Fractional Hermite-Hadamard type inequalities for co-ordinated prequasiinvex functions

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Abstract

Some new Ostrowski's inequalities for functions whose n-th derivatives are h-convex are established.

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Keywords: In this paper, the concept of co-ordinated prequasiinvex is introduced, some fractional Hermite-Hadamard type inequalities for functions whose modulus of the mixed derivatives lies in this novel class of functions are established.

1. INTRODUCTION

One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard integral inequality, which can be stated as follows: for every convex function f on the finite interval $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

If the function f is concave, then (1) holds in the reverse direction (see [Pečarić et al. 1992]).

In recent years, lot of efforts have been made by mathematicians and researchers to generalize the classical convexity. Hanson [Hanson 1981], introduced a new class of generalized convex functions, called invex functions, In [Ben-Israel and Mond 1986] the authors gave the concept of preinvex function which is special case of invexity. Pini [Pini 1991] introduced the concept of prequasiinvexity which generalize that of preinvex function, Noor [Noor 1994; 2005], Yang and Li [Yang and Li 2001] and Weir [Weir and Mond 1988], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

Dragomir [Dragomir 2001] introduced the concept of the convexity on the co-ordinates as follows:

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A function $f:\Delta\rightarrow\mathbb{R}$ is said to be convex on the co-ordinates on Δ, where $\Delta:=[a,b]\times[c,d]$ is a bidimensional interval in $\mathbb{R}^2$ with $a<b$ and $c<d$, if

$$\begin{aligned} f\left(tx+(1-t)u,\lambda y+(1-\lambda)v\right) \leq\ & t\lambda f(x,y)+t(1-\lambda)f(x,v) \\ & +(1-t)\lambda f(u,y)+(1-t)(1-\lambda)f(u,v) \end{aligned}$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(u,v)\in\Delta$.

Also, he proved the two-dimensional analog of (1), which can be stated as follows: For all co-ordinated convex function f on $[a,b]\times[c,d]$, we have

$$\begin{aligned} & f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right) \\ \leq\ & \tfrac{1}{2}\left(\tfrac{1}{b-a}\int_a^b f(x,\tfrac{c+d}{2})dx+\tfrac{1}{d-c}\int_c^d f(\tfrac{a+b}{2},y)dy\right) \\ \leq\ & \tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ \leq\ & \tfrac{1}{4}\left(\tfrac{1}{b-a}\left(\int_a^b f(x,c)dx+\int_a^b f(x,d)dx\right)+\tfrac{1}{d-c}\left(\int_c^d f(a,y)dy+\int_c^d f(b,y)dy\right)\right) \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{2}$$

Özdemir et al. [Özdemir et al. 2012] introduced the concept of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions as follows:

A function $f:\Delta\rightarrow\mathbb{R}$ is said to be co-ordinated quasi-convex on Δ, if

$$f\left(tx+(1-t)u,\lambda y+(1-\lambda)v\right)\leq\max\left\{f(x,y),f(u,v)\right\}$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(u,v)\in\Delta$.

A formal definition of co-ordinated quasi-convex functions is

$$f\left(tx+(1-t)u,\lambda y+(1-\lambda)v\right)\leq\max\left\{f(x,y),f(x,v),f(u,y),f(u,v)\right\}$$

for all $t,\lambda\in[0,1]$ and $(x,y),(u,v)\in\Delta$.

In [Özdemir et al. 2012] Özdemir et al. established the following Hermite-Hadamard's inequalities for differentiable co-ordinated quasi-convex functions

THEOREM 1.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial \lambda \partial t}\right|$ is quasi-convex on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\max\left\{\left|\frac{\partial^2 f}{\partial \lambda \partial t}(a,b)\right|, \left|\frac{\partial^2 f}{\partial \lambda \partial t}(c,d)\right|\right\},$$

where

$$A = \frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]\,dy. \tag{3}$$

THEOREM 1.2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial \lambda \partial t}\right|^q$, $q > 1$, is quasi-convex on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left(\max\left\{\left|\frac{\partial^2 f}{\partial \lambda \partial t}(a,b)\right|^q, \left|\frac{\partial^2 f}{\partial \lambda \partial t}(c,d)\right|^q\right\}\right)^{\frac{1}{q}},$$

where A is as defined by (3)*, and* $\frac{1}{p} + \frac{1}{q} = 1$.

THEOREM 1.3. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial \lambda \partial t}\right|^q$, $q \geq 1$, is quasi-convex on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left(\max\left\{\left|\frac{\partial^2 f}{\partial \lambda \partial t}(a,b)\right|^q, \left|\frac{\partial^2 f}{\partial \lambda \partial t}(c,d)\right|^q\right\}\right)^{\frac{1}{q}},$$

where A is as defined by (3).

In this paper we first introduce the concept of co-ordinated prequasiinvex, and then we derive some fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives lies in this new class of functions.

2. PRELIMINARIES

In this section we recall some concepts of generalized convexity and fractional calculus

DEFINITION 2.1. *[Matł oka 2013] Let K_1, K_2 be nonempty subsets of $\mathbb{R}^n$, $(u,v) \in K_1 \times K_2$. We say $K_1 \times K_2$ is invex at (u,v) with respect to η_1 and η_2, if*

$$(u + t\eta_1(x,u), v + s\eta_2(y,v)) \in K_1 \times K_2$$

holds for each $(x,y) \in K_1 \times K_2$ and $t,s \in [0,1]$.

$K_1 \times K_2$ is said to be an invex set with respect to η_1 and η_2 if $K_1 \times K_2$ is invex at each $(u,v) \in K_1 \times K_2$.

In what follows we assume that $K_1 \times K_2$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$.

DEFINITION 2.2. *[Latif and Dragomir 2013] A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be preinvex on the co-ordinates, if the following inequality*

$$\begin{aligned} f(u + \lambda\eta_1(x,u), v + t\eta_2(y,v)) \leq\ & (1-\lambda)(1-t)f(u,v) + (1-\lambda)tf(u,y) \\ & + (1-t)\lambda f(x,v) + \lambda t f(x,y) \end{aligned}$$

holds for all $t,\lambda \in [0,1]$ and $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$.

DEFINITION 2.3. *[Kilbas et al. 2006] Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x$$

respectively. Where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$, is the Gamma function and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

DEFINITION 2.4. *[Latif and Dragomir 2013] Let $f \in L([a,b] \times [c,d])$. The Riemann–Liouville integrals $J^{\alpha,\beta}_{a^+,c^+}$, $J^{\alpha,\beta}_{a^+,d^-}$, $J^{\alpha,\beta}_{b^-,c^+}$, and $J^{\alpha,\beta}_{b^-,d^-}$ of order $\alpha,\beta > 0$ with*

$a,c \geq 0$, $a<b$ and $<d$ are defined by

$$J^{\alpha,\beta}_{a^+,c^+} f(b,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (b-x)^{\alpha-1} (d-y)^{\beta-1} f(x,y)\, dydx, \tag{4}$$

$$J^{\alpha,\beta}_{a^+,d^-} f(b,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (b-x)^{\alpha-1} (y-c)^{\beta-1} f(x,y)\, dydx, \tag{5}$$

$$J^{\alpha,\beta}_{b^-,c^+} f(a,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (x-a)^{\alpha-1} (d-y)^{\beta-1} f(x,y)\, dydx, \tag{6}$$

and

$$J^{\alpha,\beta}_{b^-,d^-} f(a,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (x-a)^{\alpha-1} (y-c)^{\beta-1} f(x,y)\, dydx, \tag{7}$$

where Γ is the Gamma function, and

$$J^{0,0}_{a^+,c^+} f(b,d) = J^{0,0}_{a^+,d^-} f(b,c) = J^{0,0}_{b^-,c^+} f(a,d) = J^{0,0}_{b^-,d^-} f(a,c) = f(x,y).$$

DEFINITION 2.5. *[Sarı kaya 2014] Let $f \in L([a,b]\times[c,d])$. The Riemann–Liouville integrals $J^{\alpha}_{b^-} f(a,c)$, $J^{\alpha}_{a^+} f(b,c)$, $J^{\beta}_{d^-} f(a,c)$, and $J^{\alpha}_{c^+} f(a,d)$ of order $\alpha,\beta>0$ with $a,c \geq 0$, $a<b$, and $<d$ are defined by*

$$J^{\alpha}_{b^-} f(a,c) = \frac{1}{\Gamma(\alpha)} \int_a^b (x-a)^{\alpha-1} f(x,c)\, dx, \tag{8}$$

$$J^{\alpha}_{a^+} f(b,c) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x,c)\, dx, \tag{9}$$

$$J^{\beta}_{d^-} f(a,c) = \frac{1}{\Gamma(\beta)} \int_c^d (y-c)^{\beta-1} f(a,y)\, dy, \tag{10}$$

and

$$J^{\alpha}_{c^+} f(a,d) = \frac{1}{\Gamma(\beta)} \int_c^d (d-y)^{\beta-1} f(a,y)\, dy, \tag{11}$$

where Γ is the Gamma function.

LEMMA 2.6. *[Meftah 2019] Let $f: K \to \mathbb{R}$ be a partially differentiable function on K, if $\frac{\partial^2 f}{\partial t \partial s} \in L(K)$, then the following equality holds*

$$
\begin{aligned}
&\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A \\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c) \\
&+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a,c+\eta_2(d,c)) \\
&+ J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a),c+\eta_2(d,c)) \Big) \\
= \; &\tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \int_0^1 \int_0^1 \left(t^{\alpha} - (1-t)^{\alpha}\right) \left(s^{\beta} - (1-s)^{\beta}\right) \\
&\times \tfrac{\partial^2 f}{\partial t \partial s} (a+t\eta_1(b,a), c+s\eta_2(d,c))\, ds dt, \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
&A = \tfrac{\Gamma(\alpha+1)}{4(\eta_1(b,a))^{\alpha}} \Big(J^{\alpha}_{(a+\eta_1(b,a))^-} f(a,c+\eta_2(d,c)) + J^{\alpha}_{(a+\eta_1(b,a))^-} f(a,c) \\
&+ J^{\alpha}_{a^+} f(a+\eta_1(b,a),c+\eta_2(d,c)) + J^{\alpha}_{a^+} f(a+\eta_1(b,a),c) \Big) \\
&+\tfrac{\Gamma(\beta+1)}{4(\eta_2(d,c))^{\beta}} \Big(J^{\beta}_{(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c) + J^{\beta}_{(c+\eta_2(d,c))^-} f(a,c) \\
&+ J^{\alpha}_{c^+} f(a+\eta_1(b,a),c+\eta_2(d,c)) + J^{\alpha}_{c^+} f(a,c+\eta_2(d,c)) \Big). \qquad (13)
\end{aligned}
$$

3. MAIN RESULTS

In what follows we assume that $K = [a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)]$ be an invex subset of $\mathbb{R}^2$ with respect to η_1, η_2 where $\eta_1, \eta_2 : \mathbb{R}^2 \to \mathbb{R}$ are two bifunctions such that $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$.

We will start with the following definitions, and the lemma

DEFINITION 3.1. *A function $f: K \to \mathbb{R}$ is said to be prequasiinvex on the co-ordinates, if the following inequality*

$$f(u+\lambda\eta_1(x,u), v+t\eta_2(y,v)) \leq \max\{f(u,v), f(u+\eta_1(x,u), v+\eta_2(y,v))\}$$

holds for all $t, \lambda \in [0,1]$ and $(u,v), (x,y) \in K$.

A formal definition of co-ordinated prequasiinvex functions is given by the following definition

DEFINITION 3.2. *A function $f : K \to \mathbb{R}$ is said to be prequasiinvex on the co-ordinates, if the following inequality*

$$\begin{aligned} & f\left(u+\lambda \eta_1(x,u), v+t\eta_2(y,v)\right) \\ \leq\ & \max\left\{f(u,v), f(u,v+\eta_2(y,v)), f(u+\eta_1(x,u),v), f(u+\eta_1(x,u),v+\eta_2(y,v))\right\} \end{aligned}$$

holds for all $t,\lambda \in [0,1]$ and $(x,y),(x,v),(u,y),(u,v) \in K$.

THEOREM 3.3. *Let $f : K \to \mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$ is co-ordinated prequasiinvex function on K with respect to η_1 and η_2, then the following fractional inequality holds*

$$\begin{aligned} & \left| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A \right. \\ & + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c) \\ & + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a,c+\eta_2(d,c)) \\ & \left. + J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a),c+\eta_2(d,c)) \Big) \right| \\ \leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{(\alpha+1)(\beta+1)} \max\left\{ \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|, \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|, \right. \\ & \left. \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|, \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right| \right\}, \end{aligned}$$

where A is defined as in (13).

PROOF. From Lemma 2.6, and properties of modulus we have

$$\begin{aligned} & \left| \tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A \right. \\ & + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c) \\ & + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a,c+\eta_2(d,c)) \\ & \left. + J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a),c+\eta_2(d,c)) \Big) \right| \\ \leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \int_0^1\int_0^1 \left|t^{\alpha}-(1-t)^{\alpha}\right| \left|\lambda^{\beta}-(1-\lambda)^{\beta}\right| \\ & \times \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right| d\lambda dt \\ \leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \int_0^1\int_0^1 \left(t^{\alpha}+(1-t)^{\alpha}\right)\left(\lambda^{\beta}+(1-\lambda)^{\beta}\right) \end{aligned}$$

$$\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)\right|d\lambda dt. \tag{14}$$

Using prequasiinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, (14) gives

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\left(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\right.\\
&+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&\left.\left.+\,J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\right)\right|\\
\le\ &\frac{\eta_1(b,a)\eta_2(d,c)}{4}\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\right\}\\
&\times\left(\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(\lambda^{\beta}+(1-\lambda)^{\beta}\right)d\lambda dt\right)\\
=\ &\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha+1)(\beta+1)}\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\right\}.
\end{aligned}$$

The proof is achieved.

COROLLARY 3.4. *In Theorem 3.3 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$, we obtain the following fractional inequality*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.\\
&\left.\times\left(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right)\right|\\
\le\ &\frac{(b-a)(d-c)}{(\alpha+1)(\beta+1)}\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\right\}.
\end{aligned}$$

THEOREM 3.5. *Let $f: K\to\mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated prequasiinvex function on K with respect to η_1 and η_2, where $q>1$ with and $\frac{1}{p}+\frac{1}{q}=1$, then the following fractional inequality holds*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\left(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\right.\\
&+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))
\end{aligned}$$

$$+ J_{a^+,c^+}^{\alpha,\beta} f(a+\eta_1(b,a), c+\eta_2(d,c))\Big)\Big|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|^q\right\}\right)^{\frac{1}{q}},$$

where A is defined as in (13).

PROOF. From Lemma 2.6, properties of modulus, and Hölder inequality, we have

$$\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\Big(J_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}^{\alpha,\beta} f(a,c)$$
$$+J_{a^+,(c+\eta_2(d,c))^-}^{\alpha,\beta} f(a+\eta_1(b,a),c) + J_{(a+\eta_1(b,a))^-,c^+}^{\alpha,\beta} f(a,c+\eta_2(d,c))$$
$$+ J_{a^+,c^+}^{\alpha,\beta} f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\Big|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\int_0^1\int_0^1 t^{\alpha p}\lambda^{\beta p} d\lambda dt\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 t^{\alpha p}(1-\lambda)^{\beta p} d\lambda dt\right)^{\frac{1}{p}}\right.$$
$$\left. + \left(\int_0^1\int_0^1 (1-t)^{p\alpha}\lambda^{p\beta} d\lambda dt\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 (1-t)^{p\alpha}(1-\lambda)^{p\beta} d\lambda dt\right)^{\frac{1}{p}}\right)$$
$$\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right|^q d\lambda dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right|^q d\lambda dt\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated prequasiinvex function, we deduce

$$\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\Big(J_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}^{\alpha,\beta} f(a,c)$$
$$+J_{a^+,(c+\eta_2(d,c))^-}^{\alpha,\beta} f(a+\eta_1(b,a),c) + J_{(a+\eta_1(b,a))^-,c^+}^{\alpha,\beta} f(a,c+\eta_2(d,c))$$
$$+ J_{a^+,c^+}^{\alpha,\beta} f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\Big|$$

$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|^q\right\}\right)^{\frac{1}{q}},$$

which is the desired result.

COROLLARY 3.6. *In Theorem 3.5 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$, we obtain the following fractional inequality*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.$$
$$\left.\times\left(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right)\right|$$
$$\leq \frac{(b-a)(d-c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}$$
$$\times\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}.$$

THEOREM 3.7. *Let $f:K\to\mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated prequasiinvex function on K with respect to η_1 and η_2, and $p>1$ then the following inequality holds*

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\left(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\right.$$
$$+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))$$
$$\left.\left.+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\right)\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(1+\alpha)(1+\beta)}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\right\}\right)^{\frac{1}{q}},$$

where A is defined as in (13).

PROOF. From Lemma 2.6, properties of modulus, and power mean inequality, we have

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\left(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\right.$$
$$+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))$$

$$
\begin{aligned}
& + J^{\alpha,\beta}_{a^+,c^+} f\left(a+\eta_1(b,a), c+\eta_2(d,c)\right)\Big)\Big| \\
\leq\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\int_0^1\int_0^1 t^\alpha\lambda^\beta d\lambda dt\right)^{1-\frac{1}{q}}\right. \\
& \times\left(\int_0^1\int_0^1 t^\alpha\lambda^\beta\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 t^\alpha(1-\lambda)^\beta d\lambda dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1\int_0^1 t^\alpha(1-\lambda)^\beta\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 (1-t)^\alpha\lambda^\beta d\lambda dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1\int_0^1 (1-t)^\alpha\lambda^\beta\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 (1-t)^\alpha(1-\lambda)^\beta d\lambda dt\right)^{1-\frac{1}{q}} \\
& \left.\times\left(\int_0^1\int_0^1 (1-t)^\alpha(1-\lambda)^\beta\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}\right) \\
=\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4(1+\alpha)^{1-\frac{1}{q}}(1+\beta)^{1-\frac{1}{q}}}\left(\left(\int_0^1\int_0^1 t^\alpha\lambda^\beta\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}\right. \\
& +\left(\int_0^1\int_0^1 t^\alpha(1-\lambda)^\beta\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 (1-t)^\alpha\lambda^\beta\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\int_0^1\int_0^1 (1-t)^{\alpha}(1-\lambda)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}\Bigg).$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated prequasiinvex, we get

$$\begin{aligned}
&\Big|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\Big|\\
\le\ &\tfrac{\eta_1(b,a)\eta_2(d,c)}{4(1+\alpha)^{1-\frac{1}{q}}(1+\beta)^{1-\frac{1}{q}}}\left(\left(\int_0^1\int_0^1 t^{\alpha}\lambda^{\beta}d\lambda dt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1 t^{\alpha}(1-\lambda)^{\beta}d\lambda dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\int_0^1 (1-t)^{\alpha}\lambda^{\beta}d\lambda dt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1 (1-t)^{\alpha}(1-\lambda)^{\beta}d\lambda dt\right)^{\frac{1}{q}}\right)\\
&\times\Big(\max\Big\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,\\
&\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\Big\}\Big)^{\frac{1}{q}}\\
=\ &\tfrac{\eta_1(b,a)\eta_2(d,c)}{(1+\alpha)(1+\beta)}\Big(\max\Big\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,\\
&\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\Big\}\Big)^{\frac{1}{q}},
\end{aligned}$$

which is the desired result.

COROLLARY 3.8. *In Theorem 3.7 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$, we obtain the following fractional inequality*

$$\begin{aligned}
&\left|\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.\\
&\left.\times\left(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right)\right|\\
\le\ &\frac{(b-a)(d-c)}{(1+\alpha)(1+\beta)}\\
&\times\left(\max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q,\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

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