A remark on eigen values of signed graph

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Abstract

In this paper we present some results using eigen values of signed graph. This precipitate to find the determinant of signed graph using the number of vertices.

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1. INTRODUCTION

For all expressions and notation in graph theory the reader has to refer [4]. We consider only finite, simple graphs without self-loops.

A signed graph $\Gamma = (G, \sigma)$ is a graph whose edges are assigned by positive sign or negative sign. Where $G = (V, E)$ is called basic graph (underlined graph) of Γ and σ is a bijective mapping between the set of edges to the set of signs (positive and negative sign).

Cartwright and Harary [5] deliberates that vertices and edges of a sign graph represent persons and their relationship respectively, each of it fixes itself as positive or negative based on its nature. If two persons are friendly to each other then their relationship is positive. In a similar manner if two persons dislike each other, their relationship is negative. This signed graph network was discussed by Chartand [6]; Harary et. al. [7].

Katai and Iwai [8], Roberts [9] and Roberts and Xu [10] presented applications of signed graphs in literature, because of their extended use in modeling a multiple socio-psychological process and also because of their connections with many classical mathematical systems [3].

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The notation of *balanced* signed graph introduced by Hary and he has given characterization, asserting that, a signed graph is said to be *balanced* if and only if each cycle contains even number of negative edges.

A positive cycle in a signed graph is the product of signs of edges in a cycle is positive. If every cycle in a signed graph is positive then that signed graph is called *balanced* signed graph (see Harary [11]) otherwise it is called *unbalanced* signed graph. A *balance* of a signed graph can be detected by simple algorithm, which is developed by Harary and Kabell [16]

A marked graph of a signed graph Γ is denoted as Γ_μ whose vertices are assigned by sign $+$ or $-$, where μ is the canonical marking, therefore, Γ_μ can be defined as follows: For any vertex $v \in V(\Gamma)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

In a signed graph $\Gamma = (G, \sigma)$, for any $A \subseteq E(G)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

Switching of signed graph Γ is an operation with the help of marking μ, in which sign of each edge of Γ is changed to opposite sign, whenever sign of two end vertices of edge are opposite and such a signed graph is known as *switching signed graph* denoted as $\Gamma_\mu(\Gamma)$ and it is called Γ_μ -switched signed graph or *switched signed graph*.

R.P Abelson and Rosenberg introduced switching Sign graph for social behavioral analysis in [1] and its significance and mathematical connections may be found in [3].

If two underlying graphs G_1 and G_2 are *isomorphic* ($f : G_1 \to G_2$) then there signed graphs $\Gamma_1 = (G_1, \sigma)$ and $\Gamma_2 = (G_2, \sigma')$ are also *isomorphic*. For the underlying graphs G_1 and G_2, any edge l belongs to G_1, $\sigma(l) = \sigma'(f(l))$. Therefore Γ_1 *and* Γ_2 are *switching equivalent* which is represent as $\Gamma_1 \sim \Gamma_2$. Precisely for any marking μ , $\Gamma_\mu(\Gamma_1) \sim \Gamma_2$. But note that their underlying graphs G_1 and G_2 does not involve in any change in their adjacency.

If two signed graphs are cycle *isomorphic* then there corresponding cycles are having same sign.

T. Zaslavasky has given characterization of switching Singed graph through the following proposition.

PROPOSITION 1. *[2] Two signed graphs Γ_1 and Γ_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

The difference between positive edges and negative edges incident to a vertex v in a signed graph Γ is known as sdeg(v) that is $d_v^+ - d_v^-$ = sdeg(v), therefore, degree of a vertex in a signed graph is defined as $d = d_v^+ - d_v^-$ wherefore signed graph and there underlying graph has the same degree.

2. DEFINITIONS

2.1. *Adjacency matrix of a Signed graph*

The adjacency matrix of Γ is the $n \times n$ matrix $A = A(\Gamma)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent and } \sigma(v_i v_j) = + \\ -1, \text{ if } v_i \text{ and } v_j \text{ are adjacent and } \sigma(v_i v_j) = - \\ 0, \text{ otherwise.} \end{cases}$$

2.2. *Laplacian matrix of a Signed graph*

The *Laplacian* matrix of signed graph Γ is the $n \times n$ matrix $L(\Gamma) = D(\Gamma) - A(\Gamma)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent and } \sigma(v_i v_j) = - \\ -1, \text{ if } v_i \text{ and } v_j \text{ are adjacent and } \sigma(v_i v_j) = + \\ d(v_i) \text{ if i = j} \\ 0, \text{ otherwise.} \end{cases}$$

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots. \geq \lambda_n$ are the eigenvalues of *Laplacian* matrix of signed graph $\Gamma = (G, \sigma)$, having n vertices.

LEMMA 2. *[17] Two signed graphs $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ with their same underlying graphs are switching equivalent if and only if $L(\Gamma_1)$ and $L(\Gamma_2)$ are has the same signature.*

2.3. Antipodal signed graphs

The concept of *Antipodal graph* introduced by R.R. Singleton in 1968 defined it as *Antipodal graph* $\Theta(G)$ is a graph which has the same vertices of a graph G with the distance between the adjacent vertices are of diameter of G.

P.S.K Reddy et.al. devoloped the concept of Antipodal signed graph and gave the following characterization.

PROPOSITION 3. *[13] For any signed graph $\Gamma = (G, \sigma)$, $\Gamma \sim \Theta(\Gamma)$ if, and only if, $G = K_p$ and Γ is balanced signed graph, where K_p is a complete graph with p vertices.*

The *Laplacian* matrices $L(\Gamma, +)$ and $L(\Gamma, -)$ are all positive and all negative labeling respectively, and also $L(\Gamma, +)$ is the sign less *Laplacian* matrix of Γ which is the sum of the *diagonal matrix* and the *adjacency*$(L(\Gamma) = D(\Gamma) + A(\Gamma))$.

Yaoping Hou et. al. has given the new bounds of eigen values of signed graph by the following theorem:

THEOREM 4. *[14] Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices. Then*

$$\lambda_1 \leq 2(n-1),$$

equality holds if and only if $\Gamma \sim (K_n, -)$, where K_n is the complete graph with n vertices.

By the motivation of the above theorem we present some new results in this article.

PROPOSITION 5. *For any signed graph Γ,*

$$\sum_{i=1}^{n} \lambda_i = n(n-1) \; if \; \Gamma \sim (K_n, -)$$

PROOF. Since $\Gamma \sim (K_n, -)$, From Theorem 4 we have, $\lambda_1 = 2(n-1)$

therefore,

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i &= \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n \\ &= 2(n-1) + \lambda_2 + \lambda_3 + \ldots + \lambda_n \\ &= 2(n-1) + (n-1)(n-2) \\ &= (n-1)(2+n-2) \\ &= n(n-1)\end{aligned}$$

COROLLARY 6. *if* $\Gamma \sim (\Theta(\Gamma), -)$, *then* $\sum_{i=1}^{n} \lambda_i = n(n-1)$

PROOF. Since, *Antipodal graph* $\Theta(G) \cong K_n$ so, proof is same as Proposition 5.

LEMMA 7. *[15] If Q is the* Incident matrix *of a connected signed graph* $\Gamma = (G, \sigma)$ *then*

$$rank(Q) = \begin{cases} n-1, & if\ \ \Gamma\ \ is\ \ balanced \\ n, & if\ \ \Gamma\ \ is\ \ unbalanced. \end{cases}$$

from the above we can prove that.

LEMMA 8. *If $L(\Gamma)$ is the* Laplacian *matrix of a connected signed graph* $\Gamma = (G, \sigma)$ *then*

$$rank(L(\Gamma)) = \begin{cases} n-1, & if\ \ \Gamma\ \ is\ \ balanced \\ n, & if\ \ \Gamma\ \ is\ \ unbalanced. \end{cases}$$

PROOF. Since, $L(\Gamma) = QQ^T$

$rank(L(\Gamma)) = rank(QQ^T) = rank(Q)$ by Lemma 7 we proved.

PROPOSITION 9. *For any signed graph,* $\Gamma \sim (\Gamma, -)$, *whose underline graph is* K_n *is Unbalanced*

PROOF. we can prove by using Proposition 1

PROPOSITION 10. *For any signed graph* Γ, *and if* $\Gamma \sim (K_n, -)$,

$$rank(\Gamma) = \frac{\sum_{i=1}^{n} \lambda_i}{n-1}$$

PROOF. Since, $\Gamma \sim (\Gamma, -)$ and Γ is unbalanced, by Lemma 8,

$$rank(\Gamma) = \text{n},$$

but by Proposition 5,

$$\sum_{i=1}^{n} \lambda_i = n(n-1)$$

$$\sum_{i=1}^{n} \lambda_i = rank(\Gamma)(n-1)$$

$$rank(\Gamma) = \frac{\sum_{i=1}^{n} \lambda_i}{n-1}$$

PROPOSITION 11. *A unicyclic signed graph Γ with even vertices and $\Gamma \sim (\Gamma, -)$ is balanced.*

PROOF. Since Unicyclic graph having even number of vertices so it should have even number of edges,

hence, Γ and $(\Gamma, -)$ are cycle isomorphic and $(\Gamma, -)$ is balanced,

by Proposition 1, Γ is balanced.

COROLLARY 12. *If Γ is unicyclic signed graph with even vertices and $\Gamma \sim (\Gamma, -)$ then $det(\Gamma) = 0$*

PROPOSITION 13. *A Unicyclic signed graph $\Gamma = (G, \sigma)$ with even vertices. Then*

$$\lambda_1 \leq 2(n-2),$$

with equality if and only if $\Gamma \sim (\Gamma, -)$

PROOF. Since

$$L(\Gamma) = D(\Gamma) - A(\Gamma)$$

$$\lambda_1(\Gamma) \leq \lambda_1(D(\Gamma)) + \lambda_1(-A(\Gamma))$$

$$\leq (n-2) + (n-2)$$

$$\lambda_1(\Gamma) \leq 2(n-2)$$

If $\lambda_1 \sim (\Gamma, -)$ then clearly, $\lambda_1(\Gamma) = 2(n-2)$

conversely,If $\lambda_1(\Gamma) = 2(n-2)$ then $\lambda_1(D(\Gamma)) = \lambda_1(-A(\Gamma)) = (n-2)$ and Signature of Γ and $(\Gamma, -)$ are equal, hence $\lambda_1 \sim (\Gamma, -)$.

PROPOSITION 14. *For any unicyclic signed graph* Γ *with even number of vertices,*

$$\sum_{i=1}^{n} \lambda_i = n(n-2) \; if \; \Gamma \sim (\Gamma, -)$$

PROOF. Since $\Gamma \sim (\Gamma, -)$, From the Proposition 13 we have, $\lambda_1 = 2(n-2)$ therefore,

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i &= \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n \\ &= 2(n-2) + \lambda_2 + \lambda_3 + \ldots + \lambda_n \\ &= 2(n-2) + (n-2)(n-2) \\ &= (n-2)(2+n-2) \\ &= n(n-2) \end{aligned}$$

PROPOSITION 15. *For any Unicyclic signed graph with even number of vertices and* $\Gamma \sim (\Gamma, -)$,

$$rank(\Gamma) = \frac{\sum_{i=1}^{n} \lambda_i}{d_i} - 1$$

PROOF. Since, $\Gamma \sim (\Gamma, -)$ and Γ is balanced by Proposition 12,

but by Proposition 14 , $\sum_{i=1}^{n} \lambda_i = n(n-2)$

$$\sum_{i=1}^{n} \lambda_i = nd_i$$

by lemma 8 , $rank(\Gamma) = n-1$

therefore, $\sum_{i=1}^{n} \lambda_i = [1 + rank(\Gamma)]d_i$

$$rank(\Gamma) = \frac{\sum_{i=1}^{n} \lambda_i}{d_i} - 1$$

PROPOSITION 16. *A unicyclic signed graph* Γ *with odd vertices and* $\Gamma \sim (\Gamma, -)$

is unbalanced.

PROOF. Unicyclic graph having odd number of vertices so it has odd number of edges, so Γ and $(\Gamma, -)$ are not cycle isomorphic and $(\Gamma, -)$ is unbalanced,

by Proposition 1, Γ is unbalanced.

By using the Proposition 5 we can find the determinant of signed graph having n vertices.

THEOREM 17. *For any signed graph* Γ*, if* $\Gamma \sim (K_n, -)$ *then,*

$$\det(\Gamma) = 2(n-1)(n-2)^{n-1}$$

PROOF.

We know,

$$\det(\Gamma) = \prod_{i=1}^{n} \lambda_i$$

$$= \lambda_1 \,.\, \lambda_2 \,.\, \lambda_3 \;\; .... \;\; \lambda_n$$

$$= 2(n-1) \,.\, (n-2) \,.\, (n-2) \;\; .... \;\; (n-2)$$

$$= 2(n-1)(n-2)^{n-1}$$

Hence the proof.

REFERENCES

R. P. Abelson and M. J. Rosenberg, Symbolic psychologic:A model of attitudinal cognition, *Behav. Sci.*, 3 (1958), 1-13.

T. Zaslavsky, signed graphs, *Discrete Appl. Math.*, 4(1)(1982), 47-74.

T. Zaslavsky, A mathematical bibliography of signed and gain graphs and its allied areas, *Electronic J. Combin.*, 8(1)(1998), Dynamic Surveys (1999), No. DS8.

F. Harary, Graph Theory, Addison-Wesley Publishing Co., 1969.

D. W. Cartwright and F. Harary, Structural balance: A generalization of Heider's Theory, *Psych. Rev.*, 63(1956), 277-293.

G. T. Chartrand, *Graphs as Mathematical Models*, Prindle, Weber & Schmidt, Inc., Boston, Massachusetts 1977.

F. Harary, R. Z. Norman and D. W. Cartwright, *Structural models: An introduction to the theory of directed graphs*, Wiley Inter-Science, Inc., New York, 1965.

O. Katai and S. Iwai, Studies on the balancing, the minimal balancing, and the minimum balancing processes for social groups with planar and nonplanar graph structures, *J. Math. Psychol.*, 18(1978), 140-176.

F. S. Roberts, *Graph Theory and its Applications to Problems of Society*, SIAM, Philadelphia, PA, USA, 1978.

F. S. Roberts and Shaoji Xu, Characterizations of consistent marked graphs, *Discrete Applied Mathematics*, 127(2003), 357- 371.

F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, 2(1953), 143-146.

E. Sampathkumar, Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, 7(3) (1984), 91-93.

P. Siva Kota Reddy, B. Prashanth, and Kavitha. S.Permi, A note on antipodal signed graphs, *International J. Math. Combin.*, 1(2011),107-112.

Y. Hou , J. Li and Y. Pan (2003) On the Laplacian Eigenvalues of signed graphs, *Linear and Multilinear Algebra*,51:1, 21-30, DOI:10.1080/0308108031000053611

F. Belardo, S. K. Simič, On the Laplacian coefficients of signed graphs, *Linear Algebra and Its Applications*, 475(2015),94-113

F. Harary and J.A. Kabell, Counting balanced signed graphs using marked graphs, *Proc. Edinburgh Math. Soc.*, 24 (2)(1981),99-104.

T. Zaslavsky (1982). signed graphs. Discrete Appl. Math., 4, 4774.

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