Solutions of Volterra integral and integro-differential equations using modified Laplace Adomian decomposition method

D. RANI AND V. MISHRA

Abstract

In this paper, an effectual and new modification in Laplace Adomian decomposition method based on Bernstein polynomials is proposed to find the solution of nonlinear Volterra integral and integro-differential equations. The performance and capability of the proposed idea is endorsed by comparing the exact and approximate solutions for three different examples on Volterra integral, integro-differential equations of the first and second kinds. The results shown through tables and figures demonstrate the accuracy of our method. It is concluded here that the non orthogonal polynomials can also be used for Laplace Adomian decomposition method. In addition, convergence analysis of the modified technique is also presented.

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Keywords: Laplace transformation, Adomian decomposition method, modified Laplace Adomian decomposition method, Bernstein polynomials, Volterra integral and integro-differential equations.

1. INTRODUCTION

Substantial interest is devoted to solve nonlinear Volterra integral and integro-differential equations by many researchers and scientists due to its applications in science such as the population dynamics, spread of epidemics, semi-conductor devices [Wazwaz 2011], biological species coexisting together with increasing and decreasing rates of generating and in engineering such as heat transfer and neutron diffusion process [Bahuguna et al. 2009].

The nonlinear Volterra integral equation of the second kind is defined as [Wazwaz 2011]

$$u(x) = f(x) + \int_0^x k(x,t)F(u(t))dt \tag{1}$$

where $f(x)$ is known as source term and F is a nonlinear operator, $F(u(x))$ is a nonlinear function.

The nonlinear Volterra integro-differential equation of the first kind is given by

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[Wazwaz 2011; 2010]

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u^{(i)}(t)dt = f(x) \tag{2}$$

However, the nonlinear Volterra integro-differential equation of the second kind is defined as [Wazwaz 2011; 2010]

$$u^{(i)}(x) = f(x) + \int_0^x k(x,t)F(u(t))dt \tag{3}$$

where $u^{(i)}(x)$ denotes the ith order derivative of $u(x)$. The kernel $k(x,t)$ and the function $f(x)$ of these equations are given real-valued functions and $F(u(x))$ is a nonlinear function.

Earlier many numerical and analytical methods have been presented to solve these kinds of equations [Wazwaz 2011; 2010; Maleknejad and Najafi 2011; Maleknejad et al. 2011].

1.1. Laplace Adomian decomposition method and modifications

In recent years, several researchers have adapted Adomian decomposition method (ADM) to solve many kinds of functional equations, which was developed by Adomian in 1980. In [Adomian 1988; 1990], Adomian provided a review of decomposition method in applied mathematics. The solution in this method is considered as the summation of an infinite convergent series without using any restrictive assumptions. A theoretical foundation of Adomian method was developed in [Gabet 1994], Venkatarangan and Rajalakshmi [Venkatarangan and Rajalakshmi 1995] used modified ADM to solve equations containing radical signs. Adomian polynomials are modified by Adomian and Rach in [Adomian and Rach 1996], Luo et. al [Luo et al. 2006] studied the partial solutions on ADM for solving heat and wave equations, Hashim [Hashim 2006] applied ADM to solve linear and nonlinear boundary value problems for fourth order integro-differential equations. In [Hosseini 2006], Hosseini modified the Adomian decomposition method by expressing the source function $f(x)$ in Chebyshev polynomials and solved the nonlinear differential algebraic equations. The ADM is used to solve nonlinear Sturm-Liouville problems in [Somali and Gokmen 2007], Marwat and Asghar [Marwat and Asghar 2008] suggested a two step Adomian decomposition method for solving heat equation with variable coefficients, Liu [Liu 2009] employed Legendre polynomials to improve the Adomian decomposition method and concluded that Chebyshev and Legendre polynomials can be successfully used for ADM and comparatively Chebyshev expansion provides the better estimation . The interested reader can see the other applications and modifications of this method in [Ghazanfari and Sepahvandzadeh 2014; Evans et al. 2004; Singh and Kumar 2011; Biazar et al. 2004; Zhang and Lu 2011; Li and Wang 2009; Biazar et al. 2010; Abassy 2010; Bildik and Deniz 2015; Babolian and Biazar 2002].

Further, Khuri [Khuri 2001] developed Laplace Adomian decomposition method and applied to find the solutions of nonlinear differential equations. This method is the combination of two powerful tools, Laplace transform and Adomian decomposition method, which is used to solve extinct functional equations [Wazwaz 2010; Doan 2012]. Hence, there are numerous applications where Laplace Adomian decomposition method is used. The method is also improved and modified from different aspects by some authors [Manafianheris 2012; Kumar et al. 2014].

In this work, our aim is to modify Laplace Adomian decomposition method based on Bernstein polynomials. At the beginning of our technique, we expand the source function, i.e. $f(x)$ as Bernstein polynomials which approximate the function uniformly and then Laplace Adomian decomposition method is applied to solve Volterra integral and integro-differential equations, that gives the tremendous improved results as shown in examples. To the best of our knowledge, Bernstein polynomials is not combined with the LADM. Therefore, this is the new idea which we have used.

1.2. Bernstein polynomials

The Berstein basis polynomials which are named after Russian mathematician Sergei Bernstein, is used to approximate the functions and curves. Following are some basic definitions [Quain et al. 2011]:

DEFINITION 1.1 (Bernstein basis polynomials). *The Bernstein basis polynomials of degree n form a complete basis over the interval* $[0,1]$ *and are defined by*

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, k = 0,1,2\ldots n.$$

DEFINITION 1.2 (Bernstein polynomials). *A linear combination of Bernstein basis polynomials*

$$B_n(x) = \sum_{k=0}^{n} \beta_k B_{k,n}(x) \tag{4}$$

is called the Bernstein polynomial of degree n where β_k are Bernstein coefficients.

DEFINITION 1.3. *With f a real valued function defined and bounded on $[0,1]$, let $B_n(f)$ be the polynomial on $[0,1]$,that assigns to $f(x)$ the value*

$$B_n(f) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{5}$$

where $B_n(f)$ is the nth Bernstein polynomial for $f(x)$.

The utilizations and properties of Bernstein polynomials have gained much importance in the domain of applied mathematics, physics and computer aided-geometric designs [Farouki 2012; Farouki and Rajan 1998; Bohm et al. 1984; Bhatti and Bracken 2007]. Bernstein polynomials are the basis of approximation theory, with the help of these polynomials Weierstrass approximation theorem [Quain et al. 2011] is proved, which is given as follows:

THEOREM 1.4. *For all functions f in C[0, 1], the sequence of $B_n(f)$ converges uniformly to f, where $B_n(f)$ is defined by (5).*

Using Taylors series, if we approximate a function, curve or surface, it seems that it converges slowly and does not converge to original function. Comparatively, Bernstein polynomials are better approximation to a function. It also has some applications in optimal control theory, stochastic dynamics and in the modelling of chemical reactions [Yousefi and Behroozifar 2010]. Problems like, elliptic and hyperbolic partial differential equations have been solved using Bernstein polynomials by implementation of Galerkin and collocation approaches to determine the coefficients.

The contents of this paper are as follows: in Section 2, we will give analysis of modified LADM; Section 3 gives the convergence analysis of the method; in Section 4 we will give three examples to demonstrate the applicability of the proposed approach. In the last section, conclusions are drawn.

2. MODIFIED LAPLACE ADOMIAN DECOMPOSITION METHOD BASED ON BERNSTEIN POLYNOMIALS

In this section, we are analyzing the method developed in [Rani and Mishra 2017] for nonlinear Volterra integral equation with the difference kernel, i.e $k(x,t) = k(x-t)$ given by (1).

Adopting the standard Laplace Adomian decomposition method, firstly applying Laplace transform on both sides of (1) and with the use of linear property and convolution theorem of Laplace transform, we get

$$L[u(x)] = L[f(x)] + L[k(x-t)]L[F(u(x))] \tag{6}$$

According to the LADM technique $u(x)$ can be written as an infinite series given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{7}$$

Then writing the nonlinear term $F(u(x))$ as

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x) \tag{8}$$

where A_n's are the Adomian polynomials, given by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0} \tag{9}$$

Substituting (7) and (8) into (6), we get

$$L\left[\sum_{n=0}^{\infty} u_n(x)\right] = L[f(x)] + L[k(x-t)]L\left[\sum_{n=0}^{\infty} A_n(x)\right]$$

The linearity property of Laplace transform implies

$$\sum_{n=0}^{\infty} L[u_n(x)] = L[f(x)] + L[k(x-t)]\sum_{n=0}^{\infty} L[A_n(x)] \tag{10}$$

Now we are modifying the standard LADM, where the source term is expanded or written in the form of Bernstein polynomials with degree m given by (5). Therefore, we attain

$$\sum_{n=0}^{\infty} L[u_n(x)] = L[B_m f(x)] + L[k(x-t)]\sum_{n=0}^{\infty} L[A_n(x)] \tag{11}$$

$u(x)$ can be found by defining the following iterative scheme:

$$L[u_0(x)] = L[B_m(f(x))] \tag{12}$$

Taking inverse Laplace transform on both sides of (12), we obtain

$$u_0(x) = L^{-1}[L(B_m(f(x)))]$$

Therefore, the initial approximation depends on the Bernstein polynomials of source function, which plays a significant role in the next approximations, hence in the approximate solution of the given problem.
Similarly, we have the general relation as

$$L[u_{n+1}(x)] = L[k(x-t)]L[A_n(x)] \tag{13}$$

For determining the terms u_1, u_2, u_3 ... of infinite series we use the inverse Laplace transform to above recursive relation and $u(x)$, the approximate solution to given nonlinear Volterra integral equation can be calculated. The same process is used to solve nonlinear Volterra integro-differential equations of the first and second kinds.
The efficacy of technique is demonstrated by convergence analysis and following numerical examples.

3. CONVERGENCE ANALYSIS

The convergence analysis is presented here which demonstrate the efficiency of the above-modified technique. Considering $E = (C[J], \|.\|)$ the Banach space of all continuous functions on J, suppose that there exist a constant N such that $|k(x,t)| \leq N$, for all $(x,t)\varepsilon[0,T]^2$.
Also, we suppose that the nonlinear term satisfy the Lipschitz condition, the approximate solution of (1) by using Bernstein polynomials based MLADM, converges to the exact one if $0 < \alpha < 1$, where $\alpha = NLx$.
Let U be the exact solution and U^* be the approximate solution of (1) by taking n terms, then

$$\begin{aligned} ||U - U^*|| &= \max_{x\varepsilon J} |f(x) + \int_0^x k(x,t)F(U(t))dt - B_m(f(x)) - \int_0^x k(x,t)F(U^*(t))dt| \\ &= |f(x) - B_m(f(x))| + \left| \int_0^x k(x,t)(F(U(t)) - F(U^*(t)))dt \right| \end{aligned} \tag{14}$$

Now using the convergence theorem of Bernstein polynomials (1.4) and above given conditions in the statement, we get

$$
\begin{aligned}
||U-U^*|| \leq \varepsilon + \int_0^x |k(x,t)||F(U(t))-F(U^*(t))|dt \\
\leq \varepsilon + \int_0^x NL|(U(t))-(U^*(t))|dt \\
\leq \varepsilon + NLx \max_{x\varepsilon J}|U(t)-U^*(t)| \\
\leq \alpha||U-U^*||
\end{aligned}
\tag{15}
$$

Therefore, if $0<\alpha<1$, $\alpha = NLx$, the approximate solution converges to exact solution as $n \to \infty$.

4. NUMERICAL EXAMPLES

EXAMPLE 4.1. *Consider the following nonlinear Volterra integro-differential equation of the second kind [Wazwaz 2010]*

$$u'(x) = -2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos 2x + \int_0^x \cos(x-t)u^2(t)dt, \tag{16}$$

with initial condition $u(0)=1$, having the exact solution as $u(x)=\cos x - \sin x$.

In this example, the source term, i.e. $f(x) = -2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos 2x$. Now using the above technique, we expand $f(x)$ in the terms of Bernstein polynomials of order $m=6$

$$
\begin{aligned}
f(x) \approx 0.000507191x^6 + 0.010605381x^5 - 0.10640906x^4 - 0.06228815x^3 + \\
1.314840965x^2 - 1.742867841x - 1
\end{aligned}
\tag{17}
$$

By applying Laplace transform to both sides of (16), we get

$$L[u(x)] = \frac{1}{s} + \frac{1}{s}L[f(x)] + \frac{1}{s^2+1}L[u^2(x)] \tag{18}$$

The methodology consisting of letting the solution as an infinite series as mentioned above, we have

$$L\left[\sum_{n=0}^{\infty} u_n(x)\right] = \frac{1}{s} + \frac{1}{s}L[f(x)] + \frac{1}{s^2+1}L\left[\sum_{n=0}^{\infty} A_n(x)\right] \tag{19}$$

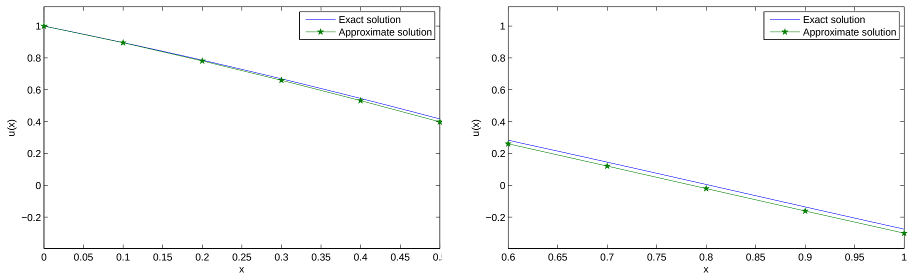

Fig. 1. Comparison of solutions in $[0,0.5]$ and $[0.6,1]$

where the nonlinear term $F(u(x)) = u^2(x)$ is decomposed in Adomian polynomials, few terms are as follows:

$$A_0 = u_0^2$$

$$A_1 = 2u_0u_1$$

$$A_2 = 2u_0u_2 + u_1^2$$

$$A_3 = 2u_0u_3 + 2u_1u_2$$

The recursive relation is obtained by comparing the terms in (19), which gives

$$L[u_0(x)] = \frac{1}{s} + \frac{1}{s}L[f(x)] \tag{20}$$

In general

$$L[u_{n+1}(x)] = \frac{1}{s^2+1}L[A_n(x)] \tag{21}$$

Employing the inverse Laplace transform on both sides of (20) and using (17), we get the value of $u_0(x)$.

Similarly (21) gives the values of $u_1(x)$, $u_2(x)$ and so on. Subsequently, one can compare the results from Figure 1, which shows that the approximate solutions are very much close to exact in the interval $[0,0.5]$ than in the interval $[0.6,1]$.

EXAMPLE 4.2. *The following nonlinear Volterra integro-differential equation of the first kind [Wazwaz 2010]*

$$\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u''(t)dt = -\frac{15}{32} + \frac{3x^2}{4} + \frac{1}{2}\cos 2x - \frac{1}{32}\cos 4x, \quad (22)$$

with initial condition $u(0) = 2$, $u'(0) = 0$ *which has the exact solution as* $u(x) = 1 + \cos 2x$.

Apply Laplace transform to both sides of (22) and using the derivative property and convolution theorem, we get

$$L\left[\int_0^x (x-t)u^2(t)dt\right] + L\left[\int_0^x (x-t)u''(t)dt\right] = L[f(x)] \quad (23)$$

By solving, we get

$$\frac{1}{s^2}L[u^2(x)] + L[u(x)] - \frac{2}{s} = L[f(x)]$$

$$L[u(x)] = \frac{2}{s} + L[f(x)] - \frac{1}{s^2}L[u^2(x)]$$

where the nonlinear term $F(u) = u^2$ is decomposed as in the previous example
Now proceeding as before, following iterative scheme is obtained:

$$L[u_0(x)] = \frac{2}{s} + L[f(x)] \quad (24)$$

In general

$$L[u_{n+1}(x)] = -\frac{1}{s^2}L[A_n(x)] \quad (25)$$

Here $f(x) = \frac{-15}{32} + \frac{3x^2}{4} + \frac{1}{2}\cos 2x - \frac{1}{32}\cos 4x$ is the source term.
By adopting the above method, we expand $f(x)$ as the Bernstein polynomials:

$$f(x) \approx -0.001381627x^6 + 0.013467998x^5 + 0.051210391x^4 + 0.02773726x^3 + 0.002551943x^2 + 0.00001698x \quad (26)$$

Applying inverse Laplace transform on both sides of (24), (25) and using the Bernstein polynomials given by (26), we get the values of $u_0(x)$, $u_1(x)$, $u_2(x)\ldots$. Therefore, we find the approximate solution. The approximate solution provides the accurate result or close to the exact solution in very few iterations that is shown in Figure 2.

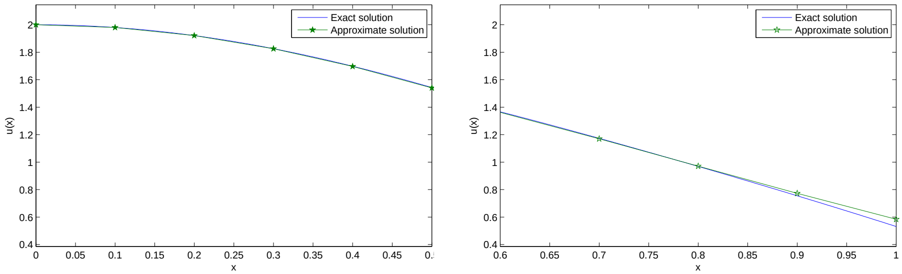

Fig. 2. Comparison of solutions in $[0,0.5]$ and $[0.6,1]$

EXAMPLE 4.3. *The nonlinear Volterra integral equation is given by*

$$u(x) = \frac{1}{4} + \frac{x}{2} + e^x - \frac{e^{2x}}{4} + \int_0^x (x-t)u^2(t)dt, \tag{27}$$

having the exact solution as $u(x) = e^x$.

The source term in (27) is $f(x) = \frac{1}{4} + \frac{x}{2} + e^x - \frac{e^{2x}}{4}$ which can be expanded in the Bernstein polynomials, here taking $m = 10$.

$$f(x) \approx -0.000000056x^{10} - 0.00000332x^9 - 0.00006387x^8 - 0.00076576x^7 - 0.00589954x^6 - 0.03027348x^5 - 0.1004593x^4 - 0.18599482x^3 - 0.05372434x^2 + 0.99820229x + 1 \tag{28}$$

Taking Laplace transform on both sides of (27), gives

$$L[u(x)] = L[f(x)] + \frac{1}{s^2}L[u^2(x)] \tag{29}$$

Now $u(x)$ can be evaluated based on Bernstein polynomials of $f(x)$ and with decomposing the nonlinear term in Adomian polynomials, which implies the relation

$$L[u_0(x)] = L[f(x)] \tag{30}$$

In general

$$L[u_{n+1}(x)] = \frac{1}{s^2}L[A_n(x)] \tag{31}$$

Substituting the approximated value of $f(x)$ from (28) in (30) and having inverse Laplace transform on both sides of (30), (31) give the values of $u_0(x)$, $u_1(x)$, $u_2(x)$,... $u_n(x)$. The sum of these terms will yield the value of truncated sum of $u(x)$. It is found that the error between exact and approximate solution is very less as shown in Figure 3 and reveals that the Bernstein polynomials based modification of LADM gives the solution in good agreement.

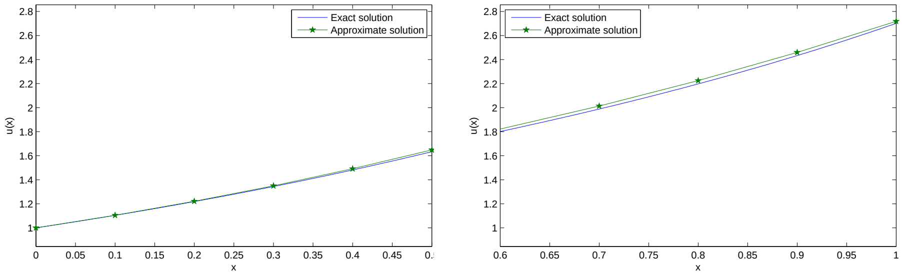

Fig. 3. Comparison of solutions in $[0, 0.5]$ and $[0.6, 1]$

Table I. Comparison of approximate solution by proposed method with exact solution of examples

	Example1		Example2		Example3	
x	Exact	Approximate	Exact	Approximate	Exact	Approximate
0	1	1	2	2	1	1
0.1	0.8952	0.8964	1.9801	1.9801	1.1052	1.1044
0.2	0.7814	0.7858	1.9211	1.9212	1.2214	1.2188
0.3	0.6598	0.6687	1.8253	1.8259	1.3499	1.3441
0.4	0.5316	0.5455	1.6967	1.6982	1.4918	1.4819
0.5	0.3982	0.4166	1.5403	1.5431	1.6487	1.6338
0.6	0.2607	0.2828	1.3624	1.3663	1.8221	1.8018
0.7	0.1206	0.1451	1.1700	1.1734	2.0138	1.9887
0.8	-0.0206	0.0049	0.9708	0.9687	2.2255	2.1978
0.9	-0.1617	-0.1361	0.7728	0.7550	2.4596	2.4335
1	-0.3012	-0.2758	0.5839	0.5314	2.7183	2.7014

The numerical results by using modified LADM based on Bernstein polynomials are also presented in Table I, which shows the performance of proposed technique.

5. CONCLUSIONS

For solving nonlinear Volterra integral and integro-differential equations a modification in standard Laplace Adomian decomposition method based on Bernstein polynomials is used here. Comparisons and analyses conclude that not only the orthogonal polynomials like Legendre, Chebyshev or Jacobi polynomials can improve the ADM, the Bernstein polynomials can also improve the source term as it is the better approximation to a function and hence the approximate solution converges to exact one as shown in the examples.

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Dimple Rani
Department of Mathematics,
Sant Longowal Institute of Engineering and Technology,
Longowal-148106 (Punjab), India
email: chawla23dimple@gmail.com

Vinod Mishra
Department of Mathematics,
Sant Longowal Institute of Engineering and Technology,
Longowal-148106 (Punjab), India
email: vinodmishra.2011@rediffmail.com