Some Fractional Calculus Results Pertaining To Mittag-Leffler Type Functions

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Abstract

In this paper, we study the generalized fractional operators pertaining to the generalized Mittag-Leffler function and multi-index Mittag-Leffler function. Some applications of the established results associated with generalized Wright function are also deduced as corollaries. The results are useful in solving the problems of science, engineering and technology where the Mittag-Leffler function occurs naturally.

Mathematics Subject Classification 2010: 26A33, 33C45, 33E12, 47G20
Key Words and Phrases: Generalized fractional calculus operators; Generalized Mittag-Leffler function; Multi-index Mittag-Leffler function; Generalized Wright function.

1. INTRODUCTION

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number, or even complex number, order of the differential operator $D = \frac{d}{dx}$ and the integration operator. The Fractional calculus has recently been applied in various areas of science, applied mathematics, engineering, bio engineering, and finance. However, many researchers remain ignorant of this field. Fractional differential equations have gained importance and popularity, mainly due to its demonstrated applications in science and engineering. In view of great importance of fractional differential equations many authors have paid attention for handling linear and non-linear fractional differential equations [1-6].

The Mittag-Leffler function was introduced by the Swedish mathematician Mittag-Leffler in 1903 [7, 8]. Mittag-Leffler function finds its applications in the solutions of fractional differential and integral equations, and they are associated with a widespread array of problem in various areas of mathematics and mathematical physics. In addition, from exponential manners, the deviations of

10.1515/jamsi-2017-0003

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physical phenomena could also be represented by physical laws via Mittag-Leffler functions. Therefore, the uses of Mittag-Leffler functions are constantly increasing, mainly in mathematics and physics. Further, in 1971, Prabhakar [9] proposed the more general Mittag-Leffler function. Many more extensions or unifications for these functions are found in large number of papers [10-15]. In this paper, we study the generalized fractional operators on the generalized Mittag-Leffler function and multi-index Mittag-Leffler function. Some applications of the established results associated with generalized Wright function are also deduced as corollaries.

2. FRACTIONAL CALCULUS OPERATORS AND GENERALIZED FRACTIONAL CALCULUS OPERATORS

The left and right-sided Riemann-Liouville fractional calculus operators are defined by Samko et al. [16, sec. 5.1] for $\alpha \epsilon C$ $(\mathbb{R}(\alpha) > 0)$

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \tag{1}$$

$$(I_{0-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt, \tag{2}$$

$$\begin{aligned}(D_{0+}^{\alpha} f)(x) &= \left(\frac{d}{dx}\right)^{[\mathbb{R}(\alpha)]+1} \left[I_{0+}^{1-\alpha+[\mathbb{R}(\alpha)]} f\right](x) \\ &= \left(\frac{d}{dx}\right)^{[\mathbb{R}(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\mathbb{R}(\alpha)])} \int_0^x \frac{f(t)}{(x-t)^{\alpha-[\mathbb{R}(\alpha)]}} dt,\end{aligned} \tag{3}$$

$$\begin{aligned}(D_{0-}^{\alpha} f)(x) &= \left(-\frac{d}{dx}\right)^{[\mathbb{R}(\alpha)]+1} \left[I_{0-}^{1-\alpha+[\mathbb{R}(\alpha)]} f\right](x), \\ &= \left(-\frac{d}{dx}\right)^{[\mathbb{R}(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\mathbb{R}(\alpha)])} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha-[\mathbb{R}(\alpha)]}} dt,\end{aligned} \tag{4}$$

where $[\mathbb{R}(\alpha)]$ is the integral of $\mathbb{R}(\alpha)$.

An exciting and valuable generalization of the Riemann-Liouville and Erdlyi-Kober fractional integral operators has been introduced by Saigo [17] in terms of Gauss hyper geometric function as given below. Let $\alpha, \beta, \gamma \epsilon C$ and $x \epsilon R_+$, then the generalized fractional integration and fractional differentiation operators associated with Gauss hyper geometric function are defined as follows:

$$\left(I_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma;\ \alpha; 1-\frac{t}{x}\right) f(t)dt;\ \mathbb{R}(\alpha) > 0, \quad (5)$$

$$\left(I_{0-}^{\alpha,\beta,\gamma} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma;\ \alpha; 1-\frac{x}{t}\right) f(t)dt; \mathbb{R}(\alpha) > 0 \quad (6)$$

$$\left(D_{0+}^{\alpha,\beta,\gamma} f\right) = \left(I_{0+}^{-\alpha,-\beta,\alpha+\gamma} f\right)(x)$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k} f\right)(x), \mathbb{R}(\alpha) > 0; k = [\mathbb{R}(\alpha)] + 1, \quad (7)$$

$$\left(D_{0-}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{0-}^{-\alpha,-\beta,\alpha+\gamma} f\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^k \left(I_{0-}^{-\alpha+k,-\beta-k,\alpha+\gamma} f\right)(x), \mathbb{R}(\alpha) > 0; k = [\mathbb{R}(\alpha)] + 1. \quad (8)$$

Operators $(5) - (8)$ reduce to that in $(1) - (4)$ as follows:

$$\left(I_{0+}^{\alpha,-\alpha,\gamma} f\right)(x) = (I_{0+}^{\alpha} f)(x), \quad (9)$$

$$\left(I_{0-}^{\alpha,-\alpha,\gamma} f\right)(x) = (I_{0-}^{\alpha} f)(x), \quad (10)$$

$$\left(D_{0+}^{\alpha,-\alpha,\gamma} f\right)(x) = (D_{0+}^{\alpha} f)(x), \quad (11)$$

$$\left(D_{0-}^{\alpha,-\alpha,\gamma} f\right)(x) = (D_{0-}^{\alpha} f)(x). \quad (12)$$

LEMMA 1. Let $\alpha, \beta, \gamma \epsilon C;\ \mathbb{R}(\alpha) > 0$ and $\rho \epsilon C$

(a) If $\mathbb{R}(\rho) > \max[0, \mathbb{R}(\beta-\gamma)]$, then

$$\left(I_{0+}^{\alpha,\beta,\gamma} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho-\beta+\gamma)}{\Gamma(\rho-\beta)\Gamma(\rho+\alpha+\gamma)} x^{\rho-\beta-1}, \quad (13)$$

(b) If $\mathbb{R}(\rho) > \max[\mathbb{R}(-\beta), \mathbb{R}(-\gamma)]$, then

$$\left(I_{0-}^{\alpha,\beta,\gamma} t^{-\rho}\right)(x) = \frac{\Gamma(\rho+\beta)\Gamma(\rho+\gamma)}{\Gamma(\rho)\Gamma(\rho+\alpha+\beta+\gamma)} x^{-\rho-\beta}. \quad (14)$$

3. LEFT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF GENERALIZED MITTAG-LEFFLER FUNCTION AND MULTI-INDEX MITTAG-LEFFLER FUNCTION

In this section we consider the left-sided generalized fractional integration formula of the generalized Mittag-Leffler function and multi-index Mittag-Leffler function.

THEOREM 3.1. If $\alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\sigma \epsilon C; p,q>0 \; and \; q \le \mathbb{R}(\alpha)+p,$ and $min(\mathbb{R}(\alpha),\mathbb{R}(\beta),\mathbb{R}(\gamma),\mathbb{R}(\delta),\mathbb{R}(\mu),\mathbb{R}(\nu),\mathbb{R}(\rho),\mathbb{R}(\sigma))>0$ and $a \epsilon R$. If the Wright function condition [14] is satisfied and $I_{0+}^{\alpha,\beta,\gamma}$ be the left-sided operator of generalized fractional integration associated with Gauss hypergeometric function, then there holds the following formula

$$\Delta = \left[I_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{\lambda}\right)\right](x) = \frac{\Gamma(\nu)}{\Gamma(\mu)}\frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{\rho-\beta-1}{}_4\psi_5$$

$$\times\left[\begin{matrix}(\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,q),(\mu,\rho)\\(\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta,\alpha),(\delta,p),(\nu,\sigma)\end{matrix}\middle| ax^{\lambda}\right] \quad (15)$$

PROOF. By using the definition of more generalized Mittag-Leffler function was defined by Shukla and Prajapati [10] and fractional integral formula (5), we hold

$$\Delta = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\ \alpha;1-\frac{t}{x}\right)$$
$$\times (t^{\rho-1})E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{\lambda}\right)dt.$$

By the using of Gauss hypergeometric series [18], series form of generalized Mittag-Leffler function was defined by Shukla and Prajapati [10], interchanging the order of integration and summations and evaluating the inner integral by the use of the known formula of Beta integral. At last by the virtue of above lemma, we have

$$\Delta = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (t^{\rho-1})(x-t)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\ \alpha;1-\frac{t}{x}\right)$$
$$\times\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{(\nu)_{\sigma n\sigma}(\delta)_{pn}\Gamma(\alpha n+\beta)}\frac{\left(at^{\lambda}\right)^n}{(n)!}dt$$

$$\Delta = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} x^{\rho-\beta-1} \sum_{n=0}^{\infty} \frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)\Gamma(\rho-\beta+\gamma+\lambda n)\Gamma(\rho+\lambda n)}{\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\beta+\alpha n)\Gamma(\rho-\beta+\lambda n)\Gamma(\rho+\alpha+\gamma+\lambda n)} \times \frac{(ax^{\lambda})^n}{(n)!}$$

or

$$\Delta = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} x^{\rho-\beta-1} {}_4\psi_5 \left[\begin{matrix} (\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,q),(\mu,\rho) \\ (\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta,\alpha),(\delta,p),(\nu,\sigma) \end{matrix} \middle| ax^{\lambda} \right],$$

this completes the proof of theorem.

If we set $\mu = \nu, \rho = \sigma$ *and* $p = 1$ then Eq. (15) reduces to the following corollary.

COROLLARY 3.1.1. The following result holds:

$$\Delta = \frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{\rho-\beta-1} {}_3\psi_4 \left[\begin{matrix} (\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,q) \\ (\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta,\alpha),(\delta,1) \end{matrix} \middle| ax^{\lambda} \right],$$

where $\alpha,\beta,\gamma,\delta,\rho \epsilon C, \mathbb{R}(\alpha) > 0, \mathbb{R}(\rho-\beta+\gamma) > 0, \lambda > 0$ and $a \epsilon R$.

If we set $p = \delta = 1$ in above result we obtain the following corollary 3.1.2

COROLLARY 3.1.2. The following result holds:

$$\Delta = \frac{1}{\Gamma(\gamma)} x^{\rho-\beta-1} {}_3\psi_3 \left[\begin{matrix} (\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,q) \\ (\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta,\alpha) \end{matrix} \middle| ax^{\lambda} \right],$$

where $\alpha,\beta,\gamma,\rho \epsilon C, \mathbb{R}(\alpha) > 0, \mathbb{R}(\rho-\beta+\gamma) > 0, \lambda > 0$ and $a \epsilon R$.

when we take $q = 1$, the above result reduced to the following corollary 3.1.3

COROLLARY 3.1.3. The following result holds:

$$\Delta = \frac{1}{\Gamma(\gamma)} x^{\rho-\beta-1} {}_3\psi_3 \left[\begin{matrix} (\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,1) \\ (\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta,\alpha) \end{matrix} \middle| ax^{\lambda} \right],$$

where $\alpha,\beta,\gamma,\rho \epsilon C, \mathbb{R}(\alpha) > 0, \mathbb{R}(\rho-\beta+\gamma) > 0, \lambda > 0$ and $a \epsilon R$.

THEOREM 3.2. If $\alpha,\beta,\gamma \epsilon C, \mathbb{R}(\alpha_j) > 0, \mathbb{R}(\beta_j) > 0$ $(j = 1 \ldots m), \mathbb{R}(k) > 0$, $\mathbb{R}\left(\sum_{j=1}^{m} \alpha_j\right) > \max[0, k-1]$, $\lambda > 0$ and $a \epsilon R$. If $I_{0+}^{\alpha,\beta,\gamma}$ be the left-sided operator of generalized fractional integration associated with Gauss hypergeometric function, then there holds the following formula

$$\left[I_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} E_{\gamma,k;\,\beta_1,\beta_2,\ldots,\beta_m}^{\alpha_1,\alpha_2,\ldots,\alpha_m} \left(at^{\lambda}\right) \right](x) = \frac{1}{\Gamma(\gamma)} x^{\rho-\beta-1} {}_3\psi_{2+m}$$

$$\times \left[\begin{matrix} (\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,k) \\ (\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta_j,\alpha_j) \end{matrix} \middle| ax^{\lambda} \right] \quad (16)$$

PROOF. Denote L.H.S. of the theorem 3.2 by Θ then

$$\Theta = \left[I_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} E_{\gamma,k;\,\beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m} \left(at^{\lambda}\right) \right](x)$$

Using the definition of multi-index Mittag-Leffler function was defined by Saxena and Nishimoto and fractional integral formula (5), we hold

$$\Theta = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\gamma;\ \alpha; 1-\frac{t}{x}\right)$$
$$\times \left(t^{\rho-1}\right) E_{\gamma,k;\,\beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m} \left(at^{\lambda}\right) dt.$$

By the using of Gauss hypergeometric series [18], series form of multi-index Mittag-Leffler function was defined by Saxena and Nishimoto [12], interchanging the order of integration and summations and evaluating the inner integral by the use of the known formula of Beta integral. At last by the virtue of above lemma, we have

$$\Theta = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x \left(t^{\rho-1}\right)(x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\gamma;\ \alpha; 1-\frac{t}{x}\right)$$
$$\times \sum_{n=0}^{\infty} \frac{(\gamma)_{kn}}{\prod_{j=1}^{m} \Gamma\left(n\alpha_j+\beta_j\right)} \frac{\left(at^{\lambda}\right)}{(n)!} dt$$

$$\Theta = \frac{x^{\rho-\beta-1}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+kn)\Gamma(\rho+\lambda n)\Gamma(\rho-\beta+\gamma+\lambda n)}{\Gamma(\rho-\beta+\lambda n)\Gamma(\rho+\alpha+\gamma+\lambda n)\prod_{j=1}^{m}\Gamma\left(n\alpha_j+\beta_j\right)} \frac{\left(ax^{\lambda}\right)}{(n)!},$$

or

$$\Theta = \frac{x^{\rho-\beta-1}}{\Gamma(\gamma)} {}_3\psi_{2+m} \left[\begin{matrix} (\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,k) \\ (\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta_j,\alpha_j) \end{matrix} \middle| ax^{\lambda} \right]$$

this completes the proof of theorem.

If we set k = q in Eq. (16) then we obtain the following corollary.

COROLLARY 3.2.1. The following result holds

$$\Theta = \frac{x^{\rho-\beta-1}}{\Gamma(\gamma)}\, {}_3\psi_{2+m}\left[\begin{matrix}(\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,q)\\(\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta_j,\alpha_j)\end{matrix}\middle| ax^{\lambda}\right]$$

$\alpha,\beta,\gamma \in C, \mathbb{R}(\alpha_j) > 0, \mathbb{R}(\beta_j) > 0\ (j = 1 \ldots m), \mathbb{R}(\gamma) > 0$, and $a \in R\ \mathbb{R}\left(\sum_{j=1}^{m}\alpha_j\right) > \max[0, q-1], \lambda > 0$.

when $q = m = 1$, then above result reduced to the following corollary.

COROLLARY 3.2.2. The following result holds

$$\Theta = \frac{x^{\rho-\beta-1}}{\Gamma(\gamma)}\, {}_3\psi_3\left[\begin{matrix}(\rho-\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,1)\\(\rho-\beta,\lambda),(\alpha+\rho+\gamma,\lambda),(\beta,\alpha)\end{matrix}\middle| ax^{\lambda}\right]$$

$\alpha,\beta,\gamma \in C, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\gamma) > 0, \lambda > 0$ and $a \in R$.

4. RIGHT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF GENERALIZED MITTAG-LEFFLER FUNCTION AND MULTI-INDEX MITTAG-LEFFLER FUNCTION

In this section we discussed the right-sided generalized fractional integration formula of the generalized Mittag-Leffler function and multi-index Mittag-Leffler function.

THEOREM 4.1. If $\alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\sigma \in C; p,q > 0\ and\ q \leq \mathbb{R}(\alpha)+p,$ and $min(\mathbb{R}(\alpha),\mathbb{R}(\beta),\mathbb{R}(\gamma),\mathbb{R}(\delta),\mathbb{R}(\mu),\mathbb{R}(\nu),\mathbb{R}(\rho),\mathbb{R}(\sigma)) > 0$ and $a \in R$. If the Wright function condition [14] is satisfied and $I_{0-}^{\alpha,\beta,\gamma}$ be the right-sided operator of generalized fractional integration associated with Gauss hypergeometric function, then there holds the following formula

$$\left[I_{0-}^{\alpha,\beta,\gamma} t^{-\alpha-\rho} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{-\lambda}\right)\right](x) = \frac{\Gamma(\nu)}{\Gamma(\mu)}\frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{-\alpha-\beta-\rho}\, {}_4\psi_5$$

$$\times\left[\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\gamma,q),(\mu,\rho)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\gamma+\rho,\lambda),(\beta,\alpha),(\delta,p),(\nu,\sigma)\end{matrix}\middle| ax^{\lambda}\right] \quad (17)$$

PROOF. Denote L.H.S. of the theorem 4.1 by Λ, then

$$\Lambda = \left[I_{0-}^{\alpha,\beta,\gamma} t^{-\alpha-\rho} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{-\lambda}\right)\right](x)$$

By using the definition of generalized Mittag-Leffler function was defined by Shukla and Prajapati [10] and fractional integral formula (6) and proceeding in the same way to the proof of theorem 3.1, we get

$$\Lambda = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma;\ \alpha; 1-\frac{x}{t}\right)$$
$$\times (t^{-\alpha-\rho}) E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}\left(at^{-\lambda}\right) dt.$$

or

$$\Lambda = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} x^{-\alpha-\beta-\rho}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)\Gamma(\alpha+\beta+\rho+\lambda n)\Gamma(\alpha+\rho+\gamma+\lambda n)}{\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\beta+\alpha n)\Gamma(\alpha+\rho+\lambda n)\Gamma(2\alpha+\beta+\gamma+\rho+\lambda n)} \frac{\left(ax^{-\lambda}\right)^n}{(n)!}$$

or

$$\Lambda = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} x^{-\alpha-\beta-\rho} {}_4\psi_5\left[\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\gamma,q),(\mu,\rho)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\gamma+\rho,\lambda),(\beta,\alpha),(\delta,p),(\nu,\sigma)\end{matrix}\middle| ax^{-\lambda}\right],$$

this completes the proof of theorem.

If we set $\mu = \nu, \rho = \sigma$ *and* $q = 1$ then Eq. (17) reduces to the following corollary.

COROLLARY 4.1.1. The following result holds:

$$\Lambda = \frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{-\alpha-\beta-\rho} {}_3\psi_4\left[\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\gamma,1)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\gamma+\rho,\lambda),(\beta,\alpha),(\delta,p)\end{matrix}\middle| ax^{-\lambda}\right]$$

where $\alpha,\beta,\gamma,\rho,\delta \epsilon C, \mathbb{R}(\alpha) > 0,, \lambda > 0$ and $a \epsilon R$.

If we set $p = \delta = 1$ in above result we obtain the following corollary 4.1.2

COROLLARY 4.1.2. The following result holds:

$$\Lambda = \frac{1}{\Gamma(\gamma)} x^{-\alpha-\beta-\rho} {}_3\psi_3\left[\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\gamma,1)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\gamma+\rho,\lambda),(\beta,\alpha)\end{matrix}\middle| ax^{-\lambda}\right]$$

where $\alpha,\beta,\gamma,\rho \epsilon C, \mathbb{R}(\alpha) > 0,, \lambda > 0$ and $a \epsilon R$.

THEOREM 4.2. If $\alpha,\beta,\gamma \epsilon C, \mathbb{R}(\alpha_j) > 0, \mathbb{R}(\beta_j) > 0$ $(j = 1 \dots m), \mathbb{R}(k) > 0$, $\mathbb{R}\left(\sum_{j=1}^m \alpha_j\right) > \max[0, k-1], \lambda > 0$ and $a \epsilon R$. If the Wright function condition [14] is satisfied and $I^{\alpha,\beta,\gamma}_{0-}$ be the right-sided operator of generalized fractional integration associated with Gauss hypergeometric function, then there holds the following formula

$$\left[I^{\alpha,\beta,\gamma}_{0-} t^{-\alpha-\rho} E^{\alpha_1,\alpha_2,\dots,\alpha_m}_{\gamma,k;\ \beta_1,\beta_2,\dots,\beta_m}\left(at^{-\lambda}\right)\right](x) = \frac{1}{\Gamma(\gamma)} x^{-\alpha-\beta-\rho} {}_3\psi_{2+m}$$

$$\times\left[\left.\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\gamma,k)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\rho+\gamma,\lambda),(\beta_j,\alpha_j)\end{matrix}\right|ax^{-\lambda}\right] \qquad (18)$$

PROOF. Denote L.H.S. of the theorem 4.2 by Ξ then

$$\Xi=\left[I_{0-}^{\alpha,\beta,\gamma}(t^{-\alpha-\rho})E_{\gamma,k;\,\beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{\lambda}\right)\right](x)$$

Using the definition of multi-index Mittag-Leffler function was defined by Saxena and Nishimoto [12] and fractional integral formula (6) and proceeding in the same way to the proof of theorem 3.2, we get

$$\Xi=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\;\alpha;1-\frac{x}{t}\right)$$
$$\times(t^{-\alpha-\rho})E_{\gamma,k;\,\beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{-\lambda}\right)dt.$$

or

$$\Xi=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-2\alpha-\beta-\rho}{}_2F_1\left(\alpha+\beta,-\gamma;\;\alpha;1-\frac{x}{t}\right)$$
$$\times\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}}{\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\frac{\left(at^{-\lambda}\right)^n}{(n)!}dt$$

or

$$\Xi=\frac{x^{-\alpha-\beta-\rho}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\beta+\rho+\lambda n)\Gamma(\alpha+\rho+\gamma+\lambda n)\Gamma(\gamma+kn)}{\Gamma(\alpha+\rho+\lambda n)\Gamma(2\alpha+\beta+\rho+\gamma+\lambda n)\prod_{j=1}^{m}\Gamma(n\alpha_j+\beta_j)}\frac{\left(ax^{-\lambda}\right)}{(n)!},$$

or

$$\Xi=\frac{x^{-\alpha-\beta-\rho}}{\Gamma(\gamma)}{}_3\psi_{2+m}\left[\left.\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\gamma,k)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\rho+\gamma,\lambda),(\beta_j,\alpha_j)\end{matrix}\right|ax^{-\lambda}\right]$$

this completes the proof of theorem.

If we set $\mathrm{k}=\mathrm{q}$ in Eq. (18) then we obtain the following corollary.

COROLLARY 4.2.1. The following result holds

$$\Xi = \frac{x^{-\alpha-\beta-\rho}}{\Gamma(\gamma)}\, {}_3\psi_{2+m}\left[\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\gamma,q)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\rho+\gamma,\lambda),(\beta_j,\alpha_j)\end{matrix}\middle|\, ax^{-\lambda}\right]$$

where $\alpha,\beta,\gamma,\rho \in C, \mathbb{R}(\alpha_j) > 0, \mathbb{R}(\beta_j) > 0\ (j = 1 \dots m), \mathbb{R}(\gamma) > 0,$ and $a \in R$ $\mathbb{R}\left(\sum_{j=1}^{m}\alpha_j\right) > \max[0, q-1], \lambda > 0.$

when $q = m = 1$, then above result reduced to the following corollary.

COROLLARY 4.2.2. The following result holds

$$\Xi = \frac{x^{-\alpha-\beta-\rho}}{\Gamma(\gamma)}\, {}_3\psi_3\left[\begin{matrix}(\alpha+\beta+\rho,\lambda),(\alpha+\rho+\gamma,\lambda),(\gamma,1)\\(\alpha+\rho,\lambda),(2\alpha+\beta+\rho+\gamma,\lambda),(\beta,\alpha)\end{matrix}\middle|\, ax^{-\lambda}\right]$$

$\alpha,\beta,\gamma \in C, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\gamma) > 0, \lambda > 0$ and $a \in R.$

5. LEFT-SIDED GENERALIZED FRACTIONAL DIFFERENTIATION OF GENERALIZED MITTAG-LEFFLER FUNCTION AND MULTI-INDEX MITTAG-LEFFLER FUNCTION

In this section we consider the left-sided generalized fractional differentiation formula of generalized Mittag-Leffler function and multi-index Mittag-Leffler function.

THEOREM 5.1. If $\alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\sigma \in C; p,q > 0$ *and* $q \le \mathbb{R}(\alpha) + p,$ and $min(\mathbb{R}(\alpha),\mathbb{R}(\beta),\mathbb{R}(\gamma),\mathbb{R}(\delta),\mathbb{R}(\mu),\mathbb{R}(\nu),\mathbb{R}(\rho),\mathbb{R}(\sigma)) > 0$ and $a \in R$. If the Wright function condition [14] is satisfied and $D_{0+}^{\alpha,\beta,\gamma}$ be the left-sided operator of generalized fractional differentiation associated with Gauss hypergeometric function, then there holds the following formula

$$\left[D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{\lambda}\right)\right](x) = \frac{\Gamma(\nu)}{\Gamma(\mu)}\frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{\rho+\beta-1}\, {}_4\psi_5$$

$$\times\left[\begin{matrix}(\rho+\alpha+\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,q),(\mu,\rho)\\(\rho+\beta,\lambda),(\rho+\gamma,\lambda),(\beta,\alpha),(\delta,p),(\nu,\sigma)\end{matrix}\middle|\, ax^{\lambda}\right] \quad (19)$$

PROOF. Denote L.H.S. of the theorem 5.1 by Φ then

$$\Phi = \left[D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{\lambda}\right)\right](x)$$

Using the definition of generalized Mittag-Leffler function was defined by Shukla and Prajapati and fractional differentiation formula (7), we get

$$\Phi = \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k} t^{\rho-1} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{\lambda}\right)\right)(x)$$

or

$$\Phi = \left(\frac{d}{dx}\right)^k \frac{x^{\alpha+\beta}}{\Gamma(-\alpha+k)} \int_0^x (x-t)^{-\alpha+k-1} {}_2F_1\left(-\alpha-\beta, -\gamma-\alpha+k;\ -\alpha+k; 1-\frac{t}{x}\right)$$
$$\times t^{\rho-1} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{\lambda}\right) dt$$

or

$$\Phi = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} x^{\rho+\beta-1}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\alpha+\beta+\gamma+\lambda n)\Gamma(\rho+\lambda n)\Gamma(\mu+\rho n)\Gamma(\gamma+qn)}{\Gamma(\rho+\beta+\lambda n)\Gamma(\rho+\gamma+\lambda n)\Gamma(\beta+\alpha n)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)} \frac{\left(ax^{\lambda}\right)^n}{(n)!}$$

$$\Phi = \frac{\Gamma(\nu)}{\Gamma(\mu)} \frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{\rho+\beta-1} {}_4\psi_5 \left[\begin{matrix} (\rho+\alpha+\beta+\gamma,\lambda), (\rho,\lambda), (\gamma,q), (\mu,\rho) \\ (\rho+\beta,\lambda), (\rho+\gamma,\lambda), (\beta,\alpha), (\delta,p), (\nu,\sigma) \end{matrix} \middle| ax^{\lambda}\right]$$

this completes the proof of theorem.

If we set $\mu = \nu, \rho = \sigma$ and $q = 1$ in Eq. (19), then we obtain the following corollary:

COROLLARY 5.1.1. The following result holds

$$\Phi = \frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{\rho+\beta-1} {}_3\psi_4 \left[\begin{matrix} (\rho+\alpha+\beta+\gamma,\lambda), (\rho,\lambda), (\gamma,1) \\ (\rho+\beta,\lambda), (\rho+\gamma,\lambda), (\beta,\alpha), (\delta,p) \end{matrix} \middle| ax^{\lambda}\right]$$

where $\alpha, \beta, \gamma, \rho, \delta \epsilon C, \mathbb{R}(\alpha) > 0,, \lambda > 0$ and $a \epsilon R$.

If we set $p = \delta = 1$ in above result we obtain the following corollary 5.1.2

COROLLARY 5.1.2. The following result holds:

$$\Phi = \frac{1}{\Gamma(\gamma)} x^{\rho+\beta-1} {}_3\psi_3 \left[\begin{matrix} (\rho+\alpha+\beta+\gamma,\lambda), (\rho,\lambda), (\gamma,1) \\ (\rho+\beta,\lambda), (\rho+\gamma,\lambda), (\beta,\alpha) \end{matrix} \middle| ax^{\lambda}\right]$$

where $\alpha, \beta, \gamma, \rho \epsilon C, \mathbb{R}(\alpha) > 0,, \lambda > 0$ and $a \epsilon R$.

THEOREM 5.2. If $\alpha, \beta, \gamma \epsilon C, \mathbb{R}(\alpha_j) > 0, \mathbb{R}(\beta_j) > 0$ $(j = 1 \ldots m), \mathbb{R}(k) > 0$, $\mathbb{R}\left(\sum_{j=1}^{m} \alpha_j\right) > \max[0, k-1], \lambda > 0$ and $a \epsilon R$. If the Wright function condition

[14] is satisfied and $D_{0+}^{\alpha,\beta,\gamma}$ be the left-sided operator of generalized fractional differentiation associated with Gauss hypergeometric function, then there holds the following formula

$$\left[D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} E_{\gamma,k;\,\beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{\lambda}\right)\right](x) = \frac{x^{\rho+\beta-1}}{\Gamma(\gamma)}\,{}_3\psi_{2+m}$$

$$\times\left[\begin{matrix}(\rho+\alpha+\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,q),(\mu,\rho)\\(\rho+\beta,\lambda),(\rho+\gamma,\lambda),(\beta,\alpha),(\delta,p),(\nu,\sigma)\end{matrix}\middle|\, ax^{\lambda}\right] \tag{20}$$

PROOF. Denote L.H.S. of the theorem 5.2 by Ω then

$$\Omega = \left[D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} E_{\gamma,k;\,\beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{\lambda}\right)\right](x)$$

Using the definition of multi-index Mittag-Leffler function was defined by Saxena and Nishimoto [12] and fractional differentiation formula (7), we get

$$\Omega = \left(\frac{d}{dx}\right)^{k}\left(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k} t^{\rho-1} E_{\gamma,k;\,\beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{\lambda}\right)\right)(x)$$

or

$$\Omega = \left(\frac{d}{dx}\right)^{k}\frac{x^{\alpha+\beta}}{\Gamma(-\alpha+k)}\int_0^x (x-t)^{-\alpha+k-1}\,{}_2F_1\left(-\alpha-\beta,-\gamma-\alpha+k;\,-\alpha+k;1-\frac{t}{x}\right)$$

$$\times\, t^{\rho-1} E_{\gamma,k;\,\beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{\lambda}\right)dt$$

or

$$\Omega = \frac{x^{\rho+\beta-1}}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\alpha+\beta+\gamma+\lambda n)\Gamma(\rho+\lambda n)\Gamma(\gamma+kn)}{\Gamma(\rho+\beta+\lambda n)\Gamma(\rho+\gamma+\lambda n)\prod_{j=1}^{m}\Gamma\left(n\alpha_j+\beta_j\right)}\frac{\left(ax^{\lambda}\right)^{n}}{(n)!}$$

$$\Omega = \frac{x^{\rho+\beta-1}}{\Gamma(\gamma)}\,{}_3\psi_{2+m}\left[\begin{matrix}(\rho+\alpha+\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,k)\\(\rho+\beta,\lambda),(\rho+\gamma,\lambda),\left(\beta_j,\alpha_j\right)\end{matrix}\middle|\, ax^{\lambda}\right]$$

this completes the proof of theorem.

If we set $k=q$ in Eq. (20), we obtain the following corollary 5.2.1

COROLLARY 5.2.1. The following result holds

$$\Omega = \frac{x^{\rho+\beta-1}}{\Gamma(\gamma)} {}_3\psi_{2+m}\left[\begin{matrix}(\rho+\alpha+\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,q)\\ (\rho+\beta,\lambda),(\rho+\gamma,\lambda),(\beta_j,\alpha_j)\end{matrix}\middle| ax^{\lambda}\right]$$

where $\alpha,\beta,\gamma,\rho \epsilon C, \mathbb{R}(\alpha_j) > 0, \mathbb{R}(\beta_j) > 0\ (j = 1 \dots m), \mathbb{R}(\gamma) > 0,$ and $a \epsilon R$ $\mathbb{R}\left(\sum_{j=1}^{m} \alpha_j\right) > \max[0, q-1]\,, \lambda > 0.$

when $q = m = 1$, then above result reduced to the following corollary.

COROLLARY 5.2.2. The following result holds

$$\Omega = \frac{x^{\rho+\beta-1}}{\Gamma(\gamma)} {}_3\psi_3\left[\begin{matrix}(\rho+\alpha+\beta+\gamma,\lambda),(\rho,\lambda),(\gamma,1)\\ (\rho+\beta,\lambda),(\rho+\gamma,\lambda),(\beta_j,\alpha_j)\end{matrix}\middle| ax^{\lambda}\right]$$

$\alpha,\beta,\gamma \epsilon C, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\gamma) > 0, \lambda > 0$ and $a \epsilon R$.

6. RIGHT-SIDED GENERALIZED FRACTIONAL DIFFERENTIATION OF GENERALIZED MITTAG-LEFFLER FUNCTION AND MULTI-INDEX MITTAG-LEFFLER FUNCTION

In this section we consider the right-sided generalized fractional differentiation formula of generalized Mittag-Leffler function and multi-index Mittag-Leffler function.

THEOREM 6.1. If $\alpha,\beta,\gamma,\delta,\mu,\nu,\rho,\sigma \epsilon C; p,q > 0\ and\ q \leq \mathbb{R}(\alpha)+p,$ and $min(\mathbb{R}(\alpha),\mathbb{R}(\beta),\mathbb{R}(\gamma),\mathbb{R}(\delta),\mathbb{R}(\mu),\mathbb{R}(\nu),\mathbb{R}(\rho),\mathbb{R}(\sigma)) > 0$ and $a \epsilon R$. If the Wright function condition [14] is satisfied and $D_{0-}^{\alpha,\beta,\gamma}$ be the right-sided operator of generalized fractional differentiation associated with Gauss hypergeometric function, then there holds the following formula

$$\left[D_{0-}^{\alpha,\beta,\gamma} t^{\alpha-\rho} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{-\lambda}\right)\right](x) = \frac{\Gamma(\nu)}{\Gamma(\mu)}\frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{\alpha+\beta+\rho} {}_4\psi_5$$

$$\times\left[\begin{matrix}(\rho-\alpha-\beta,\lambda),(\rho+\gamma,\lambda),(\gamma,q),(\mu,\rho)\\ (\rho-\alpha,\lambda),(\rho-\alpha-\beta+\gamma,\lambda),(\beta,\alpha),(\delta,p),(\nu,\sigma)\end{matrix}\middle| ax^{-\lambda}\right] \quad (21)$$

PROOF. Denote L.H.S. of the theorem 6.1 by χ then

$$\chi = \left[D_{0-}^{\alpha,\beta,\gamma} t^{\alpha-\rho} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{-\lambda}\right)\right](x)$$

Using the definition of generalized Mittag-Leffler function was defined by Shukla and Prajapati [10] and fractional differentiation formula (8), we get

$$\chi = \left(-\frac{d}{dx}\right)^k \left(I_{0-}^{-\alpha+k,-\beta-k,\alpha+\gamma} t^{\alpha-\rho} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{-\lambda}\right)\right)(x)$$

or

$$\chi = \left(-\frac{d}{dx}\right)^k \frac{1}{\Gamma(-\alpha+k)} \int_x^\infty (t-x)^{-\alpha+k-1} t^{\alpha+\beta} {}_2F_1\left(-\alpha-\beta, -\alpha-\gamma;\ -\alpha+k; 1-\frac{x}{t}\right)$$
$$\times\, t^{\alpha-\rho} E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(at^{-\lambda}\right) dt$$

or

$$\chi = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} x^{\alpha+\beta-\rho}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\rho-\alpha-\beta+\lambda n)\Gamma(\rho+\gamma+\lambda n)\Gamma(\mu+\rho n)\Gamma(\gamma+qn)}{\Gamma(\rho-\alpha+\lambda n)\Gamma(\rho-\alpha-\beta+\gamma+\lambda n)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\beta+\alpha n)} \frac{\left(ax^{-\lambda}\right)^n}{(n)!}$$

or

$$\chi = \frac{\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)} x^{\alpha+\beta-\rho}\ {}_4\psi_5\left[\begin{matrix}(\rho-\alpha-\beta,\lambda),(\rho+\gamma,\lambda),(\gamma,q),(\mu,\rho)\\(\rho-\alpha,\lambda),(\rho-\alpha-\beta+\gamma,\lambda),(\beta,\alpha),(\delta,p),(\nu,\sigma)\end{matrix}\middle| ax^{-\lambda}\right],$$

this completes the proof of theorem.

If we set $\mu=\nu, \rho=\sigma$ and $q=1$ in Eq. (21), then result reduced in following corollary:

COROLLARY 6.1.1. The following result holds

$$\chi = \frac{\Gamma(\delta)}{\Gamma(\gamma)} x^{\alpha+\beta-\rho}\ {}_3\psi_4\left[\begin{matrix}(\rho-\alpha-\beta,\lambda),(\rho+\gamma,\lambda),(\gamma,1)\\(\rho-\alpha,\lambda),(\rho-\alpha-\beta+\gamma,\lambda),(\beta,\alpha),(\delta,p)\end{matrix}\middle| ax^{-\lambda}\right]$$

where $\alpha,\beta,\gamma,\rho,\delta \in C, \mathbb{R}(\alpha)>0,,\lambda>0$ and $a \in R$.

If we set $p=\delta=1$ in above result we obtain the following corollary 6.1.2

COROLLARY 6.1.2. The following result holds:

$$\chi = \frac{1}{\Gamma(\gamma)} x^{\alpha+\beta-\rho}\ {}_3\psi_3\left[\begin{matrix}(\rho-\alpha-\beta,\lambda),(\rho+\gamma,\lambda),(\gamma,1)\\(\rho-\alpha,\lambda),(\rho-\alpha-\beta+\gamma,\lambda),(\beta,\alpha)\end{matrix}\middle| ax^{-\lambda}\right]$$

where $\alpha,\beta,\gamma,\rho \in C, \mathbb{R}(\alpha)>0,,\lambda>0$ and $a \in R$.

THEOREM 6.2. If $\alpha, \beta, \gamma \epsilon C, \mathbb{R}(\alpha_j) > 0, \mathbb{R}(\beta_j) > 0$ $(j = 1 \dots m), \mathbb{R}(k) > 0$, $\mathbb{R}(\sum_{j=1}^{m} \alpha_j) > \max[0, k-1]$, $\lambda > 0$ and $a \epsilon R$. If the Wright function condition [14] is satisfied and $D_{0-}^{\alpha,\beta,\gamma}$ be the right-sided operator of generalized fractional differentiation associated with gauss-hyper geometric function, then there holds the following formula

$$\left[D_{0-}^{\alpha,\beta,\gamma} t^{\alpha-\rho} E_{\gamma,k;\, \beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{-\lambda}\right)\right](x) = \frac{1}{\Gamma(\gamma)} x^{\alpha+\beta-\rho}\, {}_3\psi_{2+m}$$
$$\times \left[\begin{matrix} (\rho-\alpha-\beta,\lambda), (\rho+\gamma,\lambda), (\gamma,k) \\ (\rho-\alpha-\beta+\gamma,\lambda), (\rho-\alpha,\lambda), (\beta_j,\alpha_j) \end{matrix}\middle| ax^{-\lambda}\right] \tag{22}$$

PROOF. Denote L.H.S. of the theorem 6.2 by X then

$$X = \left[D_{0-}^{\alpha,\beta,\gamma} t^{\alpha-\rho} E_{\gamma,k;\, \beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{-\lambda}\right)\right](x)$$

Using the definition of multi-index Mittag-Leffler function was defined by Saxena and Nishimoto [12] and fractional differentiation formula (8), we get

$$X = \left(-\frac{d}{dx}\right)^k \left(I_{0-}^{-\alpha+k,-\beta-k,\alpha+\gamma} t^{\alpha-\rho} E_{\gamma,k;\, \beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{-\lambda}\right)\right)(x)$$

or

$$X = \left(-\frac{d}{dx}\right)^k \frac{1}{\Gamma(-\alpha+k)} \int_x^\infty (t-x)^{-\alpha+k-1} t^{\alpha+\beta} {}_2F_1\left(-\alpha-\beta, -\alpha-\gamma;\ -\alpha+k; 1-\frac{x}{t}\right)$$
$$\times t^{\alpha-\rho} E_{\gamma,k;\, \beta_1,\beta_2,\dots,\beta_m}^{\alpha_1,\alpha_2,\dots,\alpha_m}\left(at^{-\lambda}\right) dt$$

or

$$X = \frac{x^{\alpha+\beta-\rho}}{\Gamma(-\alpha+k)} \sum_{n=0}^{\infty} \frac{\Gamma(\rho-\alpha-\beta+\lambda n)\Gamma(\rho+\gamma+\lambda n)\Gamma(\gamma+kn)}{\Gamma(\rho-\alpha+\lambda n)\Gamma(\rho-\alpha-\beta+\gamma+\lambda n)\prod_{j=1}^{m}\Gamma(\beta_j,+\alpha_j n)} \frac{\left(ax^{-\lambda}\right)^n}{(n)!}$$

or

$$X = \frac{1}{\Gamma(\gamma)} x^{\alpha+\beta-\rho}\, {}_3\psi_{2+m} \left[\begin{matrix} (\rho-\alpha-\beta,\lambda), (\rho+\gamma,\lambda), (\gamma,k) \\ (\rho-\alpha-\beta+\gamma,\lambda), (\rho-\alpha,\lambda), (\beta_j,\alpha_j) \end{matrix}\middle| ax^{-\lambda}\right],$$

this completes the proof of theorem.

If we set $k = q$ in Eq. (10), we obtain the following corollary 6.2.1

COROLLARY 6.2.1. The following result holds

$$\mathrm{X} = \frac{x^{\alpha+\beta-\rho}}{\Gamma(\gamma)} {}_3\psi_{2+m}\left[\begin{matrix}(\rho-\alpha-\beta,\lambda),(\rho+\gamma,\lambda),(\gamma,q)\\(\rho-\alpha-\beta+\gamma,\lambda),(\rho-\alpha,\lambda),(\beta_j,\alpha_j)\end{matrix}\middle| ax^{-\lambda}\right],$$

where $\alpha,\beta,\gamma,\rho \in C, \mathbb{R}(\alpha_j) > 0, \mathbb{R}(\beta_j) > 0$ $(j = 1 \dots m), \mathbb{R}(\gamma) > 0,$ and $a \in R$ $\mathbb{R}\left(\sum_{j=1}^{m} \alpha_j\right) > \max[0, q-1], \lambda > 0.$

when $q = m = 1$, then above result reduced to the following corollary [19].

COROLLARY 6.2.2. The following result holds

$$\Omega = \frac{x^{\alpha+\beta-\rho}}{\Gamma(\gamma)} {}_3\psi_3\left[\begin{matrix}(\rho-\alpha-\beta,\lambda),(\rho+\gamma,\lambda),(\gamma,1)\\(\rho-\alpha-\beta+\gamma,\lambda),(\rho-\alpha,\lambda),(\beta,\alpha)\end{matrix}\middle| ax^{-\lambda}\right],$$

$\alpha,\beta,\gamma \in C, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\gamma) > 0, \lambda > 0$ and $a \in R$.

REMARK if we set $\lambda = \upsilon$ in above corollary, we arrive at the result [20].

CONCLUSIONS

In this paper, we have deduced some fractional calculus results involving generalized Mittag-Leffler function and multi-index Mittag-Leffler function. The results are general in nature and can be used to desire some new and know results having application in scientific and technological fields.

ACKNOWLEDGMENTS

The authors are very grateful to the anonymous referees for carefully reading the paper and for their constructive comments and suggestions which have improved the paper.

REFERENCES

[1] KILBAS, A.A., SRIVASTAVA, H.M., AND TRUJILLO, J.J. 2006. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematical Studies, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 204.

[2] CATTANI, C., SRIVASTAVA H.M. AND YANG X.J. 2015. *Fractional Dynamics*. Emerging Science Publishers (De Gruyter Open), Berlin and Warsaw.

[3] YANG, X.J., BALEANU, D. AND SRIVASTAVA, H.M. 2016. *Local Fractional Integral Transforms and Their Applications*. Academic Press (Elsevier Science Publishers), Amsterdam, Heidelberg, London and New York.

[4] SRIVASTAVA, H.M. 2016. *Some families of Mittag-Leffler type functions and associated operators of fractional calculus*. TWMS J. Pure Appl. Math., 7(2), 123-145.

[5] SRIVASTAVA, H.M. AND SAXENA, R.K. 2001. *Operators and fractional integration and their applications.* Appl. Math. Comput., 118, 1-52.

[6] SRIVASTAVA, H.M., KUMAR, D. AND SINGH, J. 2017. *An efficient analytical technique for fractional model of vibration equation*. Applied Mathematical Modelling 45, 192-204.

[7] MITTAG-LEFFLER, G.M. 1903. *Sur la nouvelle fonction* $E_\alpha(x)$, C.R. Acad. Sci., Paris (Ser.II), 137, 554-558.

[8] MITTAG-LEFFLER, G.M. 1905. *Sur la representation analytique d'une branche uniforme d'une fonction monogene*. Acta Math., 29, 101-181.

[9] PRABHAKAR, T.K. 1971. *A singular integral equation with a generalized Mittag-Leffler function in the kernel*. Yokohama Math. J., 19, 7-15.

[10] SHUKLA, A.K. AND PRAJAPATI, J.C. 2007. *On a generalized Mittag-Leffler function and its properties*. J. Math. Anal. Appl. 336, 797-811.

[11] SAXENA, R.K., RAM, J. AND VISHNOI, M. 2010. *Fractional differentiation and fractional integration of the generalized Mittag-Leffler function*. J. Indian Acad. Math., 32(1), 153-162.

[12] SAXENA, R.K. AND NISHIMOTO, K. 2011. *Further results on the generalized Mittag-Leffler functions of fractional calculus*. J. Fract. Calc., 40, 29-41.

[13] WRIGHT, E.M. 1935. *The asymptotic expansion of the generalized hypergeometric functions*. J. London Math. Soc. 10, 286-293.

[14] WRIGHT, E.M. 1934. *The asymptotic expansion of the generalized Bessel function*. Proc. London Math. Soc., 38(2), 257-270.

[15] KHAN, M.A. AND AHMED, S. 2013. *On some properties of the generalized Mittag-Leffler function*. Springer Plus a Springer Open Journal, doi: 10.1186/2193-1801-2-337, 2.

[16] SAMKO, S.G., KILBAS, A.A. AND MARICHEV, O.I. 1993. *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Yverdon, Switzerland.

[17] SAIGO, M. 1978. *A Remark on integral operators involving the Gauss hypergeometric functions*. Math .Rep. Kyushu Univ., 11, 135-143.

[18] SRIVASTAVA, H.M. AND KARLSSON, P.W. 1985. *Multiple Gaussian hypergeometric series*. Ellis Horwood, Chichester [John Wiley and Sons], New York.

[19] AHMED, S. 2014. *On the generalized fractional integrals of the generalized Mittag-Leffler function*. Springer Plus a Springer Open Journal, 3(1), 198.

[20] CHAURASIA, V.B.L. AND PANDEY, S.C. 2010. *On the fractional calculus of the generalized Mittag-Leffler function*. Scientia, Ser. A, Math. Sci., 20, 113-122.

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Received May 2015