A novel multivariate generalized skew-normal distribution with two parameters $BGSN_{n,m}(\lambda_1,\lambda_2)$

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Abstract

In this paper we first introduce a new class of multivariate generalized asymmetric skew-normal distributions with two parameters λ_1, λ_2 that we present it by $BGSN_{n,m}(\lambda_1, \lambda_2)$, and we finally obtain some special properties of $BGSN_{n,m}(\lambda_1, \lambda_2)$.

Additional Key Words and Phrases: $BGSN_{n,m}(\lambda_1, \lambda_2)$, Multivariate distribution, Skew-normal, Conditional $BGSN_{n,m}(\lambda_1, \lambda_2)$.

1. INTRODUCTION

The skew-normal distribution introduced by Azzalini [2]. As for extensions of the distribution, many classes of distributions have been proposed. For example, Arellano-Valle [1], Sharafi and Behbodian [5] and Hasanalipour [3].

Jamalizadeh [4] considered a generalization of $SN(\lambda)$ so called two-parameters generalized skew-normal distribution defined in the following form

$$f(x;\ \lambda_1, \lambda_2) = c_n(\lambda_1, \lambda_2)\phi(x)\Phi(\lambda_1 x)\Phi(\lambda_2 x), \quad x \in R, \tag{1}$$

where $\lambda_1 \in R$, $\lambda_2 \geq 0$, and $n, m \geq 1$. This distribution denoted by $X \sim GSN(\lambda_1, \lambda_2)$.

In this paper, we want to introduce a new class of multivariate skew-normal distribution which generalizes (1) while preserving most of its properties.

in section 2, we present the definition and some interesting properties of the $BGSN_{n,m}(\lambda_1, \lambda_2)$

2. A NEW GENERALIZED MULTIVARIATE SKEW-NORMAL DISTRIBUTION

In this section we introduce the $BGSN_{n,m}(\lambda_1, \lambda_2)$ class and we present some of its properties.

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2.1. Definition and simple properties

DEFINITION 1. We say that vector (X, Y) has the multivariate skew-normal distribution with two parameters if and only if it has the following probability density function,

$$f(x,y;\ \lambda_1,\lambda_2) = c_{n,m}(\lambda_1,\lambda_2)\phi(x)\phi(y)\Phi^n(\lambda_1 x)\Phi^m(\lambda_2 y), \quad x,y\in R, \tag{2}$$

where $\lambda_1 \in R$ and $\lambda_2 \geq 0$. For future reference we denote it by $BGSN_{n,m}(\lambda_1,\lambda_2)$. The coefficient $c_{n,m}(\lambda_1,\lambda_2)$ which is a function of n, m and the parameters λ_1,λ_2 is given by

$$c_{n,m}(\lambda_1,\lambda_2) = \frac{1}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x)\phi(y)\Phi^n(\lambda_1 x)\Phi^m(\lambda_2 y)dxdy}$$
$$= \frac{1}{E\{\Phi^n(\lambda_1 U)\}E\{\Phi^m(\lambda_2 V)\}}$$

where $U, V \sim N(0,1)$.

We now present some simple properties of $BGSN_{n,m}(\lambda_1,\lambda_2)$.

1. $BGSN_{n,m}(0,0) = \phi(x)\phi(y)$.
2. $BGSN_{1,1}(\lambda_1,\lambda_2) = 4\phi(x)\phi(y)\Phi(\lambda_1 x)\Phi(\lambda_2 y)$.
3. $X \mid \{Y = y\} = c_n(\lambda_1)\phi(x)\Phi^n(\lambda_1 x) \sim SNB_n(\lambda_1)$.
4. $Y \mid \{X = x\} = c_m(\lambda_2)\phi(y)\Phi^m(\lambda_2 y) \sim SNB_m(\lambda_2)$.
5. $\lim_{\lambda_1\to\infty}\lim_{\lambda_2\to\infty} f(x,y;\ \lambda_1,\lambda_2) = 4\phi(x)\phi(y)\, I_{\{x>0,y>0\}}$.
6. $\lim_{\lambda_1\to-\infty}\lim_{\lambda_2\to-\infty} f(x,y;\ \lambda_1,\lambda_2) = 4\phi(x)\phi(y)\, I_{\{x<0,y<0\}}$.

2.2. Some important properties of $BGSN_{n,m}(\lambda_1,\lambda_2)$

In this step we introduce some special theorems of $BGSN_{n,m}(\lambda_1,\lambda_2)$ distribution.

THEOREM 1. If $X, Y, Z_1, \ldots, Z_n$ and $W_1, \ldots, W_m$ are *i.i.d* with $N(0, 1)$ distribution, then we have:

$$(X,Y) \mid \left\{Z_{(n)} \leq \lambda_1 X,\ W_{(m)} \leq \lambda_2 Y\right\} \sim BGSN_{n,m}(\lambda_1,\lambda_2) \tag{3}$$

where $Z_{(n)} = max\{Z_1, \ldots, Z_n\}$ and $W_{(m)} = max\{W_1, \ldots, W_m\}$.

PROOF. Suppose $A = (Z_{(n)} \leq \lambda_1 X,\ W_{(m)} \leq \lambda_2 Y)$. Then, we write

$$f_{(X,Y)|A}(x,z|A) = \frac{P(A \mid X = x, Y = y)f(x,y)}{P(A)}$$

$$=\frac{P\big(Z_{(n)}\le\lambda_1 X,\ W_{(m)}\le\lambda_2 Y\ \big|\ X=x,Y=y)\phi(x)\phi(y)}{P(Z_{(n)}\le\lambda_1 X,\ W_{(m)}\le\lambda_2 Y)}$$

$$=\frac{P(Z_1\le\lambda_1 x,\dots,Z_n\le\lambda_1 x,W_1\le\lambda_2 y,\dots W_m\le\lambda_2 y)\ \phi(x)\phi(y)}{P(Z_1\le\lambda_1 X,\dots,Z_n\le\lambda_1 X,W_1\le\lambda_2 Y,\dots W_m\le\lambda_2 Y)}$$
$$=c_{n,m}(\lambda_1,\lambda_2)\phi(x)\phi(y)\Phi^n(\lambda_1 x)\Phi^m(\lambda_2 y)$$

For random number generation, it is more efficient to use single variant of this result, namely to put

$$Z=(Z_1,Z_2)=\begin{cases}(X,Y) & Z_{(n)}\le\lambda_1 X,W_{(m)}\le\lambda_2 Y\\(-X,-Y) & Z_{(n)}>\lambda_1 X,W_{(m)}>\lambda_2 Y\end{cases}\tag{4}$$

This make an important point for $BGSN_{n,m}(\lambda_1,\ \lambda_2)$ distribution, comparing with acceptance-rejection method simulation of independent normal distribution.

THEOREM 2. $(U,K,U_1,\dots,U_n,V_1,\dots,V_m)\overset{D}{=}N_{n+m+2}(0,R)$ with this properties for correlation matrix (R).

$$\rho(K,U_i)=\rho(U,U_i)=\rho_1\quad i=1,2,\dots,n$$
$$\rho\big(U_i,U_j\big)=\rho_1^2\quad i\ne j,\quad i,j=1,2,\dots,n$$
$$\rho(K,V_i)=\rho(U,V_i)=\rho_2\quad i=1,2,\dots,m$$
$$\rho\big(V_i,V_j\big)=\rho_2^2\quad i\ne j,\quad i,j=1,2,\dots,m$$
$$\rho\big(U_i,V_j\big)=\rho_{1,2}\quad i=1,2,\dots,n,\ \ j=1,2,\dots,m$$

then

$$\big((U,K)|\{U_{(1)}>0,V_{(1)}>0\}\big)\sim BGSN_{n,m}(\lambda_1,\lambda_2).\tag{5}$$

where $U_{(1)}=min\{U_1,\dots,U_n\}$, $V_{(1)}=min\{V_1,\dots,V_m\}$, $\lambda_1=\frac{\rho_1}{\sqrt{1-\rho_1^2}}$ and $\lambda_2=\frac{\rho_2}{\sqrt{1-\rho_2^2}}$

PROOF. Let X, Y, Z_1, ..., Z_n and W_1, ..., W_m are *i.i.d* $N(0, 1)$ distribution, and let

$$U_1^*=\rho_1 X-\sqrt{1-\rho_1^2}Z_1\qquad V_1^*=\rho_2 Y-\sqrt{1-\rho_2^2}W_1$$
$$U_2^*=\rho_1 X-\sqrt{1-\rho_1^2}Z_2\qquad V_2^*=\rho_2 Y-\sqrt{1-\rho_2^2}W_2$$

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$$U_n^*=\rho_1 X-\sqrt{1-\rho_1^2}Z_n\qquad V_m^*=\rho_2 Y-\sqrt{1-\rho_2^2}W_m.$$

it is then easy to show that $(U, K, U_1, \dots, U_n, V_1, \dots, V_m) \stackrel{D}{=} (X, Y, U_1^*, \dots, U_n^*,\ V_1^*, \dots, V_m^*)$ therefore,

$$\begin{aligned}(U, K) \mid min(U_{(1)} > 0, V_{(1)} > 0) &\stackrel{D}{=} (X, Y) \mid \{min(U_i) > 0,\ min\ (V_i) > 0\} \\ &\stackrel{D}{=} (X, Y) \mid \{min(U_i^*) > 0,\ min\ (V_i^*) > 0\} \\ &\stackrel{D}{=} (X, Y) \mid Z_1 \le \frac{\rho_1}{\sqrt{1-\rho_1^2}} X, \dots, Z_n \le \frac{\rho_1}{\sqrt{1-\rho_1^2}} X, \\ &\qquad W_1 \le \frac{\rho_2}{\sqrt{1-\rho_2^2}} Y, \dots, W_m \le \frac{\rho_1}{\sqrt{1-\rho_1^2}} Y \\ &\stackrel{D}{=} (X, Y) \mid \{\ Z_{(n)} \le \lambda_1 X, W_{(m)} \le \lambda_2 Y\ \}\end{aligned}$$

The result in (5) follows, using theorem (1).

THEOREM 3. If $(X, Y) \sim BGSN_{n,m}(\lambda_1, \lambda_2)$, let $W_1 = X / Y$ and $W_2 = Y / X$ then $W_1^2, W_2^2 \xrightarrow{L} x_{(1)}^2$, as $\lambda_1 \to \infty$, where $x_{(1)}^2$shows chi-square random variable with one degree of freedom.

PROOF. Let $Z = W_1^2$. The density of Z is

$$\begin{aligned} f_Z(z; \lambda_1) &= \frac{1}{2\sqrt{z}} \left(f_{W_1}(\sqrt{z}) + f_{W_1}(-\sqrt{z}) \right) \\ &= \frac{1}{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z}{2}} c_n(\lambda_1) \left[\frac{\Phi^n(\lambda_1\sqrt{z}) + \Phi^n(-\lambda_1\sqrt{z})}{2} \right] \\ &= f_{x_{(1)}^2}(z)[a_n(z, \lambda_1)];\ z > 0 \end{aligned}$$

with $a_n(z, \lambda_1) = c_n(\lambda_1) \left[\frac{\Phi^n(\lambda_1\sqrt{z}) + \Phi^n(-\lambda_1\sqrt{z})}{2} \right]$.

Since $c_n(\lambda_1) \to 2$ as $\lambda_1 \to \infty$ we conclude that $a_n(z, \lambda_1) \to 1$ as $\lambda_1 \to \infty$.

Therefore, the density $f_Z(z; \lambda_1)$ converges to the distribution of $x_{(1)}^2$, $i.e.$ $Z \xrightarrow{L} x_{(1)}^2$.

PROPOSITION. If $(X, Y) \sim BGSN_{n,m}(\lambda_1, \lambda_2)$ and $Z \sim N(0, 1)$, then $\lim_{\lambda_1\to\infty}|X/Y|$ or $\lim_{\lambda_1\to\infty}|Y/X|$ and $|Z|$ are identically distributed, *i.e.* $\lim_{\lambda_1\to\infty}|X/Y| \stackrel{D}{=} |Z| \sim HN(0,1)$ where $HN(0,1)$ denotes the standard half-normal distribution.

PROOF. We know that $|Z|$ has density $2\phi(z)\, I_{\{z>0\}}$. On the other hand, by property (3) of $BGSN_{n,m}(\lambda_1, \lambda_2)$ the density $W = |X / Y|$ is

$$f_W(w) = f_{X/Y}(w) + f_{X/Y}(-w)$$

$$= c_n(\lambda_1)\phi(w)\Phi^n(\lambda_1 w) + c_n(\lambda_1)\phi(-w)\Phi^n(-\lambda_1 w)$$
$$= c_n(\lambda_1)\phi(w)\Phi^n(\lambda_1 w) + \Phi^n(-\lambda_1 w)$$
$$= c_n(\lambda_1)\phi(w)$$

Now, we can show that $c_n(\lambda_1) \to 2$ as $\lambda_1 \to \infty$, then $\lim_{\lambda_1\to\infty} | W | = 2\phi(w)$ for $w > 0$ and we have $\lim_{\lambda_1\to\infty} | W | \stackrel{D}{=} | Z |$.

THEOREM 4. The moment generating function of $(X, Y) \sim BGSN_{n,m}(\lambda_1, \lambda_2)$ is

$$M_{X,Y}(t_1, t_2) = c_{n,m}(\lambda_1, \lambda_2)e^{\frac{t_1^2+t_2^2}{2}}E\{\Phi^n\{\lambda_1(U + t_1)\}\}E\{\Phi^m\{\lambda_2(W + t_2)\}\}$$

$U, W \sim N(0, 1)$ distribution.

PROOF.

$$M_{X,Y}(t_1, t_2) = E(e^{t_1X+t_2Y})$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_{n,m}(\lambda_1, \lambda_2)e^{t_1x+t_2y}\phi(x)\phi(y)\Phi^n(\lambda_1 x)\Phi^m(\lambda_2 y)dxdy$$
$$= c_{n,m}(\lambda_1, \lambda_2)e^{\frac{t_1^2+t_2^2}{2}}E\{\Phi^n\{\lambda_1(U + t_1)\}\}E\{\Phi^m\{\lambda_2(W + t_2)\}\}$$

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