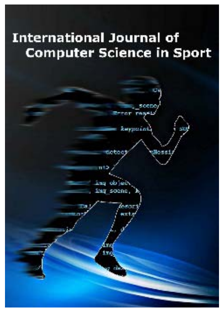
International Journal of Computer Science in Sport

Volume 16, Issue 2 (Special Edition), 2017

Journal homepage: http://iacss.org/index.php?id=30

DE GRUYTER OPEN

DOI: 10.1515/ijcss-2017-0008



How to Stay Ahead of the Pack: Optimal Road Cycling Strategies for two Cooperating Riders

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Abstract

Within road-cycling, the optimization of performance using mathematical models has primarily been performed in the individual time trial. Nevertheless, most races are 'mass-start' events in which many riders compete at the same time. In some special situations, e.g. breakaways from the peloton, the riders are forced to team up. To simulate those cooperative rides of two athletes, an extension of models and optimization approaches for individual time trials is presented. A slipstream model based on experimental data is provided to simulate the physical interaction between the two riders. In order to simulate real world behavior, a penalty for the difference in the exertion levels of the two riders is introduced. This means, that even though both riders aim to be as fast as possible as a group, neither of them should have an advantage over the other because of significantly different levels of fatigue during the ride. In our simulations, the advantage of cooperation of two equally trained athletes adds up to a time gain of about 10% compared to an individual ride.

KEYWORDS: ROAD CYCLING, SLIPSTREAM, PACING STRATEGY, OPTIMAL CONTROL

Introduction

During recent years, evaluating and optimizing pacing strategies in endurance sports like cycling, running, or cross-country skiing has become popular. Besides preparation, the strategy plays a major role in winning a race. In road cycling, one common tactical approach is that several riders break away from the peloton, to enhance their chances to win a race. For the success of such an attack, it is important when and how to break away as well as how to interact and distribute the individual resources of each rider in order to stay ahead of the pack. In this second phase, the riders are forced to work together in order to maintain a distance to the peloton until reaching the end of the race. In this study, we investigate the best strategies for two riders to finish a course using mathematical methods. Several studies have applied mathematical models to solve the corresponding problem for individual time trials, by calculating optimal pacing strategies in terms of achieving the shortest race time. However, only little work has been done so far incorporating more than one rider.

First results for individual time trial strategies in cycling were gathered by Gordon (2005). The 3-parameter critical power model of Morton (1996) and a simple mechanical model, which includes air resistance, friction, and gravitation, were used to analytically calculate strategies on simple, piecewise constant courses. An extension of this work has been provided by Dahmen et al. (2012). The more realistic mechanical model of Martin et al. (1998) which also includes inertia and bearing friction, as well as slope profiles of real world courses, have been used. Due to the higher complexity, numerical methods were applied. Since then, several studies have been published that use more sophisticated physiological models. For example, Sundström et al. (2014) compared strategies with the traditional critical power model to a more versatile 3-component model, which was introduced by Morton (1986).

Until now, only few studies consider more than one competitor. One of the first looking into strategies for two runners was Pitcher (2009). Slipstream effects in running were considered, and it was pointed out that the runner behind can take advantage of this effect and win a race by taking over shortly before the finish line. Dahmen and Saupe (2014) adapted this approach to cycling and showed that a similar strategy leads to victory in cycling on a Tour de France stage. A major limitation in both studies is that the loosing athlete does not react to the strategy of his opponent. Aftalion and Fiorini (2015) generalized Pitcher’s approach and modified the optimal control problem to allow both runners a free strategy. The optimization goal was to minimize the time of the winning runner, while considering that the loser had tried to win as well.

Wind resistance is the major force resulting in the difference in the cycling dynamics between individual rides and rides in a group. Cycling in a group can reduce the wind resistance significantly for riders in the slipstream. In a study of Barry et al. (2014), the change in wind resistance has been investigated in a wind tunnel for different position configurations of two riders. Riding close behind another rider gives the largest advantage, with a reduction of the force to overcome wind resistance of about 50%. In addition, the leading rider gains a reduction of about 5% in this configuration due to less turbulences and an artificial tailwind produced by the trailing rider. On the other hand, there is an increase in wind resistance if both cyclists are riding side by side of about 7%. If the distance between the two riders increases, Olds (1998) reported a quadratic increase of wind resistance for the rider in the slipstream up to a tire-to-tire distance of 3m. See the parabola on the right hand side of Figure 1. For larger distances, the impact of slipstream can be neglected.

In this study, we provide a model for the dynamics between two cyclists and present the optimal control problem for two cooperating riders. We will present the underlying mathematical models, the optimal control problem, and provide results of numerical simulations.

Methods

In this work, the physical model of Martin et al. (1998), to describe the equilibrium of the rider's pedal force and the forces induced by aerodynamic drag, friction, gravitation, and inertia, is used. Slipstream is modelled based on the findings of Barry et al. (2014) and Olds (1998) by smooth exponential functions. The physiological capabilities of the athlete are modelled by a dynamic version of the critical power concept introduced by Monod and Scherrer (1965) incorporating aerobic and anaerobic energy resources.

Table 1. Parameters of the mechanical model and the values that were used in the optimization.

Description	Variable	Value
riders' power output	P	derived by algorithm
speed	v	derived by algorithm
travelled distance	x	derived by algorithm
gap between riders	x_d	derived by algorithm
total mass	m	80 kg
gravity factor	g	9.81 m/s^2
slope of the course	s	0 %
friction factor	μ	0.004
wheel inertia	I_w	0.2 kgm^2
wheel radius	r_w	0.335 m
mass of inertia	M	$m + \frac{I_w}{r_w^2}$
drag coefficient	c_d	0.7
air density	ϱ	1.2 kg/m^3
cross-sectional area	A	0.4 m^2
chain efficiency	η	0.975
bearing factor	b_0	0.091 Nm
bearing factor	b_1	0.0087 Nms
bike length	l	1.8 m
minimum tire distance	$d_{\min}$	0.1 m
length of the race	x_f	5000 m

Mechanical Model

To model the relation between the power output of the rider and the resulting speed on the course, the well-known model of Martin et al. (1998) is used. It has been validated in Dahmen et al. (2011) on real world courses as well as in a laboratory simulator setup. The model is based on the equilibrium between the riders pedal power and the power induced by aerodynamic drag, rolling resistance, bearing friction, gravity, and inertia as shown in the following equation:

$$\eta P = \underbrace{\frac{1}{2} c_d \varrho A v^3}_{air\ resistance} + \underbrace{mgsv}_{gravity} + \underbrace{\mu mgv}_{rolling\ resistance} + \underbrace{b_0 v + b_1 v^2}_{bearing\ friction} + \underbrace{Mv\dot{v}}_{inertia} \tag{1}$$

with model parameters presented in Table 1.

This model describes the cycling mechanics for an individual rider. In order to account for interactions between two riders, slipstream effects need to be considered. Slipstream changes

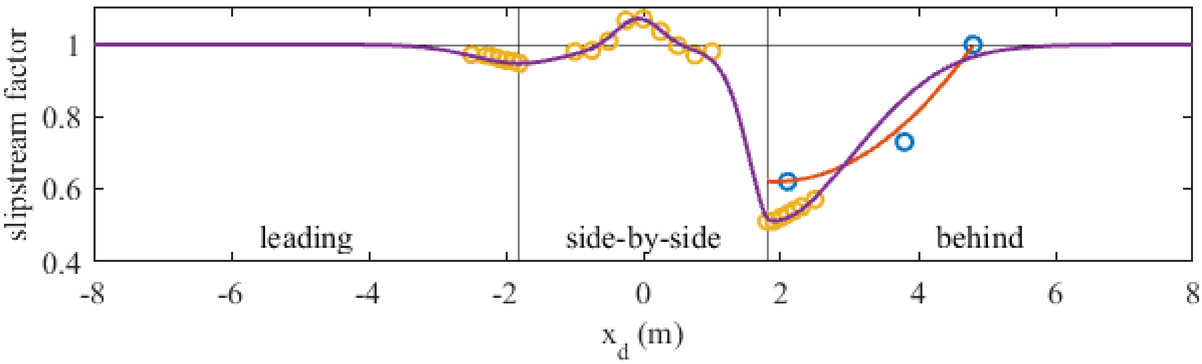

Figure 1. Slipstream factor for wind resistance as a fraction of the distance between the two cyclists, x_d. The blue dots are the measurements of Kyle (1979), and the red curve is the quadratic relation between gap and reduction in wind resistance determined by Olds (1998), based on Kyle's data. The yellow dots are the wind tunnel measurements of Barry et al. (2014). The purple curve is the wind reduction curve defined by our slipstream function $r(x_d)$.

the air resistance by a multiplicative factor $r(x_d)$ that depends on the gap x_d, which is defined as the longitudinal distance between the centers of mass of the two riders. Distances in lateral directions are neglected or implicitly covered by our slipstream formula shown below. Figure 1 shows the modelled slipstream effect together with the experimental measurements of Kyle (1979) and Barry et al. (2014) upon which our formula is based on. The slipstream formula $r(x_d)$ is described as the sum of four exponential functions in order to approximate the experimental data by a smooth function.

$$r(x_d) = e_1(x_d) + e_2(x_d) + e_3(x_d) + e_4(x_d) \tag{2}$$

with the exponential functions

$$\begin{aligned} e_1(x_d) &= -0.053 \exp(-(x_d + l)^2) \\ e_2(x_d) &= 0.076 \exp(-5(x_d + 0.07)^2) \\ e_3(x_d) &= -0.053 \exp(-(x_d - l)^2) \\ e_4(x_d) &= -0.437 \exp(-a(x_d)(x_d - l - d_{\min})^2) \end{aligned}$$

the sigmoid function

$$a(x_d) = -1.85 \tanh\big(10\,(x_d - l)\big) + 2.15$$

and model parameters as shown in Table 1. Therefore, the model formula for the cycling dynamics for two riders is given by

$$\dot{v} = \frac{1}{M}\left[\eta \frac{P}{v} - \left(\frac{1}{2} r(x_d) c_d \varrho A v^2 + mgs(x) + \mu mg + b_0 + b_1 v\right)\right] \stackrel{\text{def}}{=} F_{\text{mech}}(x, x_d, v, P) \tag{3}$$

Physiological Model

The athletes' energy expenditure throughout the race is modelled using the critical power concept introduced by Monod and Scherrer (1965). It incorporates parameters for the maximum available anaerobic energy (E_{an}) and the rate at which the rider can access the aerobic power supplies (critical power P_C). The model was originally designed to estimate time to exhaustion for rides with constant power output. Looking at the hydraulic representation shown in Figure 2, a dynamic version can easily be derived. The change of the remaining amount of energy e_{an} in the anaerobic energy vessel can be described as the difference of the amount of fluid entering the vessel (P_C) and the amount of the fluid leaving the vessel (P), and therefore is given by

$$\dot{e}_{\mathrm{an}} = P_C - P. \tag{4}$$

It is assumed, that e_{an} corresponds inversely to the level of exertion of the athlete and therefore is zero when the athlete is completely exhausted and E_{an} in a fully recovered state.

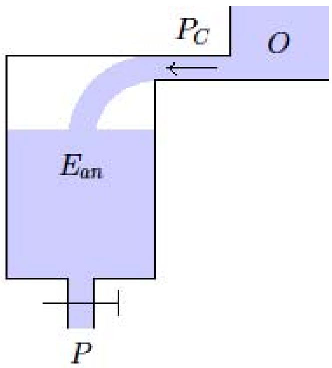

Figure 2. Hydraulic representation of the critical power model. O represents the aerobic energy resources, P_C the rate at which these energy resources can be used (critical power) and E_{an} the anaerobic energy resources. The rider's power output corresponds to the amount of fluid leaving the anaerobic energy vessel E_{an} by opening the tap in the bottom.

Optimal Control Problem

Since strategies are calculated for two cooperating riders, the objective of the optimization is to minimize the race time of the overall slower rider for a given segment on a course. Additionally, a control parameter is introduced to restrict the difference in exertion states between the two riders during the ride. To avoid singularities, regularization variables are used which penalize large variations in the power outputs of the two riders. This leads to the following optimal control problem:

Minimize the cost functional

$$J = T + \underbrace{\varepsilon_1 \int_0^T Q_1(t)^2 + Q_2(t)^2 dt}_{\text{regularization}} + \underbrace{\varepsilon_2 \int_0^T \left(\frac{e_{an,1}(t)}{E_{an,1}} - \frac{e_{an,2}(t)}{E_{an,2}} \right)^2 dt}_{\substack{\text{limitation of differences} \\ \text{in exertion states}}} \tag{4}$$

subject to the dynamic constraints

$$\begin{aligned}
\dot{x}_1(t) &= v_1(t)\\
\dot{x}_d(t) &= v_2(t) - v_1(t)\\
\dot{v}_1(t) &= F_{\text{mech}}\big(x_1(t), x_d(t), v_1(t), P_1(t)\big)\\
\dot{v}_2(t) &= F_{\text{mech}}\big(x_1(t) + x_d(t), -x_d(t), v_2(t), P_2(t)\big)\\
\dot{e}_{an,1}(t) &= P_{C,1} - P_1(t)\\
\dot{e}_{an,2}(t) &= P_{C,2} - P_2(t)\\
\dot{P}_1(t) &= Q_1(t)\\
\dot{P}_2(t) &= Q_2(t)
\end{aligned}$$

the path constraints

$$\begin{aligned}
0 \leq\ & e_{an,1}(t) \leq E_{an,1}\\
0 \leq\ & e_{an,2}(t) \leq E_{an,2}
\end{aligned}$$

and the boundary conditions

$$\begin{aligned}
x_1(0) &= 0\\
x_1(T) &= x_f\\
x_d(0) &= 0\\
x_d(T) &\geq 0\\
v_1(0) &= 0.1\\
v_2(0) &= 0.1\\
e_{an,1}(0) &= E_{an,1}\\
e_{an,2}(0) &= E_{an,2}
\end{aligned}$$

The boundary conditions define that both riders start at the same starting point at the beginning of the track. The race finishes when Rider 1 crosses the finish line and Rider 2 is side by side or ahead. Both riders should start from a standing position. However, since the mechanical model includes a division by the speed, speeds exactly zero are not a feasible option. Therefore, the starting speed is set to 0.1 m/s. The last two conditions specify that both riders are fully recovered at the beginning of the race. Furthermore, there are no restrictions on the number of position changes and the time one rider stays in the lead. This is an outcome of the optimization algorithm.

The optimal control problem is solved numerically by the state-of-the-art optimal control solver GPOPS-II (Patterson and Rao, 2014). Since the algorithm reacts sensitive to the choice of the initial guess, different starting configurations are generated and an optimal solution is calculated for each of them. The best of these solutions is used.

Results

In the following, simulation results on a 5km-long, flat track are shown. At first, the optimal strategy for two equally strong riders as well as details of the overtake process are provided. In the following, the influence of the penalty on large differences of the exertions of the two riders on the optimal strategy is investigated. This parameter is crucial to get solutions with a realistic behavior. In the end, the consequences of the fitness of the riders on the strategy are studied.

Strategies for Two Riders

Figure 3 shows the optimal strategy for our basic case: Both athletes are equally trained with a critical power of 300W and an anaerobic work capacity of 25000J. The weight for the penalty for differences in the remaining anaerobic capacity of the two riders ε_2 is set to 1, while the regularization weight ε_1 is set to 10^{-7}. The overall strategy can be described similar to optimal strategies for individual time trials: A short starting phase with maximum power output to reach the target speed as fast as possible, followed by the main phase in which the target speed is maintained constant. In a short finishing phase, due to complete exertion of the athlete, the speed slightly drops.

Looking at the gap between the two riders it is observed, that Rider 1 (blue) takes over the lead right after the start while Rider 2 (red) gets into the 'sweet-spot' position of around 1.9m behind. After 31 seconds, the riders change position for the first time. More details on this process are provided in the next section. Overall, they need to change position seven times in a periodic pattern to achieve the minimum total race time. Thereby the remaining anaerobic work capacity diminishes roughly linear over the whole race with alternating periods of depletion (leading position) and reconstruction (trailing position).

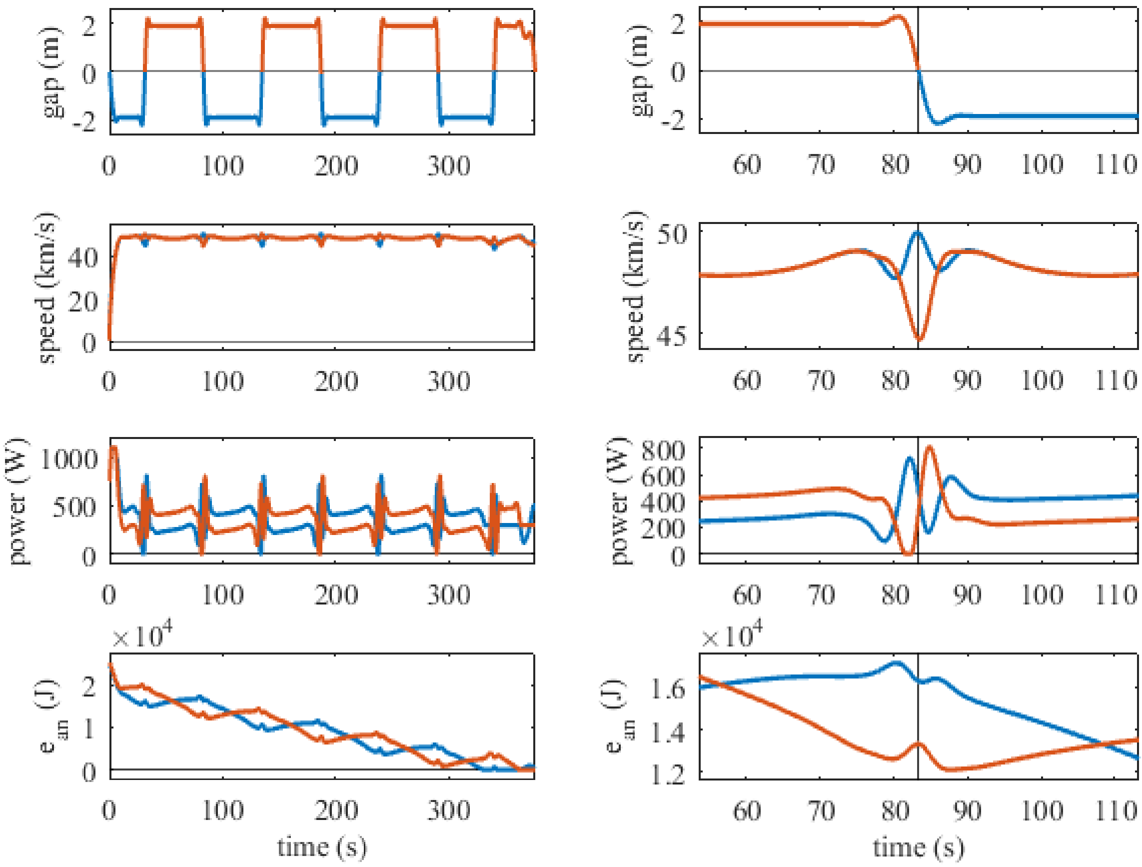

Figure 3: Optimal strategy for two equally strong riders with P_C=300W, E_an=25000J and ε_2=1. The left figure represents the strategy over the whole flat course of 5km length while the right figure gives a detailed view of the overtake process at 83 seconds, where Rider 1 (blue) overtakes Rider 2 (red). The top graph shows the gap between the two riders, blue indicates that the first rider is in lead while red indicates that the second rider is in lead. The following three graphs show the speed, power and remaining anaerobic work capacity for each rider during the race. According to the gap graph, Rider 1 is represented by the blue curve and Rider 2 by the red.

The total race time sums up to 6 minutes and 16 seconds. The time one riders spends in the leading position is nearly balanced with Rider 2 (red) leading 51 percent and Rider 1 (blue) leading 49 percent of the time. The optimal race-time for one individual rider on the same course, under the same conditions is 6 minutes and 54 seconds. That means that compared to an individual time trial, the race time improves by around 10% if two equally strong riders work together.

Change of Leading Position

A detailed view of the overtake process is shown in Figure 3. Rider 2 (red) is leading and Rider 1 (blue) is taking over. Preparing the overtake process, first both riders increase their speed equally, then Rider 1 (blue) reduces the speed slightly to enlarge the gap right before taking over the lead. This enables Rider 1 to accelerate into the draft and gain speed while Rider 2 (red) reduces the speed. Therefore, a very fast change of position is performed, which uses the slipstream in an efficient way. Rider 1 (blue) reduces the speed after passing, while Rider 2 (red) accelerates to get back to a common speed. On the way back to a stable riding configuration, we observe a similar behavior as in the initial phase: The trailing rider is falling behind further than the sweet spot and then accelerates into the emerged gap. After that maneuver, both riders have the same speed and keep the gap with the perfect slipstream distance.

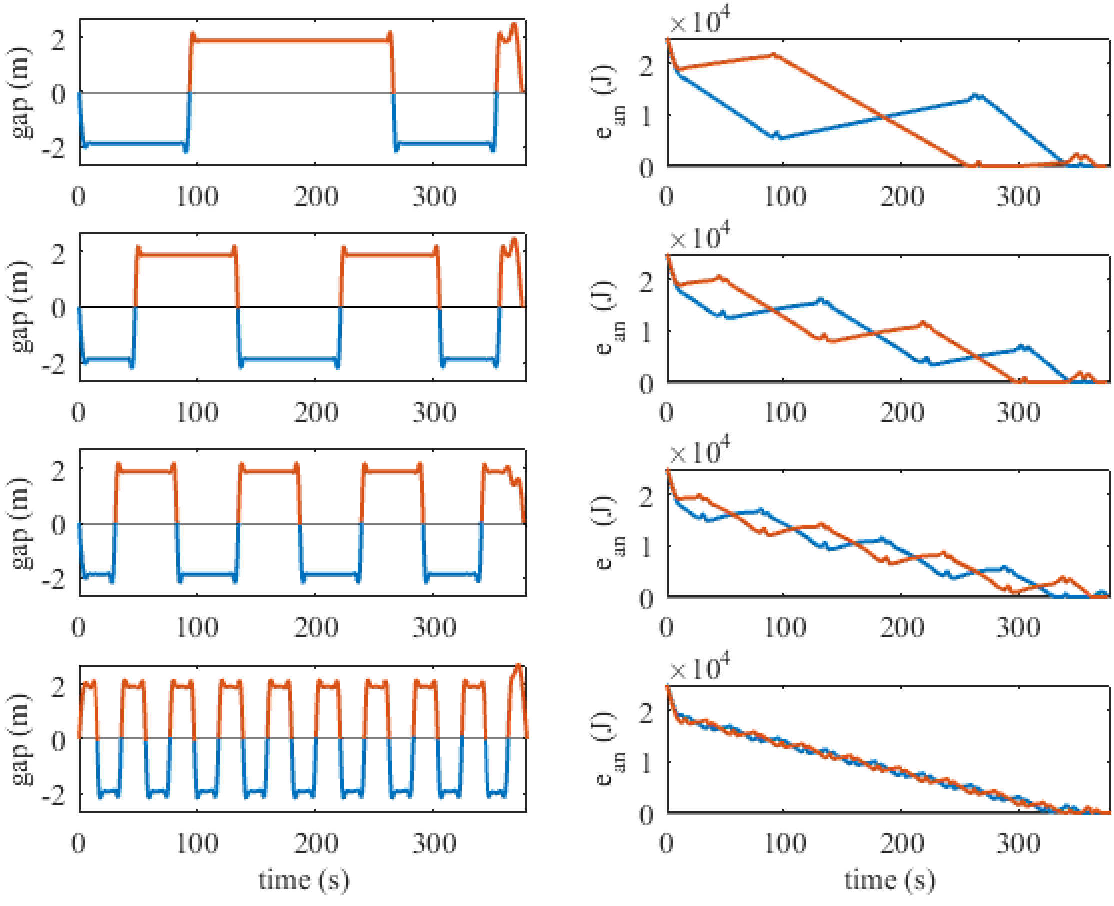

Figure 4: This figure shows the impact of the penalty of differences in exertion state. From a low influence on the top to a high influence on the bottom, the weight ε_2 is equal to 0.01, 0.1, 1 and 10. The graphs on the left show the optimal gap between the two riders and the graphs on the right show the corresponding anaerobic work capacity of Rider 1 (blue) and Rider 2 (red).

Influence of Penalizing Differences in the Fatigue State

In the previous subsections, the weight for the penalty for differences in the remaining anaerobic work capacity of the two riders was set to one. Figure 4 shows how the strategy changes, if we change this parameter. In the top graph, it is set to 0.01, which is also the smallest reasonable value since the strategy does not change if it is further decreased. Which means this corresponds to the optimal strategy if the penalty term is neglected.

Rider 1 (blue) starts the race in the lead. After around one fourth of the ride, Rider 2 (red) takes over and remains in the lead until full exhaustion. Then, Rider 1 (blue) takes over again and likewise stays in the lead until full exhaustion. At that point, Rider 2 (red) is slightly recovered again and able to lead until the end. In contrast to the previous results, it is beneficial to distribute the leading periods irregularly in this scenario. Rider 1 (blue) has two leading periods of medium length while Rider 2 (red) has one long and one very short leading period. Nevertheless, in total both rides share the lead nearly evenly by 48.4% (blue) and 51.6% (red) of the race time.

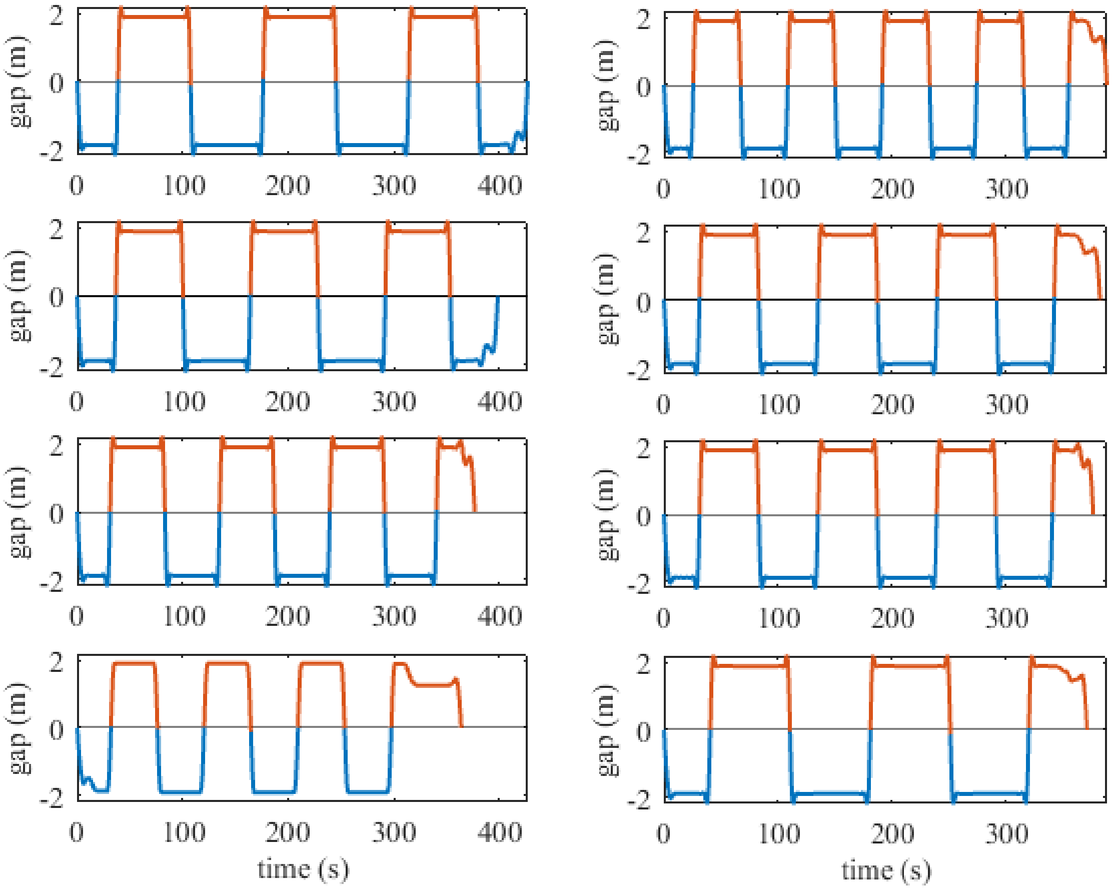

Figure 5: Optimal gap for equally trained athletes with different properties. In the left column the athletes have the same anaerobic work capacity of 25000J but a different critical power of 200W, 250W, 300W and 350W from top to bottom. In the right column, the athletes have the same critical power of 300W but a different anaerobic capacity of 15000J, 20000J, 25000J and 30000J from top to bottom.

In this scenario, we have a high discrepancy in the remaining anaerobic capacity throughout the race. While one rider is highly exhausted, the other one is nearly recovered. By increasing ε_2, the exertion states of the two riders are forced to stay closer together, which induces more position changes: From top to bottom the number of position changes increases, while the remaining anaerobic work capacity curves stay closer together. In the extreme case in the bottom, the position changes constantly and both riders are on a comparable exertion level throughout the race.

Since with increasing ε_2 the problem is more restricted, the optimal race-time increases slightly with ε_2. From 6 minutes 15 seconds in the unrestricted case, over 6 minutes 15 seconds in the second, and 6 minutes 16 seconds in the third, up to 6 minutes 19 seconds in the last case.

Differences in Critical Power and Anaerobic Capacity

In this section, the influence of an identically increased critical power or anaerobic capacity in both athletes is shown. Obviously, the total race-time decreases if we have stronger athletes, in the case of a higher critical power as well as a larger anaerobic capacity. Figure 5 shows the optimal patterns of position changes for different critical power values in the left and different anaerobic work capacities in the right.

For critical power values of 200W and 250W, the pattern of position changes is the same, while the time of each leading phase is reduced due to the reduced overall race-time. Interestingly, if the critical power is increased to 300W or 350W, an additional position change is advantageous and the leading phases get even shorter. In case of riders with a critical power of 350W, we do not have the small gap enlargements before and after the overtake process, and we have a relatively long finishing turn of Rider 2 (red) compared to the earlier, regular phases. This also leads to a difference in the leading time of each rider. While for 200W, 250W and 300W the leading times are nearly equally distributed 50±1%, for 350W, Rider 2 (red) is in the lead for 55% of the total race time.

In the case of variations in the anaerobic capacity, the results are opposite to the findings of critical power variations. While we get more position changes and shorter leading phases for stronger riders in terms of a higher critical power, we get less position changes and longer leading periods for stronger riders in terms of a larger anaerobic capacity. The leading times are nearly equally distributed with 48.7±0.3% for Rider 1 (blue).

Discussion

The simulations show that significant improvements in the total race time are achieved if two riders cooperate in an optimal way. The improvement of the total race time of about 10% for two equally strong riders is explained by exploiting slipstream effects. Since over 90% of the riders' power output is needed to overcome air resistance at high speeds (Martin et al. 1998), slipstream plays a major role in the outcome of a mass-start race. In the perfect position, riding right behind each other, both riders experience a positive effect due to reduced air resistance. Additionally, while the leading rider has to work harder, the trailing rider can recover and save energy for his next turn in front. Besides the position changes and the end of the race where both riders are exhausted, they consistently stay right behind each other to get the most out of the reduction of air resistance due to the slipstream.

In case of the naive extension of the individual time trial optimal control problem without the penalty for differences in the exertion state, the results show very few changes of position and therefore long turns for each rider. Nevertheless, in real races it can be observed that positions are changed permanently, in breakaways as well as team time trials. One explanation for this different behavior can be found in the exertion states of the two riders during the race. At the end of each leading phase, the difference is quite large, up to the situation where one is completely exhausted while the other has decent resources left. This can lead to a major disadvantage in a race situation. In case of a breakaway from the peloton, after working together, at some point the two riders have to compete against each other again. If at this point, one of the riders is significantly more fatigued, he/she has reduced chances to win the race. Therefore, none of the riders want to stay in the lead too long, which induces more position changes. For this reason, the additional penalty in the cost function was introduced. The results then behave as proposed, whereby increasing the influence of the penalty results in a decrease in the difference in the remaining anaerobic capacity of the two riders and more position changes occur. This way, depending on how much risk the riders are willing to take, the optimal number of position changes and therefore the optimal strategy can be determined.

Before the overtake process, an enlargement of the gap is observed. This may seem detrimental, because the trailing rider leaves the perfect slipstream position earlier than necessary. However, this gives the trailing rider the advantage to accelerate into the draft and take this additionally gained speed into the following part, where he/she is out of the slipstream and faces the full wind resistance. Martin et al. (2007) observed a similar behavior for sprinting applications. Starting 1 meter behind gave the second rider the advantage to win the finishing sprint with the same power output as the first rider. Since the acceleration and final speed in a finishing sprint is much higher as in the overtake process, we have a much smaller effect in our proposed strategy. A similar explanation holds for the gap enlargement shortly after the overtake process. Before changing the position, the leading rider reduces his speed in order to let the trailing rider pass as quickly as possible. Therefore, when the riders switch position, the rider whom is overtaken has a lower speed and has to accelerate to catch up to the speed of the other rider. This is again supported by accelerating into the draft.

Looking at the behavior of athletes initiating position changes, their approach seems to be much simpler than the results of our optimization indicate. Mainly, the trailing rider keeps his/her speed while the leading rider slows down to let the other rider pass. One reason for the disagreement between mathematical and practical approach could be, that the solution of the mathematical optimizer is too complex to be applied in practice, since it requires fast, highly synchronized changes of power output. Additionally, in practical situations often more than two riders are working together in a group, especially in team time trials where position changes can be trained and optimized very well. In this case, the dynamics within the group differ from those of two riders and position changes closer to what can be observed in reality seem beneficial. Indeed, preliminary pilot data from similar optimizations for a group of three riders support this hypothesis (data not yet published).

One point that cannot be understood to the extent possible in this study are the contradictory results when physiological parameters are changed. Changing the training status of the athletes either by changing the critical power or by changing the anaerobic work capacity affected the very nature of the strategy. One supposed few, long turns for weaker riders and more, short turns for stronger riders, while the other supposed exactly the opposite. This leads to the conclusion, that not only the absolute values of critical power and anaerobic work capacity are important for the strategy, but also the relationship between those two parameters. Further investigation of this relationship would also be beneficial for training purposes, to get the right balance between endurance and interval training, in order to increase critical power and anaerobic capacity, respectively.

Conclusions

In this article, existing models and optimization approaches for individual time trials were extended to simulate and improve cooperative rides of two athletes. A slipstream model based on experimental data has been provided to simulate the interaction between the two riders. The introduction of a control over the difference in the exertion states of the two riders allowed implicitly adjusting the number of position changes as well as the risk coming with one of the riders being significantly more fatigued. The simulations showed that the cooperation holds a considerable advantage for the two riders, which then for example is needed to stay in front of the peloton in a mass start race. On the other hand, the simulations also showed that the results behave not necessarily in an obvious manner if we introduce variations in the physiological parameters of the riders.

Future work will include empirical data for a closer investigation of practical approaches for strategies of two cooperating riders. Comparing successful breakaways in real races will give more insight in the quality of the simulations. The focus will be on the pattern of position changes and the practical realization of the overtake process. On the other hand, the theoretical considerations may help to analyze and improve racing in practice and increase the chance for a successful breakaway. Another important application are team time trials. Since nowadays teams of six riders compete in time trials against each other, the problem has to be extended from two riders to six riders in order to find an optimal strategy for one group. Besides handling the higher complexity of the mathematical problem, also a suitable slipstream model for several riders is needed to simulate and optimize such rides in general.

Acknowledgment

This work has been supported by a research grant of the Deutsche Forschungsgemeinschaft (DFG) for the project "Powerbike - Model-based optimization for road cycling." (SA 449/12-1)

We thank the anonymous reviewers for their very helpful comments and suggestions.

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