cdot \left(\frac{\Gamma\left(\frac{1}{2}-2b\right)}{\Gamma\left(\frac{1}{2}-2b+a\right)} + \frac{\sqrt{\pi}}{\Gamma\left(a+\frac{1}{2}\right)}\right) \qquad \left(\Re(b) < \frac{1}{4}\right) \tag{1.10}$$

and

$$ {}_4F_3\left[\begin{array}{r} a, a+\frac{1}{2}, b, b+\frac{1}{2}; \\ \frac{3}{2}, a-b+\frac{3}{2}, a-b+1; \end{array} 1\right] = \frac{2^{-2a}\,\Gamma(2a-2b+2)}{(2a-1)(2b-1)\Gamma\left(a-2b+\frac{3}{2}\right)}$$

$$\cdot \left(\frac{\Gamma\left(\frac{3}{2}-2b\right)}{\Gamma(a-2b+1)} - \frac{\sqrt{\pi}}{\Gamma(a)}\right) \qquad \left(\Re(b) < \frac{3}{4}\right). \tag{1.11}$$

Here, and in what follows, *exceptional* parameter values that would render either side of a result undefined or invalid are tacitly excluded. Moreover, by using suitable adjustment of parameters in the summation theorems (1.10) and (1.11), we can derive the four useful summation theorems (1.12) to (1.15) as detailed below.

$$
{}_4F_3\left[\begin{array}{r} a, a+\frac{1}{4}, a+\frac{1}{2}, a+\frac{3}{4}; \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; \end{array} 1\right] = \frac{2^{-2a-1}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-a\right)}\left(\frac{\Gamma(-2a)}{\Gamma(-a)} + \frac{\sqrt{\pi}}{\Gamma\left(a+\frac{1}{2}\right)}\right)
$$

$$
= 2^{-4a-2} + 2^{-2a-1}\ \cos(a\pi) \qquad (\Re(a) < 0)\,, \tag{1.12}
$$

$$
{}_4F_3\left[\begin{array}{r} a, a+\frac{1}{4}, a+\frac{1}{2}, a+\frac{3}{4}; \\ \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; \end{array} 1\right] = \frac{1}{1-4a}\left[2^{-4a-1} + 2^{-2a-\frac{1}{2}}\ \cos\left(a\pi + \frac{\pi}{4}\right)\right] \tag{1.13}
$$

$$
\left(\Re(a) < \frac{1}{4}\right),
$$

$$
{}_4F_3\left[\begin{array}{r} a, a+\frac{1}{4}, a+\frac{1}{2}, a+\frac{3}{4}; \\ \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; \end{array} 1\right] = \frac{1}{(2a-1)(4a-1)}\left[2^{-4a} - 2^{-2a}\ \sin(a\pi)\right] \tag{1.14}
$$

$$
\left(\Re(a) < \frac{1}{2}\right)
$$

and

$$
{}_4F_3\left[\begin{array}{r} a, a+\frac{1}{4}, a+\frac{1}{2}, a+\frac{3}{4}; \\ \frac{5}{4}, \frac{3}{2}, \frac{7}{4}; \end{array} 1\right] = \frac{3}{(2a-1)(4a-1)(4a-3)}
$$

$$
\cdot\left[2^{-2a+\frac{1}{2}}\ \cos\left(a\pi - \frac{\pi}{4}\right) - 2^{-4a+1}\right] \qquad \left(\Re(a) < \frac{3}{4}\right). \tag{1.15}
$$

The present investigation is motivated by the potential need for reduction formulas for hypergeometric functions in two and more variables (see, for details, [2], [3] and [4]; see also [8]). With this purpose in view, we make use of series rearrangement techniques, in conjunction with the above (known or easily derivable) hypergeometric summation theorems, in order to derive a number of general double series identities and hypergeometric reduction formulas involving the Srivastava-Daoust double hypergeometric function which is the case $n = 2$ of the definition (1.3) (see also [1]).

2. A FAMILY OF GENERAL DOUBLE SERIES IDENTITIES

In this section, we state each of the following main results of this paper.

Theorem 1. *Let* $\{\Omega_n\}_{n=0}^{\infty}$ *be a bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega_{m+3n} \frac{x^{m+3n}}{m!\,(3n)!} = \frac{2}{3} \sum_{m=0}^{\infty} \Omega_m \frac{x^m}{m!} \left[2^{m-1} + \cos\left(\frac{m\pi}{3}\right)\right], \tag{2.1}$$

provided that each of the series involved is absolutely convergent.

Theorem 2. *Let* $\{\Omega_n\}_{n=0}^{\infty}$ *be a bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega_{m+3n} \frac{x^{m+3n}}{m!\,(3n+1)!} = \Omega_0 + \frac{2}{3} \sum_{m=1}^{\infty} \Omega_m \frac{x^m}{(m+1)!} \left[2^{m} + \cos\left(\frac{(1-m)\pi}{3}\right)\right], \tag{2.2}$$

provided that each of the series involved is absolutely convergent.

Theorem 3. *Let* $\{\Omega_n\}_{n=0}^{\infty}$ *be a bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega_{m+3n} \frac{x^{m+3n}}{m!\,(3n+2)!} = \frac{\Omega_0}{2} + \frac{\Omega_1\ x}{2} + \frac{2}{3} \sum_{m=2}^{\infty} \Omega_m \frac{x^m}{(m+2)!} \left[2^{m+1} + \cos\left(\frac{(2-m)\pi}{3}\right)\right], \tag{2.3}$$

provided that each of the series involved is absolutely convergent.

Theorem 4. *Let* $\{\Omega_n\}_{n=0}^{\infty}$ *be a bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega_{m+4n} \frac{x^{m+4n}}{m!\,(4n)!} = \Omega_0 + \frac{\Omega_1\ x}{2} + \frac{x}{2} \sum_{m=0}^{\infty} \Omega_{m+1} \frac{(2x)^m}{(m+1)!} + \sqrt{2}\ x^3 \sum_{n=0}^{2} \frac{(\sqrt{2}\ x)^n}{(n+3)!} \cos\left(\frac{(n+3)\pi}{4}\right) \sum_{m=0}^{\infty} \Omega_{4m+n+3} \frac{(-4x^4)^m}{(n+4)_{4m}}, \tag{2.4}$$

provided that each of the series involved is absolutely convergent.

Theorem 5. *Let* $\{\Omega_n\}_{n=0}^{\infty}$ *be a bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega_{m+4n} \frac{x^{m+4n}}{m!\,(4n+1)!} = \frac{1}{2} \sum_{m=0}^{\infty} \Omega_m \frac{(2x)^m}{(m+1)!} + \frac{1}{\sqrt{2}} \sum_{n=0}^{2} \frac{(\sqrt{2}\ x)^n}{(n+1)!} \cos\left(\frac{(n-1)\pi}{4}\right) \sum_{m=0}^{\infty} \Omega_{4m+n} \frac{(-4x^4)^m}{(n+2)_{4m}}, \tag{2.5}$$

provided that each of the series involved is absolutely convergent.

Theorem 6. *Let $\{\Omega_n\}_{n=0}^{\infty}$ be a bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega_{m+4n} \frac{x^{m+4n}}{m!\,(4n+2)!} = \sum_{m=0}^{\infty} \Omega_m \frac{(2x)^m}{(m+2)!}$$

$$+\sqrt{2}\,x \sum_{n=0}^{2} \frac{(\sqrt{2}\,x)^n}{(n+3)!} \sin\left(\frac{(n+1)\pi}{4}\right) \sum_{m=0}^{\infty} \Omega_{4m+n+1} \frac{(-4x^4)^m}{(n+4)_{4m}}, \tag{2.6}$$

provided that each of the series involved is absolutely convergent.

Theorem 7. *Let $\{\Omega_n\}_{n=0}^{\infty}$ be a bounded sequence of arbitrary complex numbers. Then*

$$\sum_{m,n=0}^{\infty} \Omega_{m+4n} \frac{x^{m+4n}}{m!\,(4n+3)!} = 2\sum_{m=0}^{\infty} \Omega_m \frac{(2x)^m}{(m+3)!} - \frac{\Omega_0}{6}$$

$$-2^{3/2}\,x^2 \sum_{n=0}^{2} \frac{(\sqrt{2}\,x)^n}{(n+5)!} \cos\left(\frac{(n+3)\pi}{4}\right) \sum_{m=0}^{\infty} \Omega_{4m+n+2} \frac{(-4x^4)^m}{(n+6)_{4m}}, \tag{2.7}$$

provided that each of the series involved is absolutely convergent.

3. DERIVATIONS OF THEOREMS 1 TO 7

Demonstration of Theorems 1, 2 and 3. Consider the first member of the assertion (2.1). By applying such identities as (1.5) and (1.6), in conjunction with series rearrangement techniques (see, for details, [15, Chapter 2]), we get

$$\sum_{m,n=0}^{\infty} \Omega_{m+3n} \frac{x^{m+3n}}{m!\,(3n)!} = \sum_{m=0}^{\infty} \Omega_m \sum_{n=0}^{[m/3]} \frac{x^m}{(m-3n)!\,(3n)!}$$

$$= \sum_{m=0}^{\infty} \Omega_m \frac{x^m}{m!} \sum_{n=0}^{[m/3]} \frac{\left(-\frac{m}{3}\right)_n \left(-\frac{m-1}{3}\right)_n \left(-\frac{m-2}{3}\right)_n}{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{(-1)^n}{n!}$$

$$= \sum_{m=0}^{\infty} \Omega_m \frac{x^m}{m!}\, {}_3F_2\left[\begin{matrix} -\frac{m}{3}, -\frac{m-1}{3}, -\frac{m-2}{3}; \\ \frac{1}{3}, \frac{2}{3}; \end{matrix} -1\right], \tag{3.1}$$

where we have also used the definition (1.2) with $A-1=B=2$.

Now, using the summation theorem (1.7), we find from (3.1) that

$$\sum_{m,n=0}^{\infty} \Omega_{m+3n} \frac{x^{m+3n}}{m!\,(3n)} = \frac{2}{3} \sum_{m=0}^{\infty} \Omega_m \frac{x^m}{m!} \left[2^{m-1} + \cos\left(\frac{m\pi}{3}\right)\right], \tag{3.2}$$

which is precisely the second member of the double series identity (2.1) asserted by Theorem 1.

In the same manner as detailed above, by applications of the hypergeometric summation theorems (1.8) and (1.9), we can derive the assertions (2.2) and (2.3) of Theorems 2 and 3, respectively.

Demonstration of Theorem 4. By using the series rearrangement technique, we first rewrite the left-hand side of the assertion (2.4) as follows:

$$\begin{aligned}\sum_{m,n=0}^{\infty} \Omega_{m+4n} \frac{x^{m+4n}}{m!\,(4n)!} &= \sum_{m=0}^{\infty} \sum_{n=0}^{[m/4]} \Omega_m \frac{x^m}{(m-4n)!\,(4n)!} \\ &= \sum_{m=0}^{\infty} \Omega_m \frac{x^m}{m!} \sum_{n=0}^{[m/4]} \frac{\left(-\frac{m}{4}\right)_n \left(-\frac{m-1}{4}\right)_n \left(-\frac{m-2}{4}\right)_n \left(-\frac{m-3}{4}\right)_n}{n!\,\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n} \\ &= \sum_{m=0}^{\infty} \Omega_m \frac{x^m}{m!}\, {}_4F_3\left[\begin{array}{r} -\frac{m}{4}, -\frac{m-1}{4}, -\frac{m-2}{4}, -\frac{m-3}{4}; \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; \end{array} 1\right] \\ &= \Omega_0 + \sum_{m=1}^{\infty} \Omega_m \frac{x^m}{m!}\, {}_4F_3\left[\begin{array}{r} -\frac{m}{4}, -\frac{m-1}{4}, -\frac{m-2}{4}, -\frac{m-3}{4}; \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; \end{array} 1\right], \end{aligned} \tag{3.3}$$

under the appropriate convergence conditions associated with the hypergeometric summation theorem (1.12).

Next, by applying the hypergeometric summation theorem (1.12) in the last member of (3.3), we obtain

$$\begin{aligned}\sum_{m,n=0}^{\infty} \Omega_{m+4n} \frac{x^{m+4n}}{m!\,(4n)!} &= \Omega_0 + \sum_{m=0}^{\infty} \Omega_{m+1} \frac{x^{m+1}}{(m+1)!} \left[2^{m-1} + 2^{(m-1)/2} \cos\left(\frac{(m+1)\pi}{4}\right)\right] \\ &=: \Omega_0 + \frac{1}{4} \sum_{m=0}^{\infty} \Omega_{m+1} \frac{(2x)^{m+1}}{(m+1)!} + \frac{1}{\sqrt{2}} \sum_{m=0}^{\infty} \Theta(m). \end{aligned} \tag{3.4}$$

Finally, since

$$\cos\left(\frac{(m+1)\pi}{4}\right) = 0 \qquad \text{when} \qquad m = 4j+1 \qquad (j \in \mathbf{N}_0), \tag{3.5}$$

it is fairly easy to observe from (3.4) that

$$\begin{aligned}\sum_{m,n=0}^{\infty} \Omega_{m+4n} \frac{x^{m+4n}}{m!\,(4n)!} &= \Omega_0 + \frac{1}{4} \sum_{m=0}^{\infty} \Omega_{m+1} \frac{(2x)^{m+1}}{(m+1)!} \\ &+ \frac{1}{\sqrt{2}} \left(\Theta(0) + \sum_{m=0}^{\infty} \Theta(4m+2) + \sum_{m=0}^{\infty} \Theta(4m+3) + \sum_{m=0}^{\infty} \Theta(4m+4)\right) \end{aligned}$$

$$= \Omega_0 + \frac{1}{4} \sum_{m=0}^{\infty} \Omega_{m+1} \frac{(2x)^{m+1}}{(m+1)!} + \frac{1}{\sqrt{2}} \left[\Theta(0) + \sum_{n=0}^{2} \left(\sum_{m=0}^{\infty} \Theta(4m+n+2) \right) \right]$$

$$= \Omega_0 + \frac{\Omega_1 \, x}{2} + \frac{x}{2} \sum_{m=0}^{\infty} \Omega_{m+1} \frac{(2x)^m}{(m+1)!}$$

$$+ \sqrt{2} \, x^3 \sum_{n=0}^{2} \sum_{m=0}^{\infty} \Omega_{4m+n+3} \cos\left(\frac{(4m+n+3)\pi}{4} \right) \frac{(\sqrt{2}\, x)^{4m+n}}{(4m+n+3)!}$$

$$= \Omega_0 + \frac{\Omega_1 \, x}{2} + \frac{x}{2} \sum_{m=0}^{\infty} \Omega_{m+1} \frac{(2x)^m}{(m+1)!}$$

$$+ \sqrt{2} \, x^3 \sum_{n=0}^{2} \cos\left(\frac{(n+3)\pi}{4} \right) \frac{(\sqrt{2}\, x)^n}{(n+3)!} \sum_{m=0}^{\infty} \Omega_{4m+n+3} \frac{(-4x^4)^m}{(n+4)_{4m}}, \tag{3.6}$$

which evidently proves the assertion (2.4) of Theorem 4.

Demonstration of Theorems 5, 6 and 7. By appropriately applying the hypergeometric summation theorems (1.13), (1.14) and (1.15), we can similarly derive the assertions (2.5), (2.6) and (2.7) of Theorems 5, 6 and 7, respectively.

4. REDUCTION FORMULAS FOR DOUBLE HYPERGEOMETRIC FUNCTIONS

In Theorems 1 to 7, we set

$$\Omega_n = \frac{[(a_A)]_n}{[(b_B)]_n} = \frac{(a_1)_n (a_2)_n \cdots (a_A)_n}{(b_1)_n (b_2)_n \cdots (b_B)_n} \qquad (n \in \mathbf{N}_0) \tag{4.1}$$

and appropriately simplify each member of the resulting assertions. We are thus led to the following reduction formulas for the Srivastava-Daoust hypergeometric function in two variables:

$$F^{A:0;0}_{B:0;2} \left(\begin{array}{l} [(a_A):1,3]: \text{——} ; \text{————} ; \\ [(b_B):1,3]: \text{——} ; \left[\frac{1}{3}:1\right], \left[\frac{2}{3}:1\right]; \end{array} x, \frac{x^3}{27} \right)$$

$$= \frac{1}{3} \, {}_AF_B \left[\begin{array}{l} (a_A); \\ (b_B); \end{array} 2x \right] + \frac{2}{3} \sum_{n=0}^{2} \frac{[(a_A)]_n}{[(b_B)]_n} \frac{x^n}{n!} \cos\left(\frac{n\pi}{3} \right)$$

$$\cdot {}_{3A+1}F_{3B+3} \left[\begin{array}{c} \Delta[3;(a_A)+n], 1; \\ \Delta[3;(b_B)+n], \Delta(3;1+n); \end{array} - 3^{3(A-B-1)} \, x^3 \right], \tag{4.2}$$

$$F^{A:0;0}_{B:0;2} \left(\begin{array}{l} [(a_A):1,3]: \text{——} ; \text{————} ; \\ [(b_B):1,3]: \text{——} ; \left[\frac{2}{3}:1\right], \left[\frac{4}{3}:1\right]; \end{array} x, \frac{x^3}{27} \right)$$

$$= 1+\frac{2x}{3}\ \frac{[(a_A)]}{[(b_B)]}\ {}_{A+1}F_{B+1}\left[\begin{matrix}(a_A)+1,1;\\ \\ (b_B)+1,3;\end{matrix}\ 2x\right]+\frac{2}{3}\sum_{n=0}^{2}\frac{[(a_A)]_{n+1}}{[(b_B)]_{n+1}}\ \frac{x^{n+1}}{(n+2)!}\ \cos\left(\frac{n\pi}{3}\right)$$

$$\cdot {}_{3A+1}F_{3B+3}\left[\begin{matrix}\Delta[3;(a_A)+n+1],1;\\ \\ \Delta[3;(b_B)+n+1],\Delta(3;3+n);\end{matrix}\ -3^{3(A-B-1)}\ x^3\right], \tag{4.3}$$

$$F^{A:0;0}_{B:0;2}\left(\begin{matrix}[(a_A):1,3]: \ \text{---}\ ; \ \text{---------}\ ; \\ \\ [(b_B):1,3]: \ \text{---}\ ; \left[\frac{4}{3}:1\right],\left[\frac{5}{3}:1\right];\end{matrix}\ x,\frac{x^3}{27}\right)$$

$$= 1+x\ \frac{[(a_A)]}{[(b_B)]}\ +\frac{4x^2}{9}\ \frac{[(a_A)][(a_A)+1]}{[(b_B)][(b_B)+1]}\ {}_{A+1}F_{B+1}\left[\begin{matrix}(a_A)+2,1;\\ \\ (b_B)+2,5;\end{matrix}\ 2x\right]$$

$$+\frac{4}{3}\sum_{n=0}^{2}\frac{[(a_A)]_{n+2}}{[(b_B)]_{n+2}}\ \frac{x^{n+2}}{(n+4)!}\ \cos\left(\frac{n\pi}{3}\right)$$

$$\cdot {}_{3A+1}F_{3B+3}\left[\begin{matrix}\Delta[3;(a_A)+n+2],1;\\ \\ \Delta[3;(b_B)+n+2],\Delta(3;5+n);\end{matrix}\ -3^{3(A-B-1)}\ x^3\right], \tag{4.4}$$

$$F^{A:0;0}_{B:0;3}\left(\begin{matrix}[(a_A):1,4]: \ \text{---}\ ; \ \text{-------------}\ ; \\ \\ [(b_B):1,4]: \ \text{---}\ ; \left[\frac{1}{4}:1\right],\left[\frac{1}{2}:1\right],\left[\frac{3}{4}:1\right];\end{matrix}\ x,\frac{x^4}{256}\right)$$

$$= 1+\frac{x}{2}\ \frac{[(a_A)]}{[(b_B)]}+\frac{x}{2}\ \frac{[(a_A)]}{[(b_B)]}\ {}_{A+1}F_{B+1}\left[\begin{matrix}(a_A)+1,1;\\ (b_B)+1,2;\end{matrix}\ 2x\right]$$

$$+\frac{\sqrt{2}\ x^3}{6}\ \frac{[(a_A)]_3}{[(b_B)]_3}\sum_{n=0}^{2}\frac{[(a_A)+3]_n}{[(b_B)+3]_n}\ \frac{(\sqrt{2}\ x)^n}{(4)_n}\ \cos\left(\frac{(n+3)\pi}{4}\right)$$

$$\cdot {}_{4A+1}F_{4B+4}\left[\begin{matrix}\Delta[4;(a_A)+n+3],1;\\ \\ \Delta[4;(b_B)+n+3],\Delta(4;n+4);\end{matrix}\ -4^{4(A-B)-3}\ x^4\right], \tag{4.5}$$

$$F^{A:0;0}_{B:0;3}\left(\begin{matrix}[(a_A):1,4]: \ \text{---}\ ; \ \text{-------------}\ ; \\ \\ [(b_B):1,4]: \ \text{---}\ ; \left[\frac{1}{2}:1\right],\left[\frac{3}{4}:1\right],\left[\frac{5}{4}:1\right];\end{matrix}\ x,\frac{x^4}{256}\right)$$

$$= \frac{1}{2}\ {}_{A+1}F_{B+1}\left[\begin{matrix}(a_A),1;\\ \\ (b_B),2;\end{matrix}\ 2x\right]+\frac{1}{\sqrt{2}}\ \sum_{n=0}^{2}\frac{[(a_A)]_n}{[(b_B)]_n}\ \frac{(\sqrt{2}\ x)^n}{(n+1)!}\ \cos\left(\frac{(n-1)\pi}{4}\right)$$

$$\cdot {}_{4A+1}F_{4B+4}\left[\begin{matrix}\Delta[4;(a_A)+n],1;\\ \\ \Delta[4;(b_B)+n],\Delta(4;n+2);\end{matrix}\ -4^{4(A-B)-3}\ x^4\right], \tag{4.6}$$

$$F^{A:0;0}_{B:0;3}\left(\begin{array}{l}[(a_A):1,4]:\ \text{——}\ ;\ \text{————————}\ ;\\ \\ [(b_B):1,4]:\ \text{——}\ ;\left[\frac{3}{4}:1\right],\left[\frac{5}{4}:1\right],\left[\frac{3}{2}:1\right];\end{array} x,\frac{x^4}{256}\right)$$

$$= {}_{A+1}F_{B+1}\left[\begin{array}{l}(a_A),1;\\ \\ (b_B),3;\end{array} 2x\right]+2\sqrt{2}\,x\ \frac{[(a_A)]}{[(b_B)]}\sum_{n=0}^{2}\frac{[(a_A)+1]_n}{[(b_B)+1]_n}\ \frac{(\sqrt{2}\,x)^n}{(n+3)!}\ \sin\left(\frac{(n+1)\pi}{4}\right)$$

$$\cdot {}_{4A+1}F_{4B+4}\left[\begin{array}{c}\Delta[4;(a_A)+n+1],1;\\ \\ \Delta[4;(b_B)+n+1],\Delta(4;n+4);\end{array} -4^{4(A-B)-3}\ x^4\right] \tag{4.7}$$

and

$$F^{A:0;0}_{B:0;3}\left(\begin{array}{l}[(a_A):1,4]:\ \text{——}\ ;\ \text{————————}\ ;\\ \\ [(b_B):1,4]:\ \text{——}\ ;\left[\frac{5}{4}:1\right],\left[\frac{3}{2}:1\right],\left[\frac{7}{4}:1\right];\end{array} x,\frac{x^4}{256}\right)$$

$$= 2\ {}_{A+1}F_{B+1}\left[\begin{array}{l}(a_A),1;\\ \\ (b_B),4;\end{array} 2x\right]-1-12\sqrt{2}\,x^2\ \frac{[(a_A)]_2}{[(b_B)]_2}$$

$$\cdot\sum_{n=0}^{2}\frac{[(a_A)+2]_n}{[(b_B)+2]_n}\ \frac{(\sqrt{2}\,x)^n}{(n+5)!}\ \cos\left(\frac{(n+3)\pi}{4}\right)$$

$$\cdot {}_{4A+1}F_{4B+4}\left[\begin{array}{c}\Delta[4;(a_A)+n+2],1;\\ \\ \Delta[4;(b_B)+n+2],\Delta(4;n+6);\end{array} -4^{4(A-B)-3}\ x^4\right], \tag{4.8}$$

where, just as already indicated in (1.2) and (4.1),

$$[(a_A)]=\prod_{j=1}^{A}(a_j) \qquad \text{and} \qquad [(a_A)]_n=\prod_{j=1}^{A}(a_j)_n \qquad (n\in\mathbf{N}_0).$$

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