Some Cubic Rank Transmuted Distributions

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Abstract

In this article, we introduce some examples of cubic rank transmuted distributions proposed by Granzatto et al. (2017). The statistical aspects of the introduced distributions such as probability density functions, hazard rate functions and reliability functions are studied. The maximum likelihood estimation method is used in order to estimate the parameters of interest. Finally, real data examples are applied for the illustration of these distributions.

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1. INTRODUCTION

In order to obtain more flexible statistical models, generalization of the well-known distributions have been widely used. Firstly, Amoroso (1925) introduced the generalized gamma distribution in order to model the distribution of income rate. Since then various authors have discussed the generalizations of the distributions. Good (1953), for example, proposed the inverse Gaussian distribution. Ljubo (1965), Pickands (1975) and Hoskings and Wallis (1987) made generalization of Pareto distribution. The generalized beta of the first and second kind was introduced by McDonald (1984) to study the distribution of income.

Shaw and Buckley (2007) proposed a new generalization method called transmution mapping. According to them a ranking quadratic transmutation (QRT) map is

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, |\lambda| < 1 \tag{1}$$

where $G(x)$ is the cumulative distribution function (cdf) of the base distribution. It should be noted that, when $\lambda = 0$, the new distribution becomes the original distribution.

This method have been used by many researchers to obtain new distributions, see Aryal and Tsokos (2011), Aryal (2013), Elbatal and Aryal (2013) and Merovci (2013). Recently, Granzatto et al. (2017) introduced a new family of transmuted distributions, the cubic rank transmutation (CRT) map distribution and to demonstrate the usefulness

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of this method CRT Weibull and log-logistic distribution are used in their article. This new method enables to fit complex data sets with bimodal hazard rates. The cdf and the probability density function (pdf) of a CRT distribution are given

$$F(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1)[G(x)]^2 + (1-\lambda_2)[G(x)]^3 \tag{2}$$

$$f(x) = g(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1-\lambda_2)[G(x)]^2] \tag{3}$$

respectively. Here , $\lambda_1 \in [0,1]$ $\lambda_2 \in [-1,1]$ and $g(x)$ is the pdf of the base distribution. The proofs and the further details can be found in Granzatto et al. (2017).

In this paper, we are motivated to generate a new family of the distributions in order to get more flexible fitting. For this reason further examples of CRT distributions are introduced. The rest of the paper organizes as follows, in Section 2-4, we offer Frechet, Gumbel and Gombertz distributions which are commonly used as life-time distributions in survival analysis. The cubic rank transmutation method is applied to these distributions and some mathematical and statistical properties of these new distributions are derived. The maximum likelihood estimations of the parameters of interest are obtained. In Section 5, real data examples, which were previously studied with Frechet, Gumbel and Gombertz distribution are fitted into the cubic rank transmuted version of the base distributions. A conclusion is given at the end of this paper.

2. CUBIC RANK TRANSMUTED FRECHET DISTRIBUTION

Generalized extreme value (GEV) distribution covers the well known probability distributions developed within extreme value theory and it combines Gumbel, Frechet and Weibull families. It is proposed by Jenkinson (1955) in order to model extreme values based on Fisher-Tippet theorem. The class of GEV distributions is very flexible, since it can be represented by single shape parameter (ξ) which controls the tail behaviour with three different distribution families. If $\xi = 0$, then the distribution has thin tail behaviour and is called Gumbel type distribution. When $\xi > 0$, then the distribution has fat tail and is called Frechet type distribution which includes well known fat tailed distribution such as Pareto, Student-t and Cauchy. Finally, if $\xi < 0$, the distribution class converts to Weibull which has short tail behaviour and includes uniform and beta distribution.

A random variable X is said to have a Frechet distribution with parameters $\mu > 0$ and $\sigma > 0$ if its pdf is given by

$$g(x) = \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha} e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} \tag{4}$$

The cdf of Frechet distribution is

$$G(x) = e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} \tag{5}$$

Mahmoud and Mandouh (2013) introduced the QRT Frechet distribution and studied its statistical properties. Now using (2) the cdf of cubic rank transmuted Frechet (CRTF) distribution with parameters μ, σ, λ_1 and λ_2 takes the form

$$F(x) = \lambda_1 e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + (\lambda_2 - \lambda_1)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2 + (1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^3 \tag{6}$$

and the pdf of CRT Frechet distribution becomes

$$f(x) = \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha} e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\left(\lambda_1 + 2(\lambda_2 - \lambda_1)e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2\right) \tag{7}$$

Figure 1 shows the pdfs of the CRT Frechet distributions for different λ_1 and λ_2 values.

It can be seen from the Figure 1, for some special λ_1 and λ_2 values the distribution become the bimodal distribution.

The hazard rate function for the CRT Frechet distribution is given by

$$h(x) = \frac{\frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha} e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\left(\lambda_1 + 2(\lambda_2 - \lambda_1)e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2\right)}{1 - \lambda_1 e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + (\lambda_2 - \lambda_1)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2 + (1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^3} \tag{8}$$

Figure 2 illustrates some of the possible shapes of the hazard function of a CRT Frechet distribution for selected values of the parameters λ_1 and λ_2.

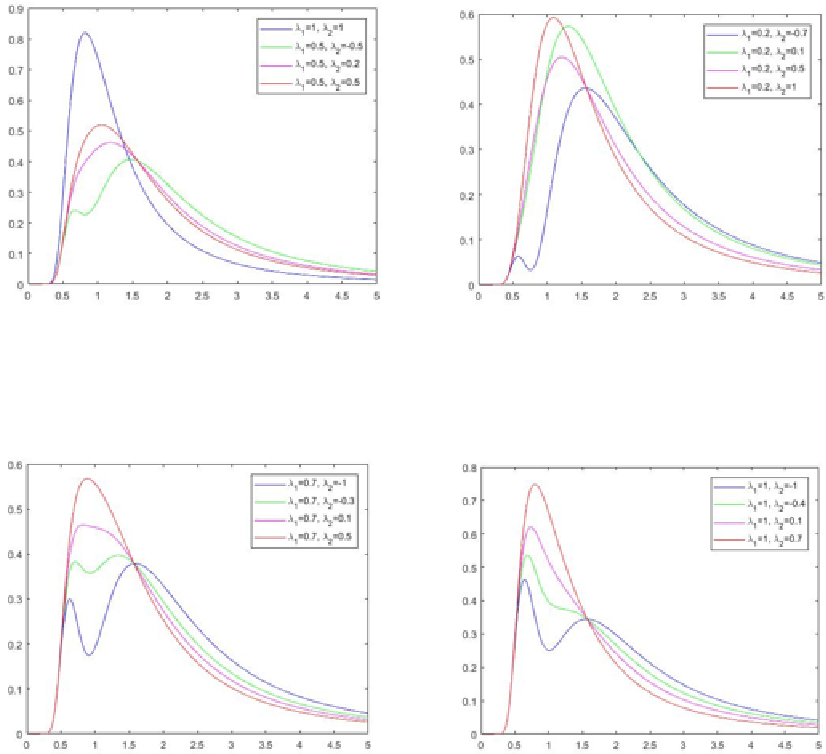

Fig. 1. The pdfs of CRT Frechet distribution, $\alpha = 2$, $\sigma = 1$.

The moments of the proposed distribution can be found easily by using the following integration

$$E(X^k) = \int x^k \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha} e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} \left(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2\right)dx \quad (9)$$

Taking $t = \left(\frac{x}{\sigma}\right)^{-\alpha}$, we can obtain the general formula of the moments of the distribution as

$$E(X^k) = \Gamma\left(1-\frac{k}{\alpha}\right)\left[\lambda_1 + (\lambda_2-\lambda_1)2^{1/\alpha} + (1-\lambda_2)3^{1/\alpha}\right]\sigma \quad (10)$$

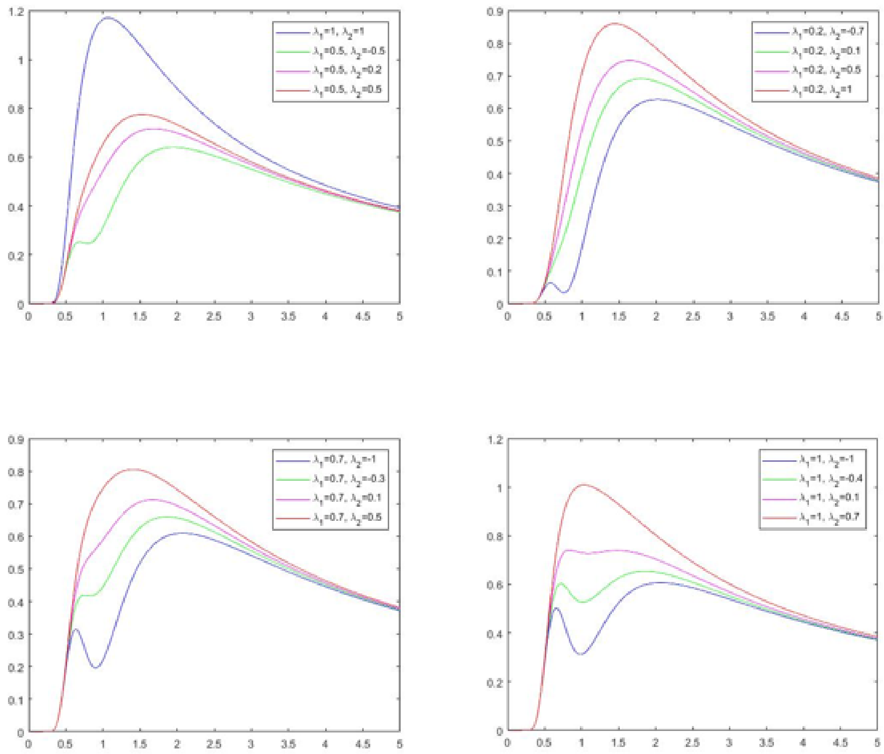

Fig. 2. The hazard rate functions of CRT Frechet distribution, $\alpha = 2$, $\sigma = 1$.

For generating random numbers from the distribution, one can use the method of inversion. After simple calculation this yields

$$x = \sigma\Big\{-ln\Big[\Big(q+\big(q^2+(r-p^2)^3\big)^{1/2}\Big)^{1/3}+\Big(q-\big(q^2+(r-p^2)^3\big)^{1/2}\Big)^{1/3}+p\Big]\Big\}^{-1/\alpha} \tag{11}$$

where $p = -\frac{\lambda_2-\lambda_1}{3(1-\lambda_2)}$, $q = p^3 + \frac{\lambda_1(\lambda_2-\lambda_1)+3u(1-\lambda_2)}{6(1-\lambda_2)^2}$, $r = \frac{\lambda_1}{3(1-\lambda_2)}$ and u is uniformly distributed random variable.

Suppose $X_1, X_2, ..., X_n$ are random samples from a CRT Frechet distribution defined in (7), then the likelihood function is given by

$$L = \Big(\frac{\alpha}{\sigma}\Big)^n e^{\left(-\sum_{i=1}^n \left(\frac{x_i}{\sigma}\right)^{-\alpha}\right)} \prod_{i=1}^n \Big(\frac{x_i}{\sigma}\Big)^{-1-\alpha} \prod_{i=1}^n \Big(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{x_i}{\sigma}\right)^{-\alpha}\right)} + 3(1-\lambda_2)\Big[e^{\left(-\left(\frac{x_i}{\sigma}\right)^{-\alpha}\right)}\Big]^2\Big) \tag{12}$$

and the log-likelihood function is

$$lnL = nln(\alpha) - nln(\sigma) - \sum_{i=1}^{n} (\frac{\sigma}{x_i})^{\alpha} - (1+\alpha)\sum_{i=1}^{n} ln\Big(\frac{x_i}{\sigma}\Big) + \sum_{i=1}^{n} ln\Big(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\Big[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\Big]^2\Big) \tag{13}$$

By differentiating the log-likelihood function with respect to the unknown parameters and equating them to zero, we obtain the following likelihood equations.

$$\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n} (\frac{\sigma}{x_i})^{\alpha} ln(\frac{\sigma}{x_i}) + \sum_{i=1}^{n} (\frac{\sigma}{x_i}) \\
&+ \sum_{i=1}^{n} \frac{2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}(\frac{\sigma}{x_i})^{\alpha} ln(\frac{\sigma}{x_i}) + 3(1-\lambda_2)\Big[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\Big]^2 (\frac{\sigma}{x_i})^{\alpha} ln(\frac{\sigma}{x_i})}{\Big(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\Big[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\Big]^2\Big)} \\
\frac{\partial \ln L}{\partial \sigma} &= -\frac{n}{\sigma} - \sum_{i=1}^{n} \frac{\sigma}{x_i}\Big(\frac{\sigma}{x_i}\Big)^{\alpha} + (\alpha+1)\frac{n}{\sigma} \\
&+ \sum_{i=1}^{n} \frac{2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\alpha\Big(\frac{\sigma}{x_i}\Big)^{\alpha-1}\frac{1}{x_i} + 6(1-\lambda_2)[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\Big]^2 \alpha\Big(\frac{\sigma}{x_i}\Big)^{\alpha-1}\frac{1}{x_i}}{\Big(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\Big[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\Big]^2\Big)} \\
\frac{\partial \ln L}{\partial \lambda_1} &= \sum_{i=1}^{n} \frac{1 - 2e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}}{\Big(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\Big[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\Big]^2\Big)} \\
\frac{\partial \ln L}{\partial \lambda_2} &= \sum_{i=1}^{n} \frac{2e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} - 3[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\Big]^2}{\Big(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\Big[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\Big]^2\Big)}
\end{aligned} \tag{14}$$

Solutions of these equations are called ML estimates. However, the equations must be solved with numerical methods such as Newton Raphson or iteratively reweighting algorithm.

3. CUBIC RANK TRANSMUTED GUMBEL DISTRIBUTION

A random variable X is said to have a Gumbel distribution with parameters if its pdf and cdf is given by

$$g(x) = \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma} + e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \tag{15}$$

and

$$G(x) = e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \tag{16}$$

respectively.

Now using (2) the cdf of cubic rank transmuted Gumbel (CRT Gumble) distribution with parameters is

$$F(x) = \lambda_1 e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + (\lambda_2 - \lambda_1)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^2 + (1-\lambda_2)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^3 \tag{17}$$

and the pdf of CRT Gumbel distribution takes the form

$$f(x) = \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma} + e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \left\{\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^2\right\} \tag{18}$$

Figure 3 and Figure 4 show the pdfs and the hazard rate functions of the CRT Gumbel distribution for representative λ values respectively.

The hazard rate function for the distribution is

$$h(x) = \frac{\frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma} + e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \left\{\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^2\right\}}{1 - \left\{\lambda_1 e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + (\lambda_2 - \lambda_1)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^2 + (1-\lambda_2)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^3\right\}} \tag{19}$$

The moments of CRT Gumbel distribution can be found as

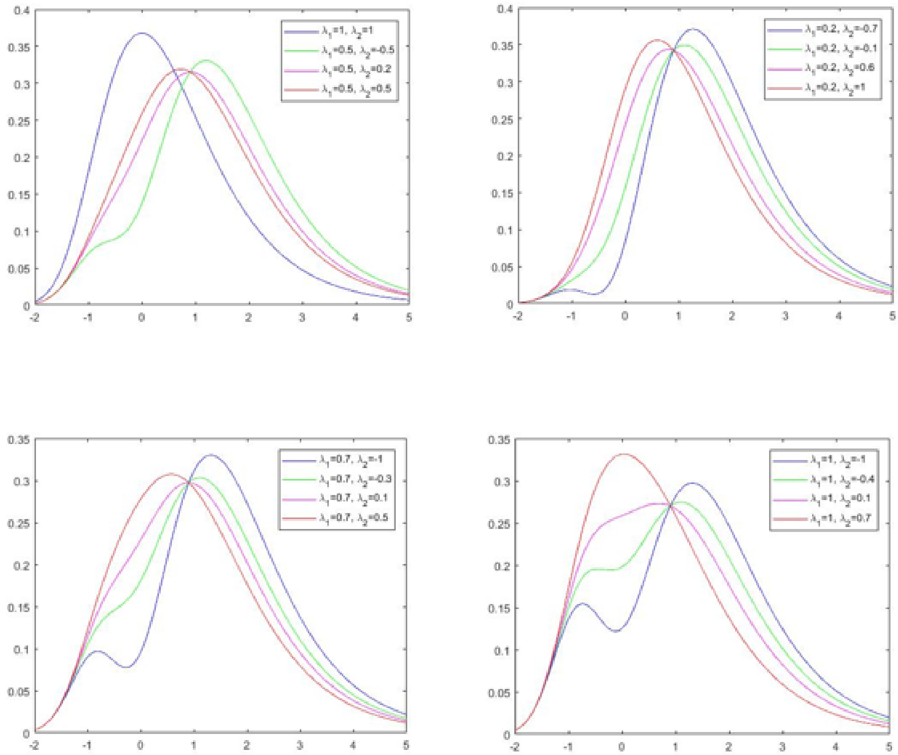

Fig. 3. The pdfs of CRT Gumbel distribution, $\mu = 0$, $\sigma = 1$.

$$E(X^k) = \int_0^\infty x^k \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma} + e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \Big\{ \lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\Big]^2 \Big\} dx \tag{20}$$

By taking $y = exp\left(-\frac{x-\mu}{\sigma}\right)$ the moments can be obtained like

$$E(X^k) = \sum_{i=0}^{n} (-1)^n \binom{n}{i} \sigma^i \mu^{n-i} \Big[\lambda_1 \frac{\partial i}{\partial \nu^i} \Gamma(\nu) + 2(\lambda_2 - \lambda_1) \frac{\partial i}{\partial \nu^i} (2^{-\nu}\Gamma(\nu)) + 3(1-\lambda_2) \frac{\partial i}{\partial \nu^i} (3^{-\nu}\Gamma(\nu)) \Big] |_{\nu=1} \tag{21}$$

For generation random numbers the following formula can be used.

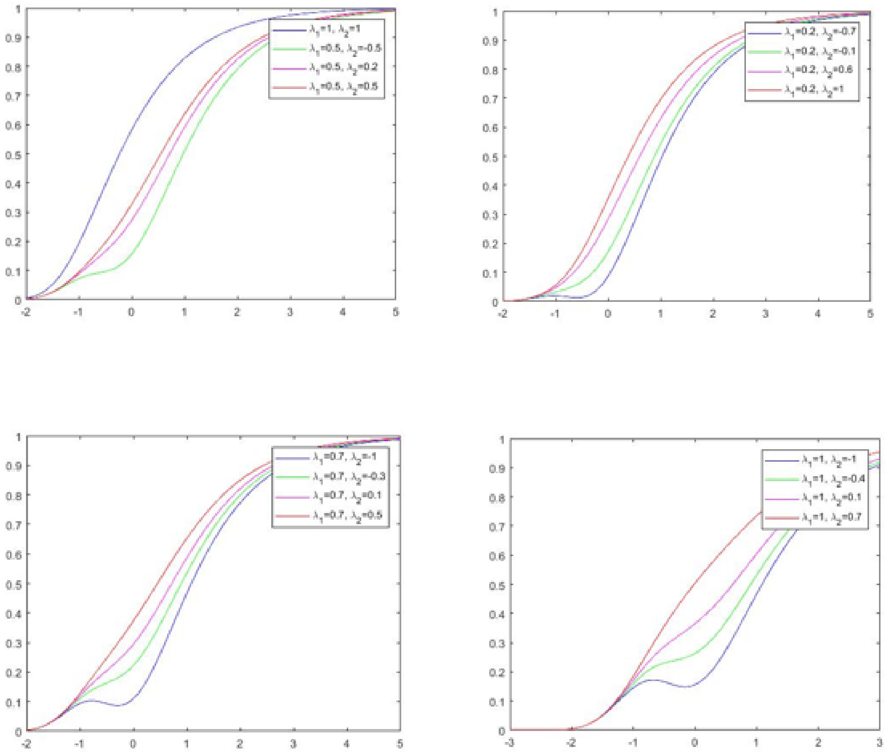

Fig. 4. The hazard rate functions of CRT Gumbel distribution, $\mu = 0$, $\sigma = 1$.

$$x = \mu - \sigma\Big(ln\Big\{-ln\Big[\Big(q + (q^2 + (r - p^2)^3)^{1/2}\Big)^{1/3} + \Big(q - (q^2 + (r - p^2)^3)^{1/2}\Big)^{1/3} + p\Big]\Big\}\Big) \tag{22}$$

where $p = -\frac{\lambda_2 - \lambda_1}{3(1-\lambda_2)}$, $q = p^3 + \frac{\lambda_1(\lambda_2 - \lambda_1) + 3u(1-\lambda_2)}{6(1-\lambda_2)^2}$, $r = \frac{\lambda_1}{3(1-\lambda_2)}$ and u is uniformly distributed random variable.

Now, in order to obtain the ML estimators of the parameters, the likelihood function

$$L = \sigma^{-n} e^{-\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma} + e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} \prod_{i=1}^{n}\Big\{\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2\Big\} \tag{23}$$

and the log-likelihood function

$$lnL = -nln(\sigma) - \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma} + e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right) + \sum_{i=1}^{n} ln\Big\{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2\Big\} \tag{24}$$

can be obtained respectively. And the likelihood equations are

$$\begin{aligned}
\frac{\partial \ln L}{\partial \mu} &= n\mu - \sum_{i=1}^{n} e^{-\left(\frac{x_i-\mu}{\sigma}\right)} \\
&+ \sum_{i=1}^{n} \frac{2(\lambda_2-\lambda_1)e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}e^{\left(-\frac{x-\mu}{\sigma}\right)} + 6(1-\lambda_2)\left(e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right)^2 e^{\left(-\frac{x-\mu}{\sigma}\right)}}{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2} \qquad (25) \\
\frac{\partial \ln L}{\partial \sigma} &= -n + \sum_{i=1}^{n}\frac{x_i-\mu}{\sigma} - \sum_{i=1}^{n} e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\frac{x_i-\mu}{\sigma} \\
&+ \sum_{i=1}^{n} \frac{2(\lambda_2-\lambda_1)e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}e^{\left(-\frac{x-\mu}{\sigma}\right)}\frac{x_i-\mu}{\sigma} + 6(1-\lambda_2)\left(e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right)^2 e^{\left(-\frac{x-\mu}{\sigma}\right)}\frac{x_i-\mu}{\sigma}}{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2} \\
\frac{\partial \ln L}{\partial \lambda_1} &= \sum_{i=1}^{n} \frac{1 - 2e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}}{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2} \\
\frac{\partial \ln L}{\partial \lambda_2} &= \sum_{i=1}^{n} \frac{2e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} - 3\Big[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\Big]^2}{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2}
\end{aligned}$$

By equating them to zero and solving the equations the ML estimators of the unknown parameters can be obtained.

4. CUBIC RANK TRANSMUTED GOMPERTZ DISTRIBUTION

The Gompertz distribution has been widely used in actuarial sciences especially in calculation of adult deaths. The pdf and the cdf of Gompetz distribution are given

$$g(x) = \alpha\beta e^{\alpha x} e^{\beta} exp\Big(-\beta e^{\alpha x}\Big) \tag{26}$$

$$G(x) = 1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big) \tag{27}$$

respectively.

Following the idea of (2), the CRT Gompertz distribution is obtained as follows,

$$\begin{aligned} F(x) = \lambda_1\Big(1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big) + (\lambda_2 - \lambda_1)\Big[1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big]^2 \\ +3(1-\lambda_2)\Big[1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big]^3 \end{aligned} \tag{28}$$

and the corresponding pdf is defined

$$\begin{aligned} f(x) = \alpha\beta e^{\alpha x} e^{\beta} exp\Big(-\beta e^{\alpha x}\Big)\Big[\lambda_1 + 2(\lambda_2 - \lambda_1)\Big(1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big) \\ +3(1-\lambda_2)\Big[1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big]^2\Big] \end{aligned} \tag{29}$$

Figure 5 shows different pdfs of CRT Gompertz distribution for plausible alternatives of λ_1 and λ_2.

The hazard rate function for the CRT Gombertz distribution is given by

$$h(x) = \frac{\alpha\beta e^{\alpha x} e^{\beta} e^{(-\beta e^{\alpha x})}\Big[\lambda_1 + 2(\lambda_2 - \lambda_1)\Big(1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big) + 3(1-\lambda_2)\Big[1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big]^2\Big]}{1 - \lambda_1\Big(1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big) + (\lambda_2 - \lambda_1)\Big[1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big]^2 + 3(1-\lambda_2)\Big[1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big]^3} \tag{30}$$

The possible shapes of hazard rate functions can be seen from Figure 6.

The moment generating function of CRT Gombertz distribution is

$$\begin{aligned} M_X(t) = \int_0^\infty e^{tx}\alpha\beta e^{\alpha x} e^{\beta} exp\Big(-\beta e^{\alpha x}\Big)\Big[\lambda_1 + 2(\lambda_2 - \lambda_1)\Big(1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big) \\ +3(1-\lambda_2)\Big[1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big]^2\Big]dx \end{aligned} \tag{31}$$

By taking $y = \beta\left(e^{\alpha x} - 1\right)$ and $z = \frac{y+\beta}{\beta}$ then, the mgf of CRT Gombertz distribution becomes

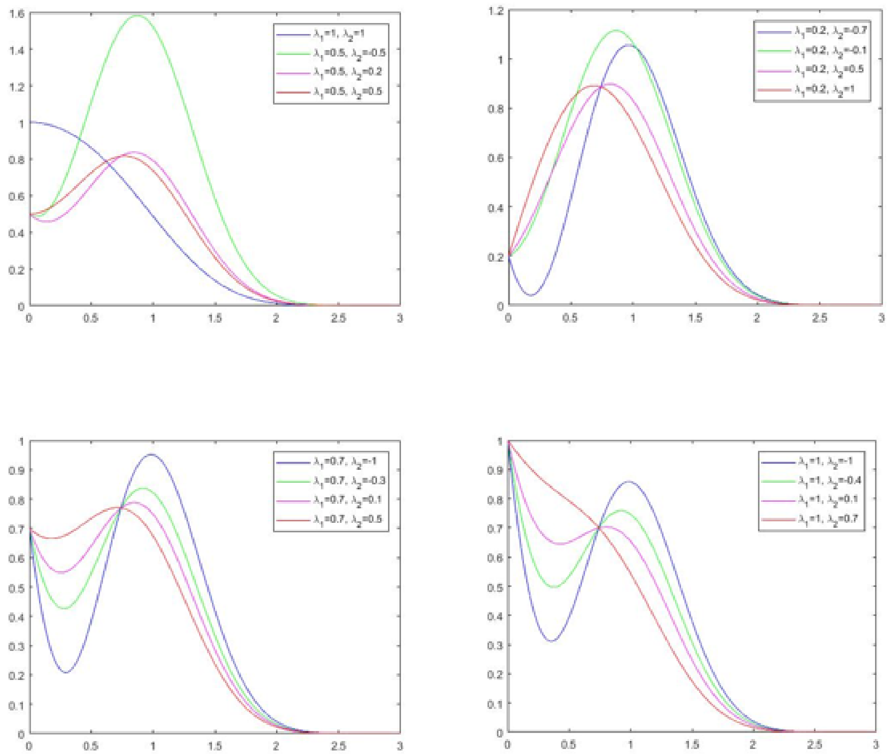

Fig. 5. The pdfs of CRT Gompertz distribution, $\alpha = 1$, $\beta = 1$.

$$\begin{aligned} M_X(t) = \beta^{-t/\alpha}\lambda_1 e^{\beta}\Big[\Gamma\Big(\frac{t}{\beta}+1\Big) - \sum_{i=0}^{\infty}\frac{(-1)^i\beta^{t/\alpha+1+i}}{i!(t/\alpha+1+i}\Big] + (2\beta)^{-t/\alpha}(\lambda_2-\lambda_1)e^{2\beta} \\ \Big[\Gamma\Big(\frac{t}{\beta}+1\Big) - \sum_{i=0}^{\infty}\frac{(-1)^i(2\beta)^{t/\alpha+1+i}}{i!(t/\alpha+1+i}\Big] + (3\beta)^{-t/\alpha}(1-\lambda_2)e^{3\beta} \\ \Big[\Gamma\Big(\frac{t}{\beta}+1\Big) - \sum_{i=0}^{\infty}\frac{(-1)^i(3\beta)^{t/\alpha+1+i}}{i!(t/\alpha+1+i}\Big] \end{aligned} \tag{32}$$

For generation random numbers from the distribution, one can use the method of inversion. After simple calculation this yields

$$x = \frac{1}{\alpha} ln\Big\{\frac{-ln\Big[1+\Big(q+(q^2+(r-p^2)^3)^{1/2}\Big)^{1/3}+\Big(q-(q^2+(r-p^2)^3)^{1/2}\Big)^{1/3}+p\Big]}{\beta}+1\Big\} \tag{33}$$

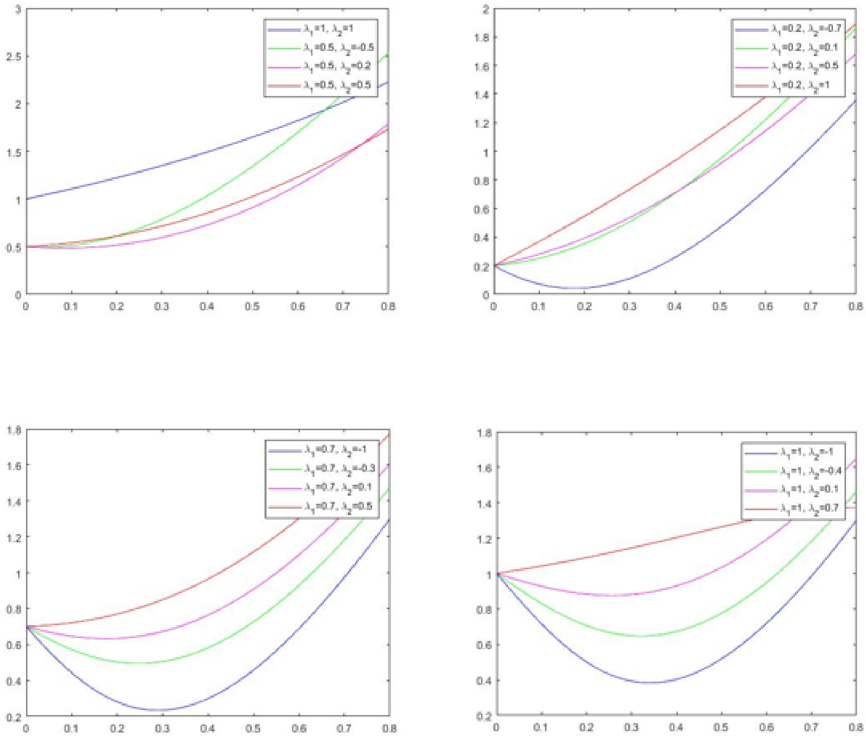

Fig. 6. The hazard rate functions of CRT Gompertz distribution, $\alpha = 1$, $\beta = 1$.

where $p = -\frac{\lambda_2-\lambda_1}{3(1-\lambda_2)}$, $q = p^3 + \frac{\lambda_1(\lambda_2-\lambda_1)+3u(1-\lambda_2)}{6(1-\lambda_2)^2}$, $r = \frac{\lambda_1}{3(1-\lambda_2)}$ and u is uniformly distributed random variable.

Suppose $X_1, X_2, ..., X_n$ are random samples from a CRT Gompertz distribution defined in (27), then the likelihood function is given by

$$L = \alpha^n \beta^n e^{\alpha \sum_{i=1}^n x_i} e^{\beta n} e^{-\beta \sum_{i=1}^n e^{\alpha x_i}} \prod_{i=1}^n \Big[\lambda_1 + 2(\lambda_2-\lambda_1)\Big(1-exp\Big(-\beta\big(e^{\alpha x_i}-1\big)\Big)\Big) + 3(1-\lambda_2)\Big[1-exp\Big(-\beta\big(e^{\alpha x_i}-1\big)\Big)\Big]^2\Big] \quad (34)$$

and the log-likelihood function is

$$lnL = nln(\alpha) + nln(\beta) + \alpha\sum_{i=1}^n x_i + \beta n - \beta\sum_{i=1}^n e^{\alpha x_i} + 3(1-\lambda_2)\Big[1-exp\Big(-\beta\big(e^{\alpha x_i}-1\big)\Big)\Big]^2\Big] + \sum_{i=1}^n ln\Big[\lambda_1 + 2(\lambda_2-\lambda_1)\Big(1-exp\Big(-\beta\big(e^{\alpha x_i}-1\big)\Big)\Big) \quad (35)$$

By differentiating the log-likelihood function with respect to the unknown parameters and equating them to zero, we obtain the following likelihood equations.

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i e^{\alpha x_i} \\
&+ \frac{2(\lambda_2-\lambda_1)\beta x_i e^{\alpha x_i}\left(e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right) + 6(1-\lambda_2)\beta x_i e^{\alpha x_i}\left(e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right)\left(1-e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right)}{\left[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) + 3(1-\lambda_2)\left[1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right]^2\right]} \quad (36) \\
\frac{\partial \ln L}{\partial \beta} &= \frac{n}{\beta} + n - \sum_{i=1}^{n} e^{\alpha x_i} \\
&+ \frac{2(\lambda_2-\lambda_1) e^{\alpha x_i}\left(e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right) + 6(1-\lambda_2)\beta x_i e^{\alpha x_i}\left(e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right)\left(1-e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right)}{\left[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) + 3(1-\lambda_2)\left[1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right]^2\right]} \\
\frac{\partial \ln L}{\partial \lambda_1} &= \sum_{i=1}^{n} \frac{1-2\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right)}{\left[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) + 3(1-\lambda_2)\left[1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right]^2\right]} \\
\frac{\partial \ln L}{\partial \lambda_2} &= \sum_{i=1}^{n} \frac{2\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) - 3\left[1-exp\left(-\beta\left(e^{\alpha x}-1\right)\right)\right]^2}{\left[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) + 3(1-\lambda_2)\left[1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right]^2\right]}
\end{aligned}
$$

Solutions of these equations are called ML estimates. However, the equations must be solved with numerical methods such as Newton Raphson or iteratively reweighting algorithm.

5. APPLICATION

In this section, we applied each of the new distribution families to the real data for demonstrating the behaviour of the distributions. Determining an appropriate model from a population problem has been widely discussed by several authors. However, Nelson (1982) suggested that *after fitting the general model to the data, then one seeks to find which special case is suitable*. For this reason, we used the data sets taken from the literature which has been fitted by the original distributions.

5.1. Wind speed data

The data used for the present study were obtained from a yearly published book at Permerhatian Cuaca Harian Pusat Pengajian Sosial, Pembangunan and Persekitaran (PPSPP), Fakulti Sains Sosial, Kemanusiaan (FSSK), Universiti Kebangsaan Malaysia (UKM) during the year 2004 to 2006, Zaharim et al. (2009). This data was collected from Malaysia and wind speeds were observed every 10 seconds and averaged over 5 minutes period. The 5-minutes averaged data were further averaged over one hour. At the end of each hour, the hourly mean wind speed was calculated and stored sequentially in a permanent memory.

Elbatal et al. (2014) fit the data into the Frechet (F), Transmuted Frechet (TF) and Transmuted Exponentiated Frechet (TEF) distributions. We also propose CRT Frechet (CRTF) distribution and Table (1) shows the comparison results based on the estimated parameter values.

Table I. Estimated Parameters of Frechet, Transmuted Frechet, Transmuted Exponentiated Frechet and Cubic Rank Transmuted Frechet Distributions.

Distribution	*ParameterEstimates*					*Log − Likelihood*
	β	θ	α	λ_1	λ_2	
F	1.922	1.024	–	–	–	−19.43
TF	2.014	2.581	–	0.746	–	−11.44
TEF	1.913	3.481	9.88	0.380	–	−6.65
$CRTF$	1.887	3.041	–	0.591	0.115	−5.97

5.2. Water Quality Data

This water quality data were obtained from the Department of Chemistry, Gauhati University. Various water quality parameters were estimated for the project entitled Assessment of Toxic Element in Water of Semi-Under Area of Assam and Investigation of the Disease Related Contaminants during 2009 for three administration sub-divisions of Nogaon district of Assam, India. Deka et al (2017) proposed Transmuted Exponentiated Gumbel (TEG) for this data set and compared the results with Gumbel (G) and Transmuted Gumbel (TG) distributions. We fit the data into the CRT Gumbel (CRTG) distribution and the results are given in Table (2).

Table II. Estimated Parameters of Gumbel, Transmuted Gumbel, Transmuted Exponentiated Gumbel and Cubic Rank Transmuted Gumbel Distributions.

Distribution	*ParameterEstimates*					*Log − Likelihood*
	β	θ	α	λ_1	λ_2	
G	1.063	0.769	–	–	–	−40.80
TG	1.001	0.855	–	0.711	–	−40.14
TEG	0.259	0.185	0.181	0.530	–	−39.70
$CRTG$	0.715	0.625	–	0.856	0.211	−38.95

5.3. Failure Data

Abdul-Maniem and Seham (2015) used the data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. For this data set Gombertz (Go) and Transmuted Gombertz (TGo) distributions were proposed in this paper. We, now propose CRT Gombertz (CRTGo) distribution for the failure rate data and the results are shown in Table (3).

Table III. Estimated Parameters of Gombertz, Transmuted Gombertz, Transmuted Exponentiated Gombertz and Cubic Rank Transmuted Gombertz Distributions.

$Distribution$			$ParameterEstimates$			$Log-Likelihood$
	β	θ	α	λ_1	λ_2	
Go	0.121	3.385	–	–	–	−87.20
TGo	0.187	1.148	–	0.819	–	−64.25
$TEGo$	0.895	3.128	0.521	0.985	–	−63.25
$CRTGo$	0.135	1.568	–	0.851	0.119	−61.13

6. CONCLUSION

In this paper, we introduce some examples of cubic rank transmutation mapping. Frechet, Gumbel and Gombertz distributions are used as the base distribution. The properties of these distributions such as the density functions, the medians, hazard rate functions and the quantile functions are examined. Also, maximum likelihood estimations are obtained. In the application section of the paper, real data set examples are used to illustrate better fit than the distributions which have been used before. For all real data sets introduced distributions provides better fittings than the corresponding distributions.

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