Study of Incomplete Elliptic Integrals Pertaining to ${}_p\psi_q$ Function

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Abstract

Elliptic-type integral plays a major role in the study of different problems of physics and technology including fracture mechanics. Many papers have been written for various families of elliptic-type integrals. Due to their applications here, we are presenting an organized study of certain generalized family of incomplete elliptic integral. The obtained results are basic in nature have various generalizations. While using the fractional integral operator of Riemann-Liouville type, we found several obvious hyper geometric representations. Which are further used to originate many definite integrals relating to their modules and amplitude of elliptic type generalized incomplete integrals.

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1. INTRODUCTION AND DEFINITIONS

The incomplete elliptic integrals having a keen interest of mathematician form a long time. In this way Legendre's normal form of incomplete elliptic integrals of the first and second kind are given [1-6]:

$$\mathrm{F}\,(\phi,\mathrm{k}) \;=\; \int\limits_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1-\mathrm{k}^2\sin^2\theta}} \;=\; \int\limits_0^{\sin\phi} \frac{\mathrm{dt}}{\sqrt{\left(1-\mathrm{t}^2\right)\left(1-\mathrm{k}^2\mathrm{t}^2\right)}}, \left(\left|\mathrm{k}^2\right| < 1 ; 0 \le \phi \le \frac{\pi}{2}\right) \quad (1)$$

and

$$E\,(\phi,k) \;=\; \int_0^{\phi} \sqrt{1-k^2 sin^2\theta}\, d\theta \;=\; \int_0^{sin\,\phi} \frac{\sqrt{\left(1-k^2t^2\right)}}{\sqrt{\left(1-t^2\right)}}\, dt, \qquad \left(\left|k^2\right| < 1 ; 0 \le \phi \le \frac{\pi}{2}\right), \quad (2)$$

with |k| modulus and amplitude ϕ.
In this paper, we take necessary constraint $\left|k^2\right| < 1$ rather than $0 \le k < 1$. Here the amplitude ϕ may attend complex values. Specially, when $\phi = \frac{\pi}{2}$, the equations (1) and (2) provides the corresponding complete elliptic integrals. It is very useful in radiation physics, nuclear technology fracture mechanics etc. (see [7-17]).
We have generalized elliptic function of third kind [6]

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$$R\left(\phi,k,\xi;\alpha,\gamma\right)=\int_0^{\phi}\frac{1}{\left(1+\xi\sin^2\theta\right)^{\alpha}\left(1-k^2\sin^2\theta\right)^{1/2-\gamma}}\,d\theta, \tag{3}$$

$$\mathrm{R}\left(\phi,\mathrm{k},\xi;\alpha,\gamma\right)=\int_0^{\sin\phi}\frac{1}{\left(1+\xi\,\mathrm{v}^2\right)^{\alpha}\sqrt{\left(1-\mathrm{v}^2\right)}\left(1-\mathrm{k}^2\mathrm{v}^2\right)^{1/2-\gamma}}\,\mathrm{dv},\quad \begin{pmatrix}\left|\mathrm{k}^2\right|<1;0\le\phi\le\frac{\pi}{2};\\ \gamma\in\mathrm{C},\alpha\ge 0\end{pmatrix} \tag{4}$$

where ξ is elliptic characteristic and $\xi > -1$.
Also we have elliptic function

$$\mathrm{I}\left(\phi,\mathrm{k},\xi;\gamma\right)=\int_0^{\phi}\frac{1}{\left(1+\xi\sin^2\theta\right)\left(1-\mathrm{k}^2\sin^2\theta\right)^{1/2-\gamma}}\,\mathrm{d}\theta, \tag{5}$$

$$\mathrm{I}\left(\phi,\mathrm{k},\xi;\gamma\right)=\int_0^{\sin\phi}\frac{1}{\left(1+\xi\,\mathrm{v}^2\right)\sqrt{\left(1-\mathrm{v}^2\right)}\left(1-\mathrm{k}^2\mathrm{v}^2\right)^{1/2-\gamma}}\,\mathrm{dv},\quad \left(\left|\mathrm{k}^2\right|<1;0\le\phi\le\frac{\pi}{2};\gamma\ge 0\right) \tag{6}$$

It is seen that by assigning some particular values of α, γ and ϕ, the above defined results reduce into known elliptic integral (see [2,6-17]).
The multivariable hyper geometric function defined by Srivastava & Daoust ([16-17])

$$\begin{aligned}&\mathrm{F}^{\mathrm{p}:\mathrm{q}_1;\ldots;\mathrm{q}_\mathrm{r}}_{l:\mathrm{m}_1;\ldots;\mathrm{m}_\mathrm{r}}\left[\begin{matrix}\left(\mathrm{a_j};\alpha'_\mathrm{j},\ldots\alpha_\mathrm{j}^{(\mathrm{r})}\right)_{1,\mathrm{p}}:\left(\mathrm{c}'_\mathrm{j},\gamma'_\mathrm{j}\right)_{1,\mathrm{q}_1};\ldots;\left(\mathrm{c}_\mathrm{j}^{(\mathrm{r})},\gamma_\mathrm{j}^{(\mathrm{r})}\right)_{1,\mathrm{q}_\mathrm{r}};\\ \left(\mathrm{b_j};\beta'_\mathrm{j},\ldots\beta_\mathrm{j}^{(\mathrm{r})}\right)_{1,l}:\left(\mathrm{d}'_\mathrm{j},\delta'_\mathrm{j}\right)_{1,\mathrm{m}_1};\ldots;\left(\mathrm{d}_\mathrm{j}^{(\mathrm{r})},\delta_\mathrm{j}^{(\mathrm{r})}\right)_{1,\mathrm{m}_\mathrm{r}};\end{matrix}\ \mathrm{z}_1,\ldots,\mathrm{z}_\mathrm{r}\right]\\ &=\sum_{\mathrm{n}_1,\ldots,\mathrm{n}_\mathrm{r}=0}^{\infty}\frac{\prod_{\mathrm{j}=1}^{\mathrm{p}}\left(\mathrm{a_j}\right)_{\mathrm{n}_1\alpha'_\mathrm{j}+\ldots+\mathrm{n}_\mathrm{r}\alpha_\mathrm{j}^{(\mathrm{r})}}\prod_{\mathrm{j}=1}^{\mathrm{q}_1}\left(\mathrm{c}'_\mathrm{j}\right)_{\mathrm{n}_1\gamma'_\mathrm{j}}\cdots\prod_{\mathrm{j}=1}^{\mathrm{q}_\mathrm{r}}\left(\mathrm{c}_\mathrm{j}^{(\mathrm{r})}\right)_{\mathrm{n}_\mathrm{r}\gamma_\mathrm{j}^{(\mathrm{r})}}}{\prod_{\mathrm{j}=1}^{l}\left(\mathrm{b_j}\right)_{\mathrm{n}_1\beta'_\mathrm{j}+\ldots+\mathrm{n}_\mathrm{r}\beta_\mathrm{j}^{(\mathrm{r})}}\prod_{\mathrm{j}=1}^{\mathrm{m}_1}\left(\mathrm{d}'_\mathrm{j}\right)_{\mathrm{n}_1\delta'_\mathrm{j}}\cdots\prod_{\mathrm{j}=1}^{\mathrm{m}_\mathrm{r}}\left(\mathrm{d}_\mathrm{j}^{(\mathrm{r})}\right)_{\mathrm{n}_\mathrm{r}\gamma_\mathrm{j}^{(\mathrm{r})}}}\frac{\mathrm{z}_1^{\mathrm{n}_1}}{\mathrm{n}_1!}\cdots\frac{\mathrm{z}_\mathrm{r}^{\mathrm{n}_\mathrm{r}}}{\mathrm{n}_\mathrm{r}!},\end{aligned} \tag{7}$$

With variable and parametric constraints the above mentioned series is absolutely convergent.
We have by the definition of well-known Riemann-Liouville operator $D_z^{\mu}f(z)$ of fractional calculus (see [6,17]):

$$\begin{aligned}&\mathrm{D}_\mathrm{z}^{\lambda-\mu}\left\{\mathrm{z}^{\lambda-1}\prod_{\mathrm{j}=1}^{\mathrm{r}}\left\{\left(1-\mathrm{a_j}\,\mathrm{z}^{\mu_\mathrm{j}}\right)^{-\alpha_\mathrm{j}}\right\}\right\}\\ &=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\mathrm{z}^{\mu-1}\mathrm{F}^{1:1;\ldots;1}_{1:0;\ldots;0}\left[\begin{matrix}(\lambda;\mu_1,\ldots\mu_\mathrm{r}):(\alpha_1,1);\ldots;(\alpha_\mathrm{r},1);\\ (\mu;\mu_1,\ldots\mu_\mathrm{r}):---;\ldots;---;\end{matrix}\ \mathrm{a}_1\mathrm{z}^{\mu_1},\ldots,\mathrm{a}_\mathrm{r}\mathrm{z}^{\mu_\mathrm{r}}\right],\\ &\left[\mathrm{R}\left(\lambda\right)>0;\mu_\mathrm{j}>0\ \left(\mathrm{j}=1,\ldots,\mathrm{r}\right);\max\left\{\left|\mathrm{a}_1\mathrm{z}^{\mu_1}\right|,\ldots,\left|\mathrm{a}_\mathrm{r}\mathrm{z}^{\mu_\mathrm{r}}\right|\right\}<1\right],\end{aligned} \tag{8}$$

where the D_z^{υ} is the Riemann-Liouville fractional differintegral operator (see [3,18-20])

$$D_z^{\upsilon}\left\{f\left(z\right)\right\}=\begin{cases}\frac{1}{\Gamma(-\upsilon)}\int_0^z(z-\zeta)^{-\upsilon-1}f(\zeta)\,d\zeta, & [R(\upsilon)<0]\\ \frac{d^n}{dz^n}D_z^{\upsilon-n}\left\{f(z)\right\}, & [0\le R(\upsilon)<n:n\in N_0]\end{cases} \tag{9}$$

which shows the defining integral in (9) exists.

Equation (3) reduces the following result by using the definition (8) applying r = 3 with, $\lambda = \mu - 1 = 1$ and z = sin ϕ, we find that

$$R(\phi,k,\xi;\alpha,\gamma) = \sin\phi\, F^{1:1;1;1}_{1:0;0;0}\left[\begin{matrix}(1:2,2,2):(1/2-\gamma,1);(1/2,1);(\alpha,1);\\(2:2,2,2):\quad ;\quad ;\quad ;\end{matrix} k^2\sin^2\phi, \sin^2\phi, -\xi\sin^2\phi\right], \tag{10}$$

conditions are already defined in the (3) and (4).

In the same manner

$$I(\phi,k,\xi;\gamma) = \sin\phi\, F^{1:1;1;1}_{1:0;0;0}\left[\begin{matrix}(1;2,2,2):(1/2-\gamma,1);(1/2,1);(1,1);\\(2;2,2,2):\quad ;\quad ;\quad ;\end{matrix} k^2\sin^2\phi, \sin^2\phi, -\xi\sin^2\phi\right], \tag{11}$$

for details see [6].

By using the definition of Pochhammer symbol, we have

$$\frac{(1)_{2l+2m+2n}}{(2)_{2l+2m+2n}} = \frac{\Gamma(2l+2m+2n+1)}{\Gamma(2l+2m+2n+2)} = \frac{\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{2}\right)}\frac{\left(\frac{1}{2}\right)_{l+m+n}}{\left(\frac{3}{2}\right)_{l+m+n}} = \frac{\left(\frac{1}{2}\right)_{l+m+n}}{\left(\frac{3}{2}\right)_{l+m+n}}, \tag{12}$$

where we used the duplication formula defined bellow:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right), \tag{13}$$

with the help of above relation defined by equation (10), we can write the relation as

$$R(\phi,k,\xi;\alpha,\gamma) = \sin\phi\,.F_1\left[\frac{1}{2}:\frac{1}{2}-\gamma, \frac{1}{2}, \alpha;\frac{3}{2};k^2\sin^2\phi, \sin^2\phi, -\xi\sin^2\phi\right], \tag{14}$$

$$\left(\left|k^2\right| < 1; 0 \le \phi \le \frac{\pi}{2}; \gamma\in C, \alpha\ge 0\right),$$

and similarly $I(\phi,k,\xi;\gamma)$ can be defined in the similar manner, where F_1 denotes the particular case of the multivariable hypergeometric function given by Srivastava-Daoust for three variables defined in (7).

We know the definition of binomial expansion:

$$(1-z)^{-\lambda} = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^n, \qquad (|z|<1;\lambda\in C) \tag{15}$$

and

$$(1+z)^{-\lambda} = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(-z)^n, \qquad (|z|<1;\lambda\in C). \tag{16}$$

By the help of the binomial expansion, we can say that

$$\left(1-k^2 sin^2\theta\right)^{\gamma-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}-\gamma\right)_n}{n!} k^{2n} sin^{2n}\theta\,, \tag{17}$$

and

$$\left(1+\xi\, sin^2\theta\right)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \left(-\xi\, sin^2\theta\right)^n . \tag{18}$$

We have by the definition of Beta function B (α,β):

$$\int_0^{\pi/2} \sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta\, d\theta = \frac{1}{2} B(\alpha,\beta), \qquad \left(\min\{R(\alpha).R(\beta)\} > 0 : B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right) \tag{19}$$

with help of these formulas and relations, we can easily establish Theorem 1.

2. THEOREMS AND COROLLARIES.

THEOREM 1. If ${}_p\psi_q$ is a Wright function [21], whose series representation is given by

$${}_p\psi_q\left[{}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p);}_{(b_1,\beta_1),\ldots,(b_q,\beta_q);}x\right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j r)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j r)} \frac{x^r}{r\,!}, \tag{20}$$

where α_i and β_j $(i=1,\ldots,p; j=1,\ldots,q)$ are real and positive, and $1+\sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j > 0$.

Also consider that $\{\tau,\eta,\lambda,\mu\} \geq 0, (\tau+\eta > 0; \lambda+\mu > 0)$ and

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{n^{1/2(1+\rho)}}\right| < \infty, \qquad [\tau = 0; R(\rho) > -1] \tag{21}$$

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{n^{1+\sigma/2}}\right| < \infty, \qquad [\eta = 0; R(\sigma) > -2] \tag{22}$$

then

$$\int_0^1 k^{\rho}\left(\sqrt{(1-k^2)}\right)^{\sigma} {}_p\psi_q\left[{}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p);}_{(b_1,\beta_1),\ldots,(b_q,\beta_q);} zk^{\tau}\left(\sqrt{(1-k^2)}\right)^{\eta}\right] R\left(\phi,\zeta k^{\lambda}\left(\sqrt{(1-k^2)}\right)^{\mu},\xi;\alpha,\gamma\right) dk$$
$$= \frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)} \frac{\sin\phi}{2} B\left(\frac{\rho+1}{2},\frac{\sigma+2}{2}\right) . F^{3:n;1;1;1}_{2:n;0;0;0}\left[{}^{\left(\frac{1}{2}:0,1,1,1\right),\left(\frac{\rho+1}{2}:\frac{\tau}{2},\lambda,0,0\right),\left(\frac{\sigma+2}{2}:\frac{\eta}{2},\mu,0,0\right):}_{\left(\frac{3}{2}:0,1,1,1\right),\left(\frac{\rho+\sigma+3}{2}:\frac{\eta+\tau}{2},\lambda+\mu,0,0\right):}\right.$$
$$\left.{}^{(a_1,\alpha_1),\ldots,(a_n,\alpha_n);\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(\alpha,1);}_{(b_1,\beta_1),\ldots,(b_n,\beta_n);-;-;} z,\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right], \tag{23}$$

where $R(\rho) > -1$ *and* $|\zeta| < 1$ [*or* $|\zeta| = 1$ *and* $R(\rho+2\lambda) > -1$].

PROOF. To establish the result defined in Theorem 1, we use the values of $R\left(\phi,\zeta k^{\lambda}\kappa^{\mu},\xi;\alpha,\gamma\right)$ and ${}_p\psi_q\left[\begin{smallmatrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p);\\(b_1,\beta_1),\dots,(b_q,\beta_q);\end{smallmatrix}x\right]$ from equations (3) and (18) respectively, we get the required result after simplification by using the formulas which are defined above.

COROLLARY 1. With the help of definition new elliptic function defined in equation (5), we can establish the following result

$$\int_0^1 \mathrm{k}^{\rho}\left(\sqrt{(1-\mathrm{k}^2)}\right)^{\sigma} {}_{\mathrm{p}}\psi_{\mathrm{q}}\left[\begin{smallmatrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p);\\(b_1,\beta_1),\dots,(b_q,\beta_q);\end{smallmatrix}\mathrm{z}\mathrm{k}^{\tau}\left(\sqrt{(1-\mathrm{k}^2)}\right)^{\eta}\right]\mathrm{I}\left(\phi,\zeta\,\mathrm{k}^{\lambda}\left(\sqrt{(1-\mathrm{k}^2)}\right)^{\mu},\xi;\gamma\right)d\mathrm{k}$$
$$=\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)}\frac{\sin\phi}{2}\mathrm{B}\left(\frac{\rho+1}{2},\frac{\sigma+2}{2}\right).\mathrm{F}^{3:n;1;1;1}_{2:n;0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:0,1,1,1\right),\left(\frac{\rho+1}{2}:\frac{\tau}{2},\lambda,0,0\right),\left(\frac{\sigma+2}{2}:\frac{\eta}{2},\mu,0,0\right):\\\left(\frac{3}{2}:0,1,1,1\right),\left(\frac{\rho+\sigma+3}{2}:\frac{\eta+\tau}{2},\lambda+\mu,0,0\right):\end{smallmatrix}\right.$$
$$\left.\begin{smallmatrix}(a_1,\alpha_1),\dots,(a_n,\alpha_n);\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(1,1);\\(b_1,\beta_1),\dots,(b_n,\beta_n);-;-;\end{smallmatrix}z,\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right],\tag{24}$$

with help of Theorem 1 we can determine the above Corollary 1, by putting $\alpha=1$.

COROLLARY 2. With the help of elliptic integral of third kind (see [6]), we can establish the following result

$$\int_0^1 \mathrm{k}^{\rho}\left(\sqrt{(1-\mathrm{k}^2)}\right)^{\sigma} {}_{\mathrm{p}}\psi_{\mathrm{q}}\left[\begin{smallmatrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p);\\(b_1,\beta_1),\dots,(b_q,\beta_q);\end{smallmatrix}\mathrm{z}\mathrm{k}^{\tau}\left(\sqrt{(1-\mathrm{k}^2)}\right)^{\eta}\right]\Pi\left(\phi,\zeta\,\mathrm{k}^{\lambda}\left(\sqrt{(1-\mathrm{k}^2)}\right)^{\mu},\xi\right)d\mathrm{k}$$
$$=\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)}\frac{\sin\phi}{2}\mathrm{B}\left(\frac{\rho+1}{2},\frac{\sigma+2}{2}\right).\mathrm{F}^{3:n;1;1;1}_{2:n;0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:0,1,1,1\right),\left(\frac{\rho+1}{2}:\frac{\tau}{2},\lambda,0,0\right),\left(\frac{\sigma+2}{2}:\frac{\eta}{2},\mu,0,0\right):\\\left(\frac{3}{2}:0,1,1,1\right),\left(\frac{\rho+\sigma+3}{2}:\frac{\eta+\tau}{2},\lambda+\mu,0,0\right):\end{smallmatrix}\right.$$
$$\left.\begin{smallmatrix}(a_1,\alpha_1),\dots,(a_n,\alpha_n);\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);(1,1);\\(b_1,\beta_1),\dots,(b_n,\beta_n);-;-;\end{smallmatrix}z,\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right],\tag{25}$$

with help of Theorem 1 we can determine the above Corollary 2, by putting $\alpha=1$ and $\gamma=0$.

THEOREM 2. The following families of integrals hold true

$$\int_0^{\pi/2}\sin^{2(a-1)}\phi\cos^{2b-1}\phi\,{}_p\psi_q\left[\begin{smallmatrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p);\\(b_1,\beta_1),\dots,(b_q,\beta_q);\end{smallmatrix}zk\right]R\left(\phi,k,\xi;\alpha,\gamma\right)d\phi=\frac{1}{2}\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)}B\left(a,b\right)$$
$$F^{2:1;1;1}_{2:0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):(a_1,\alpha_1),\dots,(a_n,\alpha_n);\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(\alpha,1);\\\left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):(b_1,\beta_1),\dots,(b_n,\beta_n);-;-;-;\end{smallmatrix}zk,k^2,1,-\xi\right],\tag{26}$$

$$\left[\left|k^2\right|<1:\min\left\{R\left(a\right),R\left(b\right),R(a_i),R(b_i)\right\}>0,i=1,2,\dots;\gamma\in C\right].$$

and

$$\int_0^w x^{2(a-1)} \left(w^2 - x^2\right)^{b-1} {}_p\psi_q \left[\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\ (b_1,\beta_1),\ldots,(b_q,\beta_q);\end{smallmatrix} zk\right] R\left(arc\sin\frac{x}{w}, k, \xi; \alpha, \gamma\right) dx$$
$$= \frac{1}{2} w^{2a+2b-3} \frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)} B(a,b)$$
$$F^{2:1;1;1}_{2:0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):(a_1,\alpha_1),\ldots,(a_n,\alpha_n);\ \left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);\ (\alpha,1);\\ \left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):(b_1,\beta_1),\ldots,(b_n,\beta_n);-;-;-;\end{smallmatrix} zk,\ k^2,\ 1,\ -\xi\right], \tag{27}$$

only if the second member of each of the integral formulas defined in equations (26) and (27) occurs.

PROOF. After replacing the $R(\phi, k, \xi; \alpha, \lambda)$ from equation (10) and the value of Wright function from equation (20) in to the integral of the affirmation equation (26) of Theorem 2, if we use the trigonometric integral (19) as it is, we can find the integral formula (26) as given above.

COROLLARY 1. With help of definition of new elliptical function defined in equation (5), we can establish the following result

$$\int_0^{\pi/2} \sin^{2(a-1)}\phi \cos^{2b-1}\phi\, {}_p\psi_q \left[\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\ (b_1,\beta_1),\ldots,(b_q,\beta_q);\end{smallmatrix} zk\right] I(\phi, k, \xi; \gamma)\, d\phi = \frac{1}{2}\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)} B(a,b)$$
$$F^{2:1;1;1}_{2:0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):(a_1,\alpha_1),\ldots,(a_n,\alpha_n);\ \left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);\ (1,1);\\ \left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):(b_1,\beta_1),\ldots,(b_n,\beta_n);-;-;-;\end{smallmatrix} zk,\ k^2,\ 1,\ -\xi\right], \tag{28}$$

or

$$\int_0^{\pi/2} \sin^{2(a-1)}(\phi)\cos^{2b-1}(\phi)\, I(\phi,k,\xi;\gamma)\, d\phi$$
$$= \frac{1}{2}B(a,b)\, F^{2;1;1;1}_{2;0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(1,1);\\ \left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):-;-;-;\end{smallmatrix} k^2, 1, -\xi\right]. \tag{29}$$

This Corollary can be found with help of Theorem 2 by putting $\alpha = 1$.

COROLLARY 2. With help of elliptical integral of third kind (see[6]), we can establish the following result

$$\int_0^{\pi/2} \sin^{2(a-1)}(\phi)\cos^{2b-1}(\phi)\, \Pi(\phi,k,\xi)\, d\phi$$
$$= \frac{1}{2}B(a,b)\, F^{2;1;1;1}_{2;0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);(1,1);\\ \left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):-;-;-;\end{smallmatrix} k^2, 1, -\xi\right]. \tag{30}$$

This Corollary can be find with the help of Theorem 2 by putting $\alpha = 1$ and $\gamma = 0$.

REMARK 2. On changing the following variables

$$\phi = arc\,sin\,x \qquad and \qquad d\phi = \frac{dx}{\sqrt{1-x^2}} \qquad with\,x \in (0,1) \tag{31}$$

equation (26) can be rewritten as

$$\begin{aligned} &\int_0^1 x^{2(a-1)} \left(1-x^2\right)^{b-1} R\left(arc\,sin\,x, k, \xi\,; \alpha, \gamma\right) dx \\ &= \frac{1}{2} B\left(a,b\right) F^{2:1;1;1}_{2:0;0;0} \left[\begin{matrix} \left(\frac{1}{2}:1,1,1\right), (a:1,1,1): \ \left(\frac{1}{2}-\gamma,1\right); \left(\frac{1}{2},1\right); (\alpha,1); \\ \left(\frac{3}{2}:1,1,1\right), (a+b:1,1,1): -; -; -; \end{matrix} k^2, 1, -\xi\right], \end{aligned} \tag{32}$$

$$\left[\left|k^2\right| < 1 : \min\left\{R\left(\alpha\right), R\left(\beta\right)\right\} > 0; \gamma \in C\right]$$

Here equation (32) can be equated with equation (27) stated by Theorem 2.

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