

# Bi-Objective Bilevel Programming Problem: A Fuzzy Approach

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## Abstract:

In this paper, a likely situation of a set of decision maker's with bi-objectives in case of fuzzy multi-choice goal programming is considered. The problem is then carefully formulated as a bi-objective bilevel programming problem (BOBPP) with multiple fuzzy aspiration goals, fuzzy cost coefficients and fuzzy decision variables. Using Ranking method the fuzzy bi-objective bilevel programming problem (FBOBPP) is converted into a crisp model. The transformed problem is further solved by adopting a two level Stackelberg game theory and fuzzy decision model of Sakawa. A numerical with hypothetical values is also used to illustrate the problem.

**Additional Key Words and Phrases:** Bi-Objective Bilevel Programming Problem, Multichoice Goal Programming, Trapezoidal Fuzzy Numbers, Ranking Function, Fuzzy Decision Model

## 1. INTRODUCTION

Bilevel Programming Problem (BPP) was first introduced by Bialas et al. [1980], they called it a two level programming problem with two levels of decision makers (DM), viz., the leader or first level decision maker (FLDM) and the follower or second level decision maker (SLDM). They also developed the *Kth* algorithm to solve their problem.

A BPP can be viewed as a standard mathematical program whose constraint region has been modified to include an implicitly defined function. To see this, suppose that each of the First and Second level decision maker wishes to maximize his own objective function,  $F$  and  $f$  respectively. As in the conventional setting FLDM has control over the set  $X \subset \mathbb{R}^n$  and SLDM has control over the set  $Y \subset \mathbb{R}^m$  where  $X \cap Y = \emptyset$ . It shall be assumed that FLDM has the first choice and selects  $x \in X$ , followed by SLDM who selects  $y \in Y$ , and that each objective function jointly depends on both of these choices; that is,  $F, f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ . A further assumption is that the choice made by the FLDM may affect the set of feasible strategies available to the SLDM.

Bard [1982] defined BPP and solved the problem using grid search algorithm. There are many researchers who used different methods to get a better pareto optimal

solution for the hierarchical problem of BPP. Among them are U.P. Wen [1986], Mathieu et al. [1994], Lu et al. [2005], Shi et al. [2005], Liu [1998] etc. who solved the BPP using different approaches.

Zadeh introduced fuzzy sets as sets with boundaries that are not precise. A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. The fuzzy programming approach [Inuiguchi and Ramik [2000], Rommelfanger [1996], Slowinski and Teghem [1990]] is useful and efficient for treating a programming problem under uncertainty. While a classical and stochastic programming approach may cost a lot to obtain the exact coefficient value or distribution, fuzzy programming approach does not [Rommelfanger [2004]]. From this fact, fuzzy programming approach can be very advantageous when the coefficients are not known exactly. Fuzzy programming offers a powerful means of handling optimization problems with fuzzy parameters. Fuzzy programming has been used in different ways in the past. The fuzzy programming method for multi-objective programming problems was proposed by Zimmermann [1978] and has been advanced by Sakawa and colleagues [Sakawa [1993b]]. An interactive fuzzy satisfying method for multi objective linear programming problems and the interactive fuzzy decision making for multi-objective nonlinear programming using augmented minimax problems have been also introduced Luhandjula [1987], Sakawa [1984], Seo and Sakawa [1988]. The fuzzy programming method in which fuzziness in the decision making process is represented by using the fuzzy concept has also been studied extensively and many results have been published [Rommelfanger [1996], Teghem Jr et al. [1986]]. This method can be applied to not only the linear multi-objective but also the nonlinear multi-objective programming. There are many more authors who work on fuzzy problems some of them are Ganesan and Veeramani [2006], Allahviranloo et al. [2008], Ezzati et al. [2013] who have studied fuzzy linear program with triangular and trapezoidal fuzzy numbers

A new area of research was originated by Healy [1964] known as a multiple choice programming problem. In these problems there is a requirement to choose, among several possible combinations as an alternative to optimize an objective function subject to a set of constraints. It is a mixed binary programming in which all binary variables constitute a number of mutually exclusive choices where only one variable is to be selected. Multi-choice linear programming problem has been discussed by several authors such as Rav, Hiller and Lieberman [1990] etc. Chang [2007] proposed multichoice goal programming problem (MCGPP) in which

objectives of mathematical programming problem can have multiple choices for their goals. Again, Tabrizi et al. [2012] proposed fuzzy goal programming for solving the MCGPP with multiple fuzzy aspiration levels and developed fuzzy membership function to get a compromise solution and recently [Biswal and Acharya [2009], Biswal and Acharya [2011]] consider the problem where the right hand side parameter has multiple choices.

In this paper the author has considered a case of BPP with two objectives at each level with multiple fuzzy choices for goals for every objective. They have further considered the cost coefficients and decision variables to be symmetric trapezoidal numbers. The problem is formulated and then crisp values using ranking functions are used to solve the formulation adopting the two planner Stackelberg function [see Anandalingam [1988]], the well-known fuzzy decision model of Sakawa [1993a] for a tolerance membership function.

## 2. PRELIMINARIES

Some basic definitions which are frequently used in the theory of BPP and fuzzy set theory.

For  $x \in X \subset \mathbb{R}^n, y \in \mathbb{R}^m, F : X \times Y \rightarrow \mathbb{R}^1$  and  $f : X \times Y \rightarrow \mathbb{R}^1$ , the linear bilevel programming problem (BLPP) can be written as follows:

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y$$

$$\text{subject to } A_1 x + B_1 y \leq b_1$$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y$$

$$\text{subject to } A_2 x + B_2 y \leq b_2,$$

where  $c_1, c_2 \in \mathbb{R}^n, d_1, d_2 \in \mathbb{R}^m, b_1 \in \mathbb{R}^p, b_2 \in \mathbb{R}^q, A_1 \in \mathbb{R}^{p \times n}, B_1 \in \mathbb{R}^{p \times m}, A_2 \in \mathbb{R}^{q \times n}, B_2 \in \mathbb{R}^{q \times m}$ .

**Definition 1(a)** *Constraint region of the linear BLP problem:*

$$S = \{(x, y) : x \in X, y \in Y, A_1 x + B_1 y \leq b_1, A_2 x + B_2 y \leq b_2\}$$

The linear bilevel programming problem constraint region refers to all possible combinations of choices that the leader and the follower may make.

**Definition 1(b)** *Projection of S onto the leader's decision space:*

$$S(X) = \{x \in X : \exists y \in Y, A_1x + B_1y \leq b_1, A_2x + B_2y \leq b_2\}$$

**Definition 1(c)** Feasible set for follower after every fixed  $x \in X$ :

$$S(x) = \{y \in Y : B_2y \leq b_2 - A_2x\}$$

**Definition 1(d)** Follower's rational reaction set for  $x \in S(X)$ :

$$P(x) = \{y \in Y : y \in \arg \min[f(x, \hat{y}) : \hat{y} \in S(x)]\},$$

where  $\arg \min[f(x, \hat{y}) : \hat{y} \in S(x)] = \{y \in S(x) : f(x, y) \leq f(x, \hat{y}), \hat{y} \in S(x)\}$ .

**Definition 1(e)** Inducible region:

$$IR = \{(x, y) : (x, y) \in S, y \in P(x)\}.$$

$S$  is assumed non-empty and compact and for all decisions taken by leader, the follower has some room to respond i.e.  $P(x) \neq \emptyset$ .  $P(x)$  defines response while  $IR$  represents the set over which leader may optimize. The rational reaction set  $P(x)$  defines the response while the inducible region  $IR$  represents the set over which the leader may optimize his objective.

Thus in terms of the above notations, the linear BLP problem can be written as

$$\min_{x \in X} \{F(x, y) : (x, y) \in IR\}$$

Even with the stated assumptions, problem may not have a solution.

**Fuzzy Set:** Let  $A$  be the universe whose generic element be denoted by  $a$ . A fuzzy set  $\tilde{A}$  in  $A$  is a function  $\tilde{A} : A \rightarrow [0, 1]$ .

**Fuzzy number:** A fuzzy number  $\tilde{A}$  is a fuzzy set of the real line  $A$ , whose membership function  $\mu_{\tilde{A}}(a)$  must possess the following properties with

$$-\infty \leq a^{(1)} \leq a^{(2)} \leq a^{(3)} \leq \infty$$

$$\mu_{\tilde{A}}(a) = \begin{cases} 0 & a \leq a^{(1)}, \\ \mu_{\tilde{L}(a)} & a^{(1)} < a \leq a^{(2)} \\ \mu_{\tilde{U}(a)} & a^{(2)} < a \leq a^{(3)} \\ 0 & a > a^{(3)} \end{cases} \quad (1)$$

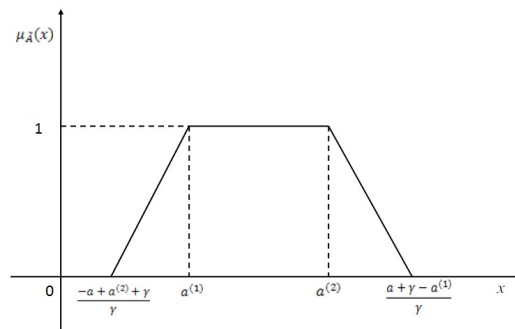
where  $\mu_{\tilde{L}}(a) : [a^{(1)}, a^{(2)}] \rightarrow [0, 1]$  is continuous and strictly increasing and  $\mu_{\tilde{U}}(a) : [a^{(2)}, a^{(3)}] \rightarrow [0, 1]$  is continuous and strictly decreasing.

**Membership function  $\mu_{\tilde{A}}(a)$ :** We frequently define function  $\tilde{A}$  and say that the fuzzy set  $A$  is characterized by its membership function  $\mu_A : A \rightarrow [0, 1]$

**Symmetric Trapezoidal Fuzzy Number (STFN):** A symmetric trapezoidal fuzzy number can be completely specified by the foursome  $\tilde{A} = (a^{(1)}, a^{(2)}, \gamma, \gamma)$  on real numbers  $\mathbb{R}$  if there exist real numbers  $a^{(1)}, a^{(2)}; a^{(1)} \leq a^{(2)}$  and  $\gamma < 0$  such that the membership function is

$$\mu_{\tilde{A}}(a) = \begin{cases} \frac{a + \gamma - a^{(1)}}{\gamma} & a^{(1)} - \gamma < a \leq a^{(1)} \\ 1 & a^{(1)} < a \leq a^{(2)}, \\ \frac{-a + a^{(2)} + \gamma}{\gamma} & a^{(2)} < a \leq a^{(2)} + \gamma \\ 0 & otherwise \end{cases} \quad (2)$$

The symmetric trapezoidal fuzzy number (STFN) is depicted in the figure below



**Ranking Function:** If  $\tilde{A} = (a^{(1)}, a^{(2)}, \gamma, \gamma)$  and  $\tilde{B} = (b^{(1)}, b^{(2)}, \beta, \beta)$  are two STFN then their product can be defined as

$$(a^{(1)}, a^{(2)}, \gamma, \gamma) \times (b^{(1)}, b^{(2)}, \beta, \beta)$$

$$\left(\frac{a^{(1)}+a^{(2)}}{2}\right)\left(\frac{b^{(1)}+b^{(2)}}{2}\right) - \omega, \left(\frac{a^{(1)}+a^{(2)}}{2}\right)\left(\frac{b^{(1)}+b^{(2)}}{2}\right) + \omega, |a^{(2)}\beta + \gamma b^{(2)}|, |a^{(2)}\beta + \gamma b^{(2)}|$$

where,  $\omega = \frac{t_1 - t_2}{2}$

$$t_1 = \min \{a^{(1)}b^{(1)}, a^{(1)}, b^{(2)}, a^{(2)}b^{(1)}, a^{(2)}b^{(2)}\}$$

$$t_2 = \max \{a^{(1)}b^{(1)}, a^{(1)}, b^{(2)}, a^{(2)}b^{(1)}, a^{(2)}b^{(2)}\}$$

### 3. FORMULATION OF BI-OBJECTIVE BILEVEL PROGRAMMING PROBLEM (BOBPP)

Again, the BPP can have more than one objective function at each level. So we consider here a linear bilevel programming problem with two objectives of maximization type at each level. Let the leader, FLDM have control over decision variable  $x \in \mathbb{R}^n$  and the follower, SLDM, have  $y \in \mathbb{R}^m$ . Moreover, the decision variables are trapezoidal fuzzy number with fuzzy cost coefficients. Cost coefficients are also fuzzy trapezoidal number. Mathematically, Bilevel Programming Problem (BPP) with two objectives of maximization type and fuzzy parameters can be written as follows

For  $\tilde{x} \in X \subset \mathbb{R}^n, \tilde{y} \in Y \subset \mathbb{R}^m$

$$\left. \begin{array}{l} \max_{\tilde{x} \in X} \quad F_1(\tilde{x}, \tilde{y}) = \tilde{c}_1\tilde{x} + \tilde{d}_1\tilde{y} \\ \quad \quad \quad F_2(\tilde{x}, \tilde{y}) = \tilde{c}_2\tilde{x} + \tilde{d}_2\tilde{y} \end{array} \right\} \quad \text{where } \tilde{y} \text{ solves}$$

$$\begin{array}{ll} \text{subject to} & A_1\tilde{x} + B_1\tilde{y} \leq b_1 \\ \max_{\tilde{y} \in Y} & f_1(\tilde{x}, \tilde{y}) = \tilde{c}_3\tilde{x} + \tilde{d}_3\tilde{y} \\ & f_2(\tilde{x}, \tilde{y}) = \tilde{c}_4\tilde{x} + \tilde{d}_4\tilde{y} \\ \text{subject to} & A_2\tilde{x} + B_2\tilde{y} \leq b_2 \\ & \tilde{x} \geq 0, \tilde{y} \geq 0. \end{array}$$

Now, in real life problem there may be situation where there are more than one choices in goals in such case the DM's of BOBPP have multiple fuzzy aspiration goals. Then BOBPP with multiple fuzzy aspiration goals may be formulated as

$$\left. \begin{array}{l} \max_{\tilde{x} \in X} \quad F_1(\tilde{x}, \tilde{y}) = \tilde{G}_1 \\ \quad \quad \quad F_2(\tilde{x}, \tilde{y}) = \tilde{G}_2 \end{array} \right\} \quad \text{where } \tilde{y} \text{ solves} \quad (3)$$

$$\begin{aligned}
& \text{subject to} & A_1\tilde{x} + B_1\tilde{y} &\leq b_1 \\
& \max_{\tilde{y} \in Y} & f_1(\tilde{x}, \tilde{y}) &= \tilde{g}_1 \\
& & f_2(\tilde{x}, \tilde{y}) &= \tilde{g}_2 \\
& \text{subject to} & A_2\tilde{x} + B_2\tilde{y} &\leq b_2 \\
& & \tilde{x} \geq 0, \tilde{y} &\geq 0.
\end{aligned}$$

where  $\tilde{G}_i, i = 1, 2$ . are fuzzy aspiration goals of FLDM and  $g_i, i = 1, 2$ . are fuzzy aspiration goals of SLDM. Since there are fuzzy aspiration goals at each objective the fuzzy bi-objective bilevel goal programming problem (FBOBPP) can be written as different cases.

**Case 1.** When there are two fuzzy aspiration goals to choose the FBOBPP will be

$$\begin{aligned}
& \max & \sum_{i=1}^2 \mu_{F_i} \\
& \text{subject to} & \mu_{F_i} \leq 1 - \left[ \frac{F_i(\tilde{x}, \tilde{y}) - G_i^{(1)}}{d_1^-} z_1 + \frac{F_i(\tilde{x}, \tilde{y}) - G_i^{(2)}}{d_2^-} (1 - z_1) \right] \\
& & \mu_{F_i} \leq 1 - \left[ \frac{G_i^{(1)} - F_i(\tilde{x}, \tilde{y})}{d_1^+} z_1 + \frac{G_i^{(2)} - F_i(\tilde{x}, \tilde{y})}{d_2^+} (1 - z_1) \right] \\
& & A_1\tilde{x} + B_1\tilde{y} \leq b_1 \\
& \max & \sum_{i=1}^2 \xi_{f_i} \\
& \text{subject to} & \xi_{f_i} \leq 1 - \left[ \frac{f_i(\tilde{x}, \tilde{y}) - g_i^{(1)}}{d_{i1}^-} k_1 + \frac{f_i(\tilde{x}, \tilde{y}) - g_i^{(2)}}{d_{i2}^-} (1 - k_1) \right] \\
& & \xi_{f_i} \leq 1 - \left[ \frac{g_i^{(1)} - f_i(\tilde{x}, \tilde{y})}{d_{i1}^+} k_1 + \frac{g_i^{(2)} - f_i(\tilde{x}, \tilde{y})}{d_{i2}^+} (1 - k_1) \right] \\
& & A_2\tilde{x} + B_2\tilde{y} \leq b_2 \\
& & \mu_{F_i} \geq 0, \xi_{f_i} \geq 0, \\
& & \tilde{x} \geq 0, \tilde{y} \geq 0; i = 1, 2; r = 1, 2, \dots, p.
\end{aligned} \tag{4}$$

**Case 2.** When there are three fuzzy aspiration goals to choose the FBOBPP will be

$$\begin{aligned}
& \max \quad \sum_{i=1}^2 \mu_{F_i} \\
\text{subject to} \quad & \mu_{F_i} \leq 1 - \left[ \frac{F_i(\tilde{x}, \tilde{y}) - G_i^{(1)}}{d_{i1}^-} z_1 z_2 + \frac{F_i(\tilde{x}, \tilde{y}) - G_i^{(2)}}{d_{i2}^-} (1 - z_1) z_2 + \right. \\
& \quad \left. + \frac{F_i(\tilde{x}, \tilde{y}) - G_i^{(3)}}{d_{i3}^-} z_1 (1 - z_2) \right] \\
& \mu_{F_i} \leq 1 - \left[ \frac{G_i^{(1)} - F_i(\tilde{x}, \tilde{y})}{d_{i1}^+} z_1 z_2 + \frac{G_i^{(2)} - F_i(\tilde{x}, \tilde{y})}{d_{i2}^+} (1 - z_1) z_2 + \right. \\
& \quad \left. + \frac{G_i^{(3)} - F_i(\tilde{x}, \tilde{y})}{d_{i3}^+} z_1 (1 - z_2) \right] \\
& A_1 \tilde{x} + B_1 \tilde{y} \leq b_1 \\
& \max \quad \sum_{i=1}^2 \xi_{f_i} \tag{5} \\
\text{subject to} \quad & \xi_{f_i} \leq 1 - \left[ \frac{f_i(\tilde{x}, \tilde{y}) - g_i^{(1)}}{d_{i1}^-} k_1 k_2 + \frac{f_i(\tilde{x}, \tilde{y}) - g_i^{(2)}}{d_{i2}^-} (1 - k_1) k_2 + \right. \\
& \quad \left. + \frac{f_i(\tilde{x}, \tilde{y}) - g_i^{(3)}}{d_{i3}^-} k_1 (1 - k_2) \right] \\
& \xi_{f_i} \leq 1 - \left[ \frac{g_i^{(1)} - f_i(\tilde{x}, \tilde{y})}{d_{i1}^+} k_1 k_2 + \frac{g_i^{(2)} - f_i(\tilde{x}, \tilde{y})}{d_{i2}^+} (1 - k_1) k_2 + \right. \\
& \quad \left. + \frac{g_i^{(3)} - f_i(\tilde{x}, \tilde{y})}{d_{i3}^+} k_1 (1 - k_2) \right] \\
& A_2 \tilde{x} + B_2 \tilde{y} \leq b_2 \\
& \mu_{F_i} \geq 0, \xi_{f_i} \geq 0, \\
& \tilde{x} \geq 0, \tilde{y} \geq 0; i = 1, 2; r = 1, 2, \dots, p.
\end{aligned}$$

where  $d_{ir}^+$  and  $d_{ir}^-$ ,  $i = 1, 2; r = 1, 2, \dots, p$  are the positive and negative deviation of  $r^{th}$  choice of fuzzy goal from the optimum value of the  $i^{th}$  objective respectively. We see a binary function is associated with these cases and we may generalize the term as a function of  $\Omega_i(z, k)$ , then the generalized FBOBPP may be written in precise as



$$\begin{aligned}
& \max \quad \sum_{i=1}^2 \mu_{F_i} \\
& \text{subject to} \quad \mu_{F_i} \leq 1 - \left[ \sum_{r=1}^m \frac{F_i(\tilde{x}, \tilde{y}) - G_i^{(r)}}{d_{ir}^-} \Omega_i(z, k) \right] \\
& \quad \mu_{F_i} \leq 1 - \left[ \sum_{r=1}^m \frac{G_i^{(r)} - F_i(\tilde{x}, \tilde{y})}{d_{ir}^+} \Omega_i(z, k) \right] \\
& \quad A_1 \tilde{x} + B_1 \tilde{y} \leq b_1 \\
& \max \quad \sum_{i=1}^2 \xi_{f_i} \tag{6} \\
& \text{subject to} \quad \xi_{f_i} \leq 1 - \left[ \sum_{r=1}^m \frac{f_i(\tilde{x}, \tilde{y}) - g_i^{(r)}}{d_{ir}^-} \Omega_{ir}(z, k) \right] \\
& \quad \xi_{f_i} \leq 1 - \left[ \sum_{r=1}^m \frac{g_i^{(r)} - f_i(\tilde{x}, \tilde{y})}{d_{ir}^+} \Omega_i(z, k) \right] \\
& \quad A_2 \tilde{x} + B_2 \tilde{y} \leq b_2 \\
& \quad \mu_{F_i} \geq 0, \xi_{f_i} \geq 0, \\
& \quad \tilde{x} \geq 0, \tilde{y} \geq 0; i = 1, 2; r = 1, 2, \dots, p.
\end{aligned}$$

where the membership function for  $i^{th}$  objective function in first level is given by

$$\mu_{F_i} = \begin{cases} 0 & \text{if } F_i(\tilde{x}, \tilde{y}) \leq G_i^{(r)} + d_{ir}^- \\ 1 - \left[ \frac{F_i(\tilde{x}, \tilde{y}) - G_i^{(r)}}{d_{ir}^-} \Omega_i(z, k) \right] & \text{if } G_i^{(r)} \leq F_i(\tilde{x}, \tilde{y}) \leq G_i^{(r)} + d_{ir}^- \\ 1 & \text{if } F_i(\tilde{x}, \tilde{y}) = G_i^{(r)}; i = 1, 2; r = 1, 2, \dots, p \\ 1 - \left[ \frac{G_i^{(r)} - F_i(\tilde{x}, \tilde{y})}{d_{ir}^+} \Omega_i(z, k) \right] & \text{if } G_i^{(r)} - d_{ir}^+ \leq F_i(\tilde{x}, \tilde{y}) \leq G_i^{(r)} \\ 0 & \text{if otherwise} \end{cases} \tag{7}$$

and membership function for  $i^{th}$  objective function in second level is

$$\xi_{f_i} = \begin{cases} 0 & \text{if } F_i(\tilde{x}, \tilde{y}) \leq g_i^{(r)} + d_{ir}^- \\ 1 - \left[ \frac{f_i(\tilde{x}, \tilde{y}) - g_i^{(r)}}{d_{ir}^-} \Omega_i(z, k) \right] & \text{if } g_i^{(r)} \leq f_i(\tilde{x}, \tilde{y}) \leq g_i^{(r)} + d_{ir}^- \\ 1 & \text{if } F_i(\tilde{x}, \tilde{y}) = g_i^{(r)}; i = 1, 2; r = 1, 2, \dots, p \\ 1 - \left[ \frac{g_i^{(r)} - f_i(\tilde{x}, \tilde{y})}{d_{ir}^+} \Omega_i(z, k) \right] & \text{if } g_i^{(r)} - d_{ir}^+ \leq f_i(\tilde{x}, \tilde{y}) \leq g_i^{(r)} \\ 0 & \text{if otherwise.} \end{cases} \tag{8}$$

#### 4. SOLUTION OF BOBPP WITH MULTIPLE FUZZY ASPIRATION GOALS

Now, we adopt the two level Stackelberg function and fuzzy decision model of Sakawa [1993a] to solve the above model of BOBPP with multiple fuzzy aspiration goals. According to this method as given by Emam [2006] the FLDM gets a satisfactory acceptable solution. Then decision variable and some goal are give to FLDM with some room for SLDM to get an optimal solution which is closest to the optimal solution of FLDM.

##### 4.1. Solution of the FLDM problem

The FLDM obtains the optimal individual solution, say  $F_i^*$  of all the objectives in upper level and solves the given problem.

Now we can get the solution of the FLDM problem by solving the following problem

$$\begin{aligned}
 & \text{Max} \quad \sum_{i=1}^2 \mu_{F_i} \\
 & \text{subject to} \quad \mu_{F_i} \leq 1 - \left[ \sum_{r=1}^p \frac{F_i(\tilde{x}, \tilde{y}) - G_i^{(r)}}{d_{ir}^-} \Omega_i(z, k) \right] \\
 & \quad \mu_{F_i} \leq 1 - \left[ \sum_{r=1}^p \frac{G_i^{(r)} - F_i(\tilde{x}, \tilde{y})}{d_{ir}^+} \Omega_i(z, k) \right] \\
 & \quad A_1 \tilde{x} + B_1 \tilde{y} \leq b_1 \\
 & \quad A_2 \tilde{x} + B_2 \tilde{y} \leq b_2 \\
 & \quad 0 \leq \mu_{F_i} \leq 1; \quad i = 1, 2, \dots, m \\
 & \quad \sum_{j=1}^n z_j \geq 0; \quad j = 1, 2, \dots, n
 \end{aligned} \tag{9}$$

whose solution is assumed to be

$$\tilde{x}^F, \tilde{y}^F, F_i^F; i = 1, 2, \dots, m.$$

##### 4.2. Solution of the SLDM problem

In similar way the SLDM solves his/her problem and denote its individual optimal solution as  $f_i^*$ . Again the solution of the SLDM's problem is obtained by solving the problem given below

$$\begin{aligned}
& \text{Max} && \sum_{i=1}^2 \xi_{f_i} \\
\text{subject to} &&& \xi_{f_i} \leq 1 - \left[ \sum_{r=1}^p \frac{f_i(\tilde{x}, \tilde{y}) - g_i^{(r)}}{d_{ir}^-} \Omega_i(z, k) \right] \\
&&& \xi_{f_i} \leq 1 - \left[ \sum_{r=1}^p \frac{g_i^{(r)} - f_i(\tilde{x}, \tilde{y})}{d_{ir}^+} \Omega_i(z, k) \right] \\
&&& A_1 \tilde{x} + B_1 \tilde{y} \leq b_1 \\
&&& A_2 \tilde{x} + B_2 \tilde{y} \leq b_2 \\
&&& 0 \leq \xi_{f_i} \leq 1; \quad i = 1, 2, \dots, m \\
&&& \sum_{j=1}^n k_j \geq 0; \quad j = 1, 2, \dots, n
\end{aligned} \tag{10}$$

whose solution is assumed to be  $\tilde{x}^S, \tilde{y}^S, F_i^S; i = 1, 2, \dots, m$

Now the solution of the FLDM and SLDM are disclosed. However, two solutions are usually different because of nature between both levels objective functions. The FLDM knows that using the optimal decisions  $\tilde{x}^F$  as a control factor for the SLDM is not practical. It is more reasonable to have some tolerance that gives the SLDM an extent feasible region to search for his/her optimal solution, and also reduce searching time or interactions.

In this way, the range of decision variables  $\tilde{x}$  should be around  $\tilde{x}^F$  with maximum tolerances  $t$  and the following membership function specify  $\tilde{x}^F$  as

$$\mu_{\tilde{x}} = \frac{\tilde{x} - (\tilde{x}^F - t)}{t} \quad \tilde{x}^F - t \leq \tilde{x} \leq \tilde{x}^F \tag{11}$$

where  $\tilde{x}^F$  is the most preferred solution.

First, the FLDM goals may reasonably consider all  $F_i \geq F_i^F, i = 1, 2, \dots, m$  absolutely acceptable and all  $F_i(\tilde{x}, \tilde{y}) < F_i' = F_i(\tilde{x}^S, \tilde{y}^S), i = 1, 2, \dots, m$  absolutely unacceptable, and that the preference with  $[F_i', F_i^F, i = 1, 2, \dots, m]$  is linearly increasing. This is due to the fact that the SLDM obtained the optimum at  $(\tilde{x}^S, \tilde{y}^S)$ , which in turn provides the FLDM the objective function values  $F_i'$ , makes any  $F_i(\tilde{x}, \tilde{y}) < F_i', i = 1, 2, \dots, m$  unattractive in practice.

The following membership functions of the FLDM can be stated as

$$\mu_i' = \begin{cases} 0 & \text{if } F_i(\tilde{x}, \tilde{y}) > F_i^F \\ 1 - \left[ \frac{F_i(\tilde{x}, \tilde{y}) - F_i'}{F_i^F - F_i'} \right] & \text{if } F_i' \leq F_i(\tilde{x}, \tilde{y}) \leq F_i^F \\ 1 & \text{if } F_i(\tilde{x}, \tilde{y}) < F_i' \end{cases} \quad i = 1, 2, \dots, m. \tag{12}$$

Second, the SLDM may be willing to build a membership function for his/her objective functions, so that he/she can rate the satisfaction of each potential solution. In this way, the SLDM has the following membership functions for his/her goals:

$$\xi'_i = \begin{cases} 0 & \text{if } f_i(\tilde{x}, \tilde{y}) > f_i^F \\ 1 - \left[ \frac{f_i(\tilde{x}, \tilde{y}) - f_i^f}{f_i^f - S'_i} \right] & \text{if } f_i^f \leq f_i(\tilde{x}, \tilde{y}) \leq f_i^S \\ 1 & \text{if } f_i(\tilde{x}, \tilde{y}) < f_i^f \end{cases} \quad i = 1, 2, \dots, m \quad (13)$$

where  $f_i^f = f_i(\tilde{x}^F, \tilde{y}^F)$

Finally, in order to generate the satisfactory solution, which is also a Pareto optimal solution with overall satisfaction for all DMs, solve the following Tchebycheff problem.

$$\begin{aligned} & \max_{\tilde{x} \in X} && \delta \\ \text{subject to} &&& \frac{\tilde{x} - (\tilde{x}^F + t_1)}{t} \geq \delta \\ &&& A_1 \tilde{x} + B_1 \tilde{y} \leq b_1 \\ &&& A_2 \tilde{x} + B_2 \tilde{y} \leq b_2 \\ &&& \xi'_i \leq 1 - \left[ \frac{F_i(\tilde{x}, \tilde{y}) - F_i^f}{F_i^F - F_i^f} \right] \\ &&& \mu'_i \leq 1 - \left[ \frac{f_i(\tilde{x}, \tilde{y}) - f_i^f}{f_i^f - S'_i} \right] \\ &&& \sum_{j=1}^n z_j \geq 0 \\ &&& \sum_{j=1}^n k_j \geq 0 \\ &&& 0 \geq \delta \geq 1 \\ &&& 0 \geq \mu_i \leq 1 \\ &&& 0 \geq \xi_i \leq 1 \\ &&& t > 0; \quad i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n \end{aligned} \quad (14)$$

where  $\delta$  is the overall satisfaction.

After solving the above equation if the FLDM is satisfied with solution then satisfactory solution is reached. Otherwise, the FLDM should provide new membership functions for the control variable and objectives to the SLDM until a satisfactory solution is reached.

## 5. NUMERICAL EXAMPLE

Consider a BOBPP with multiple fuzzy aspiration goals given below

$$\left. \begin{aligned} \max_{x \in X} F_1 &= \tilde{20}\tilde{x} + \tilde{30}\tilde{y} \approx 240, 275 \\ F_2 &= \tilde{30}\tilde{x} + \tilde{20}\tilde{y} \approx 210, 260, 290 \end{aligned} \right\} \text{ where } \tilde{y} \text{ solves}$$

$$\text{subject to } 11\tilde{x} + 12\tilde{y} \leq 130$$

$$\max_{y \in Y} f_1 = \tilde{9}\tilde{x} + \tilde{23}\tilde{y} \approx 160, 180, 200$$

$$f_2 = \tilde{10}\tilde{x} + \tilde{24}\tilde{y} \approx 160, 185$$

$$\text{subject to } 16\tilde{x} + 17\tilde{y} \leq 120$$

$$-12\tilde{x} + 9\tilde{y} \leq 35$$

$$\tilde{x} \geq 0, \tilde{y} \geq 0$$

Let the fuzzy cost coefficients and decision variables are symmetric trapezoidal fuzzy numbers where

$$\tilde{20} = 16, 20, 22, 22; \quad \tilde{30} = 15, 30, 45, 45; \quad \tilde{9} = 7, 9, 12, 12$$

$$\tilde{23} = 20, 23, 25, 25; \quad \tilde{10} = 7, 10, 12, 12; \quad \tilde{24} = 20, 24, 28, 28$$

Thus, the above problem in terms of symmetric trapezoidal fuzzy numbers can be rewritten as

$$\begin{aligned} \max_{x \in X} F_1 &= (16, 20, 22, 22) \otimes (x_1, x_2, x_3, x_3) + \\ &\quad + (15, 30, 45, 45) \otimes (y_1, y_2, y_3, y_3) \approx 240, 275 \\ \max_{x \in X} F_2 &= (15, 30, 45, 45) \otimes (x_1, x_2, x_3, x_3) + \\ &\quad + (16, 20, 22, 22) \otimes (y_1, y_2, y_3, y_3) \approx 210, 260, 290 \end{aligned}$$

$$\begin{aligned}
&\text{subject to} \quad 11 \otimes (x_1, x_2, x_3, x_3) + 12 \otimes (y_1, y_2, y_3, y_3) \leq 130 \\
&\max_{y \in Y} f_1 = (7, 9, 12, 12) \otimes (x_1, x_2, x_3, x_3) + \\
&\quad + (20, 23, 25, 25) \otimes (y_1, y_2, y_3, y_3) \quad \approx 160, 180, 200 \\
&\max_{y \in Y} f_2 = (7, 10, 12, 12) \otimes (x_1, x_2, x_3, x_3) + \\
&\quad + (20, 24, 28, 28) \otimes (y_1, y_2, y_3, y_3) \quad \approx 160, 185 \\
&\text{subject to} \quad 16 \otimes (x_1, x_2, x_3, x_3) + 17 \otimes (y_1, y_2, y_3, y_3) \leq 120 \\
&\quad -12 \otimes (x_1, x_2, x_3, x_3) + 9 \otimes (y_1, y_2, y_3, y_3) \leq 35 \\
&\quad (x_1, x_2, x_3, x_3) \geq 0, (y_1, y_2, y_3, y_3) \geq 0
\end{aligned}$$

The symmetric trapezoidal individual optimal values of the objectives of FLDM, SLDM,  $\tilde{x}$  and  $\tilde{y}$  are given in table below

$F_1^*$	202.5463	301.8519	201.0311	201.0311
$F_2^*$	198.7609	274.6685	176.2234	176.2234
$f_1^*$	209.2438	216.5021	132.7426	132.7426
$f_2^*$	192.0062	222.9630	134.9465	134.9465

Table 1: Individual optimal values of each objective function

Now, using eq. (9) the FLDM's problem is as given below

$$\begin{aligned}
&\max \quad \mu_1 + \mu_2 \\
&\text{subject to} \quad \mu_1 \leq 1 - \left[ \frac{F_1(\tilde{x}, \tilde{y}) - 240}{F_1^* - 240} z_1 + \frac{F_1(\tilde{x}, \tilde{y}) - 275}{F_1^* - 275} (1 - z_1) \right] \\
&\quad \mu_1 \leq 1 - \left[ \frac{240 - F_1(\tilde{x}, \tilde{y})}{F_1^* - 240} z_1 + \frac{275 - F_1(\tilde{x}, \tilde{y})}{F_1^* - 275} (1 - z_1) \right] \\
&\quad \mu_2 \leq 1 - \left[ \frac{F_2(\tilde{x}, \tilde{y}) - 210}{F_2^* - 210} z_2 z_3 + \frac{F_2(\tilde{x}, \tilde{y}) - 260}{F_2^* - 260} z_2 (1 - z_3) \frac{F_2(\tilde{x}, \tilde{y}) - 290}{F_2^* - 290} (1 - z_2) z_3 \right] \\
&\quad \mu_2 \leq 1 - \left[ \frac{210 - F_2(\tilde{x}, \tilde{y})}{F_2^* - 210} z_2 z_3 + \frac{260 - F_2(\tilde{x}, \tilde{y})}{F_2^* - 260} z_2 (1 - z_3) \frac{290 - F_2(\tilde{x}, \tilde{y})}{F_2^* - 290} (1 - z_2) z_3 \right] \\
&\quad 11\tilde{x} + 12\tilde{y} \leq 130 \\
&\quad 16\tilde{x} + 7\tilde{y} \leq 120 \\
&\quad -12\tilde{x} + 9\tilde{y} \leq 35 \\
&\quad z_2 + z_3 \geq 1 \\
&\quad 1 \geq \mu_1, \mu_2 \geq 0.
\end{aligned}$$

$F_1^F$	202.5463	204.8644	228.7890	228.7890
$F_2^F$	193.9403	180.7770	274.5718	274.5718
$f_1^F$	202.5463	148.5584	125.3088	125.3088
$f_2^F$	192.0062	150.9034	134.9834	134.9834

Table 2: Values for FLDM's solution

Again, using eq. (10) the solution of the SLDM's problem can be obtained from

$$\begin{aligned}
 & \text{Max} \quad \xi_1 + \xi_2 \\
 \text{subject to} \quad & \xi_1 \leq 1 - \left[ \frac{f_1(\tilde{x}, \tilde{y}) - 160}{f_1^* - 160} k_1 k_2 + \frac{f_1(\tilde{x}, \tilde{y}) - 180}{f_1^* - 180} k_1 (1 - k_2) \frac{f_1(\tilde{x}, \tilde{y}) - 200}{f_1^* - 200} (1 - k_1) k_2 \right] \\
 & \xi_1 \leq 1 - \left[ \frac{160 - f_1(\tilde{x}, \tilde{y})}{f_1^* - 160} k_1 k_2 + \frac{180 - f_1(\tilde{x}, \tilde{y})}{f_1^* - 180} k_1 (1 - k_2) \frac{200 - f_1(\tilde{x}, \tilde{y})}{f_1^* - 200} (1 - k_1) k_2 \right] \\
 & \xi_2 \leq 1 - \left[ \frac{f_2(\tilde{x}, \tilde{y}) - 160}{f_2^* - 160} k_3 + \frac{f_2(\tilde{x}, \tilde{y}) - 185}{f_2^* - 185} (1 - k_3) \right] \\
 & \xi_2 \leq 1 - \left[ \frac{160 - f_2(\tilde{x}, \tilde{y})}{f_2^* - 160} k_3 + \frac{185 - f_2(\tilde{x}, \tilde{y})}{f_2^* - 185} (1 - k_3) \right] \\
 & 11\tilde{x} + 12\tilde{y} \leq 130 \\
 & 16\tilde{x} + 7\tilde{y} \leq 120 \\
 & -12\tilde{x} + 9\tilde{y} \leq 35 \\
 & k_1 + k_2 \geq 1 \\
 & 1 \geq \xi_1, \xi_2 \geq 0.
 \end{aligned}$$

$F_1^S$	179.7286	167.7180	319.4375	319.4375
$F_2^S$	168.6757	147.5871	263.8660	263.8660
$f_1^S$	188.7443	120.1569	196.2188	196.2188
$f_2^S$	172.6852	122.7885	211.2017	211.2017

Table 3: Values for SLDM's solution

Finally, using eq. (11), eq.(12) and eq.(13) and assuming the FLDM's control decision  $x_1^F$  around 0 with tolerance 1, the pareto-optimal solution which is satisfactory to both DM's is obtained solving the Fuzzy programming problem in eq.(14).

Now the trapezoidal symmetrical fuzzy optimal values are converted into their

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$F'_1$	179.7286	176.785	218.785	218.785
$F'_2$	168.6757	145.995	233.0375	233.0375
$f'_1$	188.7443	141.3384	130.5005	130.5005
$f'_2$	172.6852	144.6666	138.4983	138.4983

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Table 4: Trapezoidal Fuzzy values for BOBLPP

respective crisp values

i.e.  $F'_1 = 178.2568, F'_2 = 157.33535, F'_3 = 165.04135$  and  $F'_4 = 158.6756$

Now when these crisp goals have been obtained the BOBLPP becomes

$$\left. \begin{array}{lcl} \max_{x \in X} F_1 & = 20x + 30y & = 178.2568 \\ F_2 & = 30x + 20y & = 157.33535 \end{array} \right\} \text{ where } \tilde{y} \text{ solves}$$

$$\text{subject to } 11x + 12y \leq 130$$

$$\max_{y \in Y} f_1 = 9x + 23y = 165.04135$$

$$f_2 = 10x + 24y = 158.6756$$

$$\text{subject to } 16x + 17y \leq 120$$

$$-12x + 9y \leq 35$$

$$x \geq 0, y \geq 0$$

So the compromise solution finally obtained is  $(x^C, y^C) = (1.403957, 5.760832)$

$$(F_1^C, F_2^C) = (200.9041, 157.3354)$$

$$(f_1^C, f_2^C) = (145.1347, 152.2995)$$

Figure (1) illustrates pareto optimal solution of the above problem.

If the FLDM is satisfied with the above solution, then a satisfactory solution is reached. Otherwise, one should provide new membership functions for the control variable and objectives to the SLDM, who also should provide new membership function for the control variable and objectives to the SLDM until a satisfactory solution is reached.



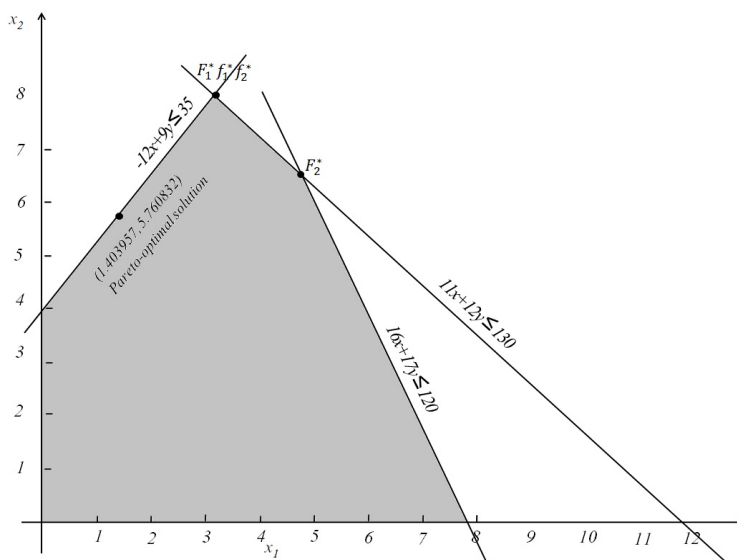


Figure 1: A Pareto Optimal Solution

## 6. CONCLUSION

In this paper the author considered a practical situation of multiple fuzzy choices of goals for decision makers, fuzzy cost coefficients and fuzzy decision variables with their own set of objectives. A two level problem with two objectives at each level and fuzzy aspiration goals is formulated as a fuzzy bi-objective bilevel programming problem (FBOBPP). The fuzzy cost coefficients and fuzzy decision variables are considered symmetric trapezoidal fuzzy numbers. Using ranking function the model is converted into a crisp model. The FBOBPP is then solved using two planner stackelberg function and fuzzy decision model with tolerance function. A comprehensive study of bi-objective bilevel programming problem in case of fuzzy parameters has been done in this paper. Finally, a pareto-optimal solution for decision maker's of bilevel problem is obtained. The problem can be further extended as a multi-objective multilevel programming problem with multiple fuzzy aspiration goals.

## REFERENCES

- T. Allahviranloo, F. H. Lofti, M. K. Kiasary, N.A. Kiani, and L. Alizadeh. Solving fully fuzzy linear programming problem by the ranking function. *Applied Mathematical Sciences*, 2(1):19–32, 2008.
- G. Anandalingam. A mathematical programming model of decentralized multi-level systems. *Journal of Operational Research Society*, 39:1021–1033, 1988.
- J.F. Bard. A grid search algorithm for the linear bilevel programming problem. *14-th Annual Meeting of American Institute for Decision Science, San Francisco, CA*, 2:256–258, 1982.
- W. Bialas, M.H. Karwan, and J Shaw. A parametric complementary pivot approach for two-level linear programming. *Technical Report, State University of New York at Buffalo, Operations Research*, 80-2, 1980.
- M. P. Biswal and S. Acharya. Transformation of a multi-choice linear programming problem. *Applied Mathematics and Computation*, 210:182–188, 2009.
- M. P. Biswal and S. Acharya. Solving multi-choice linear programming problems by interpolating polynomials. *Mathematical and Computer Modelling*, 54:1405–1412, 2011.
- C. T. Chang. Multi-choice goal programming. *Omega*, 35(4):389–396, 2007.
- O.E. Emam. A fuzzy approach for bi-level integer non-linear programming problem. *Applied Mathematics and Computation*, 172:62–71, 2006.
- R. Ezzati, E. Khorram, and R. Enayati. A algorithm to solve fully fuzzy linear programming problems using the molp problem. *Applied Mathematical Modelling*, In Press, 2013.
- K. Ganesan and P. Veeramani. Fuzzy linear programs with trapezoidal fuzzy numbers. *Annals of Operations Research*, 143:305–315, 2006.
- W. C. Healy. Multiple choice programming. operations research. *International journal of Mathematics and Applied Statistics*, 12:122–138, 1964.
- F. Hiller and G. Lieberman. *Introduction to operations research*. McGraw-Hill, New York, 1990.
- Masahiro Inuiguchi and Jaroslav Ramík. Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. *Fuzzy sets and systems*, 111(1):3–28, 2000.
- B. Liu. Stackelberg-nash equilibrium for multi-level programming with multi-follows using genetic algorithms. *Computers math applications*, 37,7:79–89, 1998.
- J. Lu, C. Shi, and G. Zhang. An extended kth-best approach for linear bilevel programming. *Applied Mathematics and Computation*, 164:843–855, 2005.
- MK Luhandjula. Multiple objective programming problems with possibilistic coefficients. *Fuzzy Sets and Systems*, 21(2):135–145, 1987.
- R. Mathieu, L. Pittard, and G. Anandalingam. Genetic algorithm based approach to bilevel linear programming. *R. A. I. R. O Recherche operationelle*, 28:1–21, 1994.
- Heinrich Rommelfanger. Fuzzy linear programming and applications. *European journal of operational research*, 92(3):512–527, 1996.
- Heinrich Rommelfanger. The advantages of fuzzy optimization models in practical use. *Fuzzy Optimization and Decision Making*, 3(4):295–309, 2004.
- M Sakawa. Interactive fuzzy goal programming for multiobjective nonlinear problems and its application to water quality management. *Control and Cybernetics*, 13(2):217–228, 1984.
- M. Sakawa. Fuzzy sets and interactive multi-objective optimization. *Plenum Press, New York*, 1993a.
- Masatoshi Sakawa. *Fuzzy sets and interactive multiobjective optimization*. Plenum New York, 1993b.
- Fumiko Seo and Masatoshi Sakawa. *Multiple criteria decision analysis in regional planning: concepts, methods and applications*. Reidel Dordrecht, The Netherlands, 1988.
- C. Shi, J. Lu, and G. Zhang. An extended kuhn-tucker approach for linear bilevel programming. *Applied Mathematics and Computation*, 162:51–63, 2005.
- Roman Slowinski and Jacques Teghem. *Stochastic vs. fuzzy approaches to multiobjective mathematical programming under uncertainty*. Kluwer Academic Publishers, 1990.

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- Behzad Bankian Tabrizi, Kamran Shahanaghi, and M. Saeed Jabalameli. Fuzzy multi-choice goal programming. *Applied Mathematical Modelling*, 36:1415–1420, 2012.
- Jea Teghem Jr, D Dufrane, M Thauvoye, and P Kunsch. Strange: an interactive method for multi-objective linear programming under uncertainty. *European Journal of Operational Research*, 26(1):65–82, 1986.
- W.F. Bialas U.P. Wen. The hybrid algorithm for solving the three-level linear programming problem. *Computers and Operations Research*, 13:367–377, 1986.
- H-J Zimmermann. Fuzzy programming and linear programming with several objective functions. *Fuzzy sets and systems*, 1(1):45–55, 1978.

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