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Added Mass of Fluid and Fundamental Frequencies of a Horizontal Elastic Circular Plate Vibrating in Fluid of Constant Depth

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Abstract

The paper deals with free vibrations of a horizontal thin elastic circular plate submerged in an infinite layer of fluid of constant depth. The motion of the plate is accompanied by the fluid motion, and thus, the pressure load on this plate results from displacements of the plate in time. The plate and fluid motions depend on boundary conditions, and, in particular, the pressure load depends on the gap between the plate and the fluid bottom. In theoretical description of this phenomenon, we deal with a coupled problem of hydrodynamics in which the plate and fluid motions are coupled through boundary conditions at the plate surfaces. This coupling leads to the so-called co-vibrating (added) mass of fluid, which significantly changes the fundamental frequencies (eigenfrequencies) of the plate. In formulation of the problem, a linear theory of small deflections of the plate is employed. At the same time, one assumes the potential fluid motion with the potential function satisfying Laplace's equation within the fluid domain and appropriate boundary conditions at fluid boundaries. In order to solve the problem, the infinite fluid domain is divided into sub-domains of simple geometry, and the solution of problem equations is constructed separately for each of these domains. Numerical experiments have been conducted to illustrate the formulation developed in this paper.

Key words: circular elastic plate, free vibrations, co-vibrating mass of fluid, eigenfrequencies

1. Introduction

In offshore engineering, we frequently deal with the problem of water wave-induced loads on structures. These loads depend on fluid flows in the vicinity of the structure, as well as on the its size, shape, rigidity and foundation. An example of such a structure is a horizontal circular plate foundation of a windmill installed in the sea coastal zone. Usually, hydrodynamic forces depend not only on water waves themselves, but also on the foundation of the plate and its orientation relative to the directions of wave propagation. In the present case of horizontal plate, placed at a small distance from the sea bottom, these forces depend also on the distance between the plate and the fluid bottom. In general, a motion of the plate is accompanied by motion of the fluid and therefore, in analyzing vibrations of the plate submerged in fluid one can speak on certain amount of fluid vibrating together with this plate. As compared to vibrations of the plate in air, this added (co-vibrating) mass of fluid changes fundamental frequencies of the plate significantly.

With respect to the above, we focus our investigations on the coupled hydrodynamic problem of a horizontal circular plate, vibrating in a layer of fluid of constant depth. In order to simplify our discussion, we confine our attention to small deflections of a simply supported plate and a potential motion of the incompressible non-viscous fluid. In theoretical investigations, we resort to approximate modeling that can describe the main features of this phenomenon. As regards vibrations of plates in contact with fluid, Solecki (1966) discussed the problem of an infinite plate floating on a water half-space. A similar problem of deformation of floating ice plates was investigated by Kerr and Palmer (1972). As far as a finite fluid body is concerned, Sawicki (1975) discussed the problem of dynamics of floating roofs of cylindrical tanks. The problem discussed in the present paper corresponds in a sense to that of Sawicki's problem, but it deals with an infinite fluid domain and fully submerged plate. Our main goal is to calculate a set of lowest eigenfrequencies of the plate, dependent on the width of the gap between the plate and the sea bottom.

2. Problem Formulation

Let us consider the three-dimensional problem of a thin elastic circular plate submerged in fluid, as shown schematically in Fig. 1. The plate is assumed to be of small thickness and its deflections are so small that in the description of the plate motion a linear theory may be applied. The motion of the plate is accompanied by the fluid motion, and thus we have the coupled problem of hydrodynamics. This coupling takes place through boundary conditions at the upper and bottom surfaces of the plate. In the present problem of plate vibrations, transverse deflections of the plate (displacements of its central plane) are governed by the following equation (e.g. Nowacki 1972):

$$\nabla^2\nabla^2 w + \frac{m_{pl.}}{D^*}\frac{\partial^2 w}{\partial t^2} = \frac{q}{D^*}, \tag{1}$$

where w is the plate deflection, q is the external load continuously distributed over the plate surface, $D^* = E\delta^3/12(1-\nu^2)$ is the flexural rigidity of the plate (δ is the plate thickness and ν is Poisson's ratio), and $\nabla^2\nabla^2$ is the bi-harmonic operator. In the case of vibrations of the plate submerged in fluid, the external load q in this equation equals the fluid pressure. In order to calculate this pressure, it is necessary to solve the coupled problem of the plate-fluid motion. To this aim, it is assumed that the fluid is non-viscous and incompressible, and its motion is potential, with the potential function $\Phi(x, y, z, t) = \Phi(r, \varphi, z, t)$ satisfying the harmonic (Laplace's) equation

$$\nabla^2\Phi = 0 \tag{2}$$

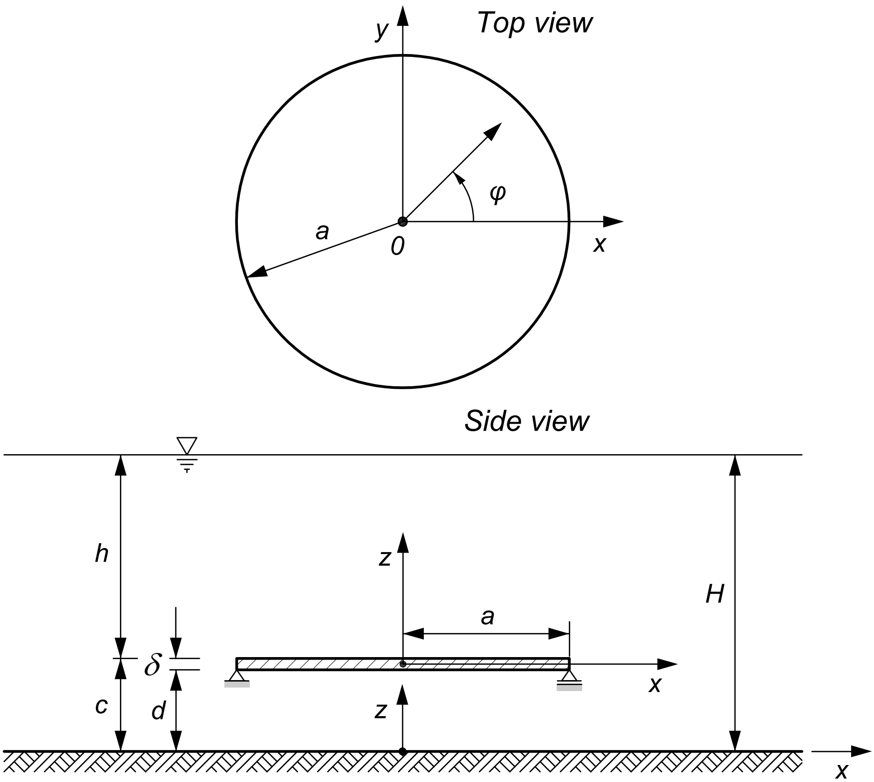

Fig. 1. Simply supported elastic circular plate submerged in an infinite layer of fluid

within the fluid domain and appropriate boundary conditions at fluid boundaries.

With respect to small vibrations of the plate, placed at a sufficiently large distance from the free fluid surface, it is reasonable to assume that the free surface is flat over the entire range of time considered (fluid pressure is constant at $z = H$), and thus

$$\Phi|_{z=H} = 0\,. \tag{3}$$

With this assumption applied, the plate-fluid system is a conservative system, i.e. its total energy remains constant during free vibrations. It means that, as in the case of free vibrations of the plate in air, it is possible to calculate a set of fundamental frequencies (eigenfrequencies) of the plate immersed in fluid. In addition to condition (3), the remaining boundary conditions for the potential function read:

$$\begin{gathered}\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = 0, \quad \Phi|_{r\to\infty} = 0, \quad \left.\frac{\partial\Phi}{\partial r}\right|_{r\to\infty} = 0,\\ \pm\frac{\partial\Phi}{\partial n} \cong \left.\frac{\partial\Phi}{\partial z}\right|_{\substack{\textit{upper and}\\ \textit{lower plate}\\ \textit{surfaces}}} = \frac{\partial w}{\partial t}.\end{gathered} \tag{4}$$

For a harmonic motion of the plate-fluid system in time, these boundary conditions should satisfy the Sommerfeld condition that no wave comes from infinity ($r \to \infty$) (no generation sources of the fluid motion exist at infinity). The index n in equations (4) denotes the outward unit vector normal to the fluid boundary at the plate surface. At end points of the plate (at $r = a$), the fluid velocity field has removable singularity. The fluid pressure is described by the formula

$$p = -\rho \frac{\partial \Phi}{\partial t} + \rho g (z - H) \ . \tag{5}$$

Substitution of this relation into equation (1) gives

$$\nabla^2 \nabla^2 w + \frac{m^*}{D^*} \left[\frac{\partial^2 w}{\partial t^2} + \frac{\rho}{m^*} \left(\left. \frac{\partial \Phi}{\partial t} \right|_{low.} - \left. \frac{\partial \Phi}{\partial t} \right|_{upp.} \right) \right] = 0, \tag{6}$$

where ρ is the fluid density, g is the gravitational acceleration, and $m^* = (\rho_{plate} - \rho_{fluid}) \cdot \delta$.

In order to construct a solution to the aforementioned problem, it is reasonable to consider, in the first step, a simpler case of free vibrations of the plate in air.

3. Free Vibrations of the Plate in Air

In accordance with the above, let us consider now the plate vibrating in air. The plate is simply supported at its perimeter. For free vibrations, $q = 0$ in equation (1), and the problem equation is reduced to the following one:

$$\nabla^2 \nabla^2 w + c^2 \frac{\partial^2 w}{\partial t^2} = 0, \tag{7}$$

where

$$c^2 = \frac{\rho_{plate} \delta}{D^*} \ . \tag{8}$$

For a harmonic motion in time, the following substitution is made

$$w(r, \varphi, t) = W(r, \varphi) \exp(i \omega t), \tag{9}$$

where ω is the vibration frequency, and i is an imaginary unit. From substitution of this description into equation (7), one obtains

$$\nabla^2 \nabla^2 W - \lambda^4 W = \left(\nabla^2 + \lambda^2 \right) \left(\nabla^2 - \lambda^2 \right) W(r, \varphi) = 0, \tag{10}$$

where

$$\lambda^4 = (\omega \cdot c)^2 \ . \tag{11}$$

With respect to the polar system of coordinates shown in Fig. 1, the harmonic operator in equation (10) reads

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \ . \tag{12}$$

Knowing that $\lambda^2 > 0$, a solution of equation (10) is reduced to solutions of the following two equations:

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \varphi^2} + \lambda^2 W = 0 \tag{13}$$

and

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \varphi^2} - \lambda^2 W = 0. \tag{14}$$

In order to find solutions of these equations, we resort to the Fourier method of separations of variables, i.e.

$$W(r, \varphi) = R(r) \cdot \Theta(\varphi) \ . \tag{15}$$

Substitution of this equation into equation (13) and simple manipulations give

$$\begin{aligned} &\frac{\partial^2 \Theta}{\partial \varphi^2} + m^2 \Theta = 0, \\ &\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \left(\lambda^2 - \frac{m^2}{r^2}\right) R = 0. \end{aligned} \tag{16}$$

It should be stressed that for the circular plate considered, the solution must be periodic in φ, and therefore m is an integer greater than zero. The case $m = 0$ is also admissible. Separation of variables in equation (14) leads to the same equation for $\Theta(\varphi)$ (first equation in 16), but now, instead of the second equation of (16), we have the following one:

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \left(\lambda^2 + \frac{m^2}{r^2}\right) R = 0. \tag{17}$$

The second equation of (16) and equation (17) are Bessel equations for $R(r)$ (McLachlan 1964). The general solution of the second equation of (16) reads

$$R(r) = A \cdot J_m(\lambda r) + B \cdot Y_m(\lambda r). \tag{18}$$

where A and B are constants, and J_m and Y_m are Bessel functions of the first and second kind of order m, respectively. At the same time, the solution of equation (17) assumes the form

$$R(r) = C \cdot I_m(\lambda r) + D \cdot K_m(\lambda r), \tag{19}$$

where C and D are constants, and I_m and K_m are modified Bessel functions of the first and second kind of order m. The solution of the first equation of (16) is represented by trigonometric functions

$$\Theta(\varphi) = E \cdot \cos(m\varphi) + F \cdot \sin(m\varphi), \tag{20}$$

where E and F are constants.

Without loss of generality, we may confine our attention to a single term in the above solution, say $\cos(m\varphi)$, which is an even function in φ. And thus, the general solution of equation (10) may be written in the following form:

$$W(r,\varphi) = \sum_{m=0,1,\dots} \cos(m\varphi)\left[A^m J_m(\lambda r) + B^m Y_m(\lambda r) + C^m I_m(\lambda r) + D^m K_m(\lambda r)\right], \quad (21)$$

where A^m, ..., D^m are constants of this solution. Bessel functions of the second kind in this equation are going to infinity when radius goes to zero, and therefore, these functions should be cancelled out. Finally, the general solution of the problem considered is reduced to the following one:

$$W(r,\varphi) = \sum_{m=0,1,\dots} \cos(m\varphi)\left[A^m J_m(\lambda r) + B^m \frac{I_m(\lambda r)}{I_m(\lambda a)}\right] . \quad (22)$$

This solution should satisfy boundary conditions at the support of the plate at $r = a$ (see Fig. 1). For the simply supported plate, its deflection W at this support and the associated bending moment M_{rr} should be equal to zeros. Accordingly, the following conditions hold (Nowacki 1972):

$$\begin{aligned} &W|_{r=a,\varphi} = 0, \\ &M_{rr}|_{r=a,\varphi} = -D^* \left[\frac{\partial^2 W}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \varphi^2}\right)\right]\Bigg|_{r=a,\varphi} = 0, \end{aligned} \quad (23)$$

where ν is Poisson's ratio. From the first condition in these relations and equation (22), it follows that

$$B^m = -A^m J_m(\lambda a) . \quad (24)$$

Substituting equations (22) and (24) into the second condition of (23) and making simple manipulation, we arrive at the homogeneous equation

$$A^m \left\{\frac{d^2 J_m(z)}{dz^2} + \frac{\nu}{z}\frac{dJ_m(z)}{dz} - \frac{J_m(z)}{I_m(z)}\left[\frac{d^2 I_m(z)}{dz^2} + \frac{\nu}{z}\frac{dI_m(z)}{dz}\right]\right\}\Bigg|_{r=a} = 0, \quad (25)$$

where $z = \lambda r$. In order to obtain nontrivial solutions of the problem considered, it is necessary to find roots of this equation, i.e. a set of values of $z = z_k (k = 1, 2, ...)$ for which the multiplier of A^m in this equation equals zero. Knowing that (McLachlan 1964)

$$\begin{aligned} &\frac{dJ_0(z)}{dz} = -J_1(z), \quad \frac{dJ_m(z)}{dz} = \frac{1}{2}\left[J_{m-1}(z) - J_{m+1}(z)\right], \quad m = 1, 2, \dots, \\ &J_{-m}(z) = (-1)^m J_m(z), \\ &\text{and} \\ &\frac{dI_0(z)}{dz} = I_1(z), \quad \frac{dI_m(z)}{dz} = \frac{1}{2}\left[I_{m-1}(z) + I_{m+1}(z)\right], \quad m = 1, 2, \dots, \\ &I_{-m}(z) = I_m(z), \end{aligned} \quad (26)$$

the problem of nontrivial solutions of (25) is reduced to zeros of the following functions:

$$R^m(z) = z\,(J_{m-2}(z) - 2J_m(z) + J_{m+2}(z)) + 2\nu\,(J_{m-1}(z) - J_{m+1}(z)) + \\ - \frac{J_m(z)}{I_m(z)}\,[z\,(I_{m-2}(z) + 2I_m(z) + I_{m+2}(z)) + 2\nu\,(I_{m-1}(z) + I_{m+1}(z))]\,, \\ m = 0, 1, ... \tag{27}$$

In order to find a set of roots (zeros) of this equation, we resort to discrete numerical calculations. Numerical calculations are made for an assumed range of the independent variable z in this equation ($0 < z < z_{\max}$). Poisson's ratio in this equation is assumed equal to 0.3. Some of the results obtained in computations are shown in Fig. 2, where the plots illustrate distributions of the functions $R^m(z)$ for chosen numbers $m = 0, 1, ...$ The set of roots z_i^m (zeros of the functions $R^m(z)$) is set down in Table 1. It is important to note that the roots of (27) depend only on Poisson's ratio, which is assumed to be constant for all cases considered. In this way, the eigenfrequencies ω_i^m of a specified plate of given thickness δ and radius a are obtained directly by means of the general eigenfrequencies – zeros $z_i^m = \lambda_i^m a$ of the fundamental relation (27). The plate eigenfrequencies are described by the formula (compare with equations 8 and 11)

$$\omega_i^m = \left(\frac{z_i^m}{a}\right)^2 \sqrt{\frac{D^*}{(\rho \cdot \delta)_{plate}}} = (\lambda_i^m)^2 \sqrt{\frac{D^*}{(\rho \cdot \delta)_{plate}}}\,. \tag{28}$$

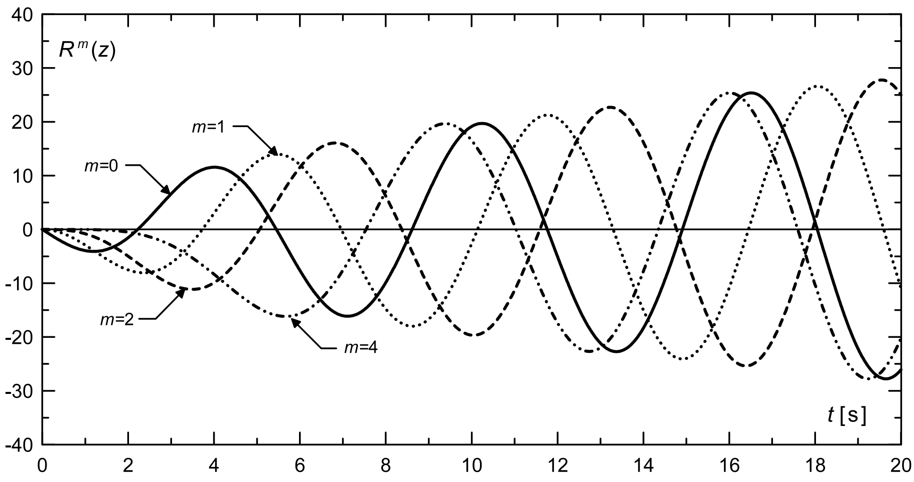

Fig. 2. Distributions of functions $R^m(z)$ for chosen values of m

Table 1. Roots z_i^m of equation (27)

i	m				
	0	1	2	3	4
1	2.2215	3.7280	5.0609	6.3211	7.5393
2	5.4515	6.9626	8.3735	9.7236	11.0318
3	8.6113	10.1377	11.5886	12.9874	14.3475
4	11.7602	13.2966	14.7716	16.2013	17.5956
5	14.9068	16.4488	17.9399	19.3910	20.8098
6	18.0512	19.5976	21.1001	22.5669	24.0042
7	21.1948	22.7444	24.2554	25.7343	27.1860
8	24.3378	25.8899	27.4076	28.8960	30.3593
9	27.4805	29.0346	30.5576	32.0538	33.5265
10	30.6230	32.1786	33.7060	35.2086	36.6893

For known values of $\lambda_i^m = z_i^m \big/ a$, equation (22) and the first condition in (23) lead to the following set of eigenfunctions of the plate vibrating in air:

$$F_i^m(r,\varphi) = \cos(m\varphi)\left[J_m\left(\lambda^i r\right) - \frac{J_m\left(\lambda^i a\right)}{I_m\left(\lambda^i a\right)} I_m\left(\lambda^i r\right)\right], \; \lambda^i = \frac{z_i^m}{a}, \qquad m = 0, 1, ..., \quad i = 1, 2, ... \tag{29}$$

It is important to note here that m and i are independent numbers. It is a simple matter to prove that for different values of λ_i^m and λ_j^n (with $i \neq j$), the functions $F_i^m(r,\varphi)$ are orthogonal within the plate domain, i.e. for arbitrary numbers $i \neq j$, the following relation holds:

$$\iint_S F_i^m F_j^n dS = \int_0^{2\pi}\int_0^a F_i^m(r,\varphi) \cdot F_j^n(r,\varphi) \cdot r \, dr d\varphi = 0. \tag{30}$$

This important property will be exploited in the further part of this research, when vibrations of the plate submerged in fluid are considered. The solution obtained is illustrated in Fig. 3, where the plots show the distribution of $F_i^m(r, \varphi = \text{const.})$ within the range $0 \le r \le a$.

4. Free Vibrations of the Plate Submerged in Fluid

With respect to equation (6), it is necessary to find the potential $\Phi(r,\varphi,z,t)$ satisfying the harmonic equation and appropriate boundary conditions. As in the previous case of vibrations in air, the steady harmonic problem is considered in which the time factor may be eliminated from equations describing the plate-fluid motion. Thus, the following substitutions are made:

$$\begin{aligned} w(r,\varphi,t) &= W(r,\varphi) \cdot \exp(i\omega t), \\ \Phi(r,\varphi,z,t) &= i\omega \cdot \phi(r,\varphi,z) \cdot \exp(i\omega t). \end{aligned} \tag{31}$$

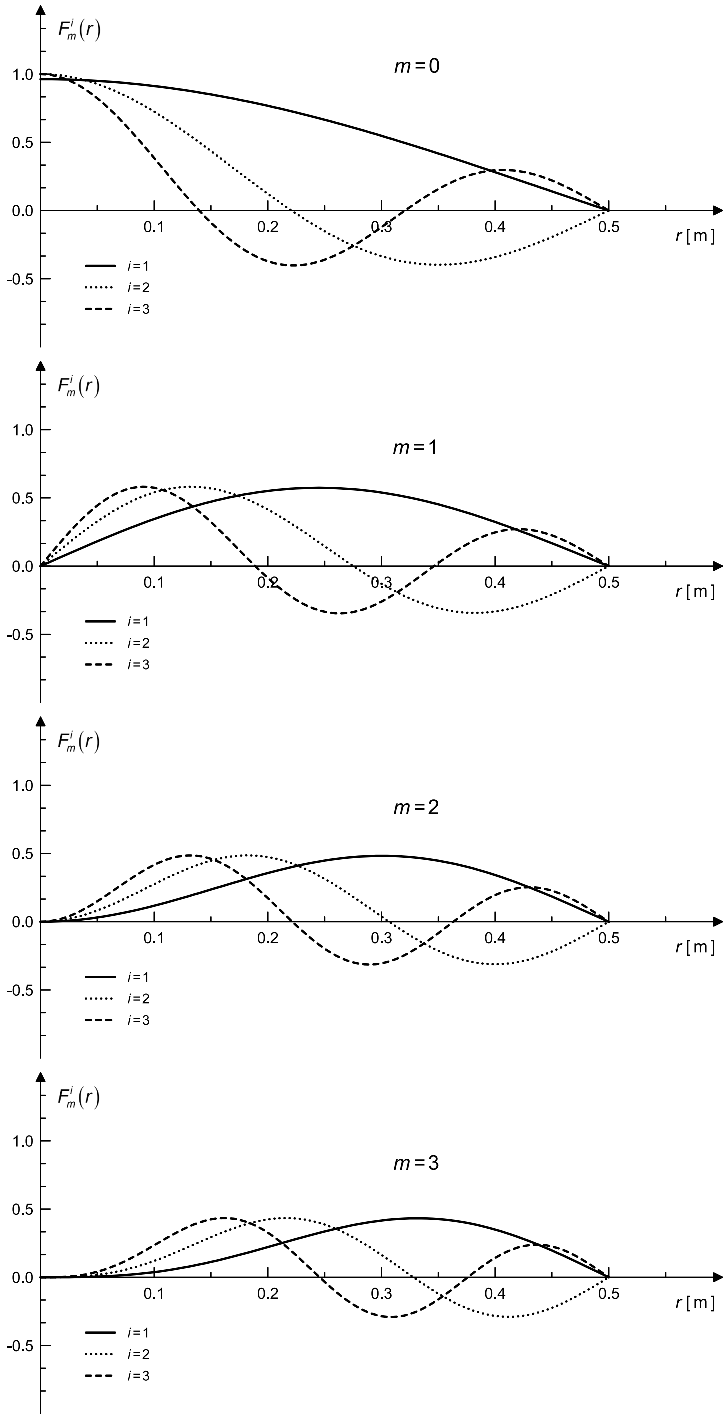

Fig. 3. Eigenfunctions of the plate vibrating in air

These equations allow us to reduce the description of the phenomenon to two space functions: $W(r,\varphi)$ and $\phi(r,\varphi,z)$. At the same time, the boundary condition at the upper and lower plate surfaces reads

$$\frac{\partial w}{\partial t} = \frac{\partial \Phi}{\partial z}\bigg|_{at\ plate\ surface} = W(r,\varphi). \tag{32}$$

In the further discussion, it is convenient to divide the fluid domain into three parts: the finite cylindrical domain below the plate, the finite domain above the plate and the infinite layer of fluid except for these finite cylindrical domains and the plate. In description of the problem within these parts, it is convenient to employ separate vertical coordinates (z variable in the potential functions). In accordance with results of the previous section, the unknown deflection $W(r,\varphi)$ of the plate is expressed in terms of the eigenfunctions $F_i^m(r,\varphi)$ obtained for the plate vibrating in air. Thus, in place of equation (22), we have

$$W(r,\varphi) = \sum_{i=1,2,\ldots}\ \sum_{m=0,1,\ldots} A_i^m \cdot F_i^m(r,\varphi)\ . \tag{33}$$

In the further discussion, however, another description of this deflection will also be employed, which corresponds directly to equation (22):

$$W(r,\varphi) = \sum_{i=1,2,\ldots}\ \sum_{m=0,1,\ldots} \cos(m\varphi)\cdot\left[A_i^m J_m(\lambda_i r) + B_i^m \frac{I_m(\lambda_i r)}{I_m(\lambda_i a)}\right]\ . \tag{34}$$

This formula is more convenient in describing boundary conditions at the plate. In order to save space and make our discussion clear, we omit the summation with respect to 'i' in our further description of the problem considered. Thus, keeping in mind this summation, and with respect to boundary conditions at the fluid sub-domains, the associated potential functions are expressed in the following forms:
– lower fluid domain

$$\begin{aligned}\phi(r,\varphi,z) = E_0 + \sum_{m=0,1,\ldots} \cos m\varphi \cdot \Bigg[A^m \frac{\cosh\lambda z}{\lambda\sinh\lambda d}\cdot J_m(\lambda r) + \\ -B^m \frac{\cos\lambda z}{\lambda\sin\lambda d}\frac{I_m(\lambda r)}{I_m(\lambda a)} + \sum_{n=1,2,\ldots} E_n^m \cos k_n z\cdot\frac{I_m(k_n r)}{I_m(k_n a)}\Bigg],\ k_n = \frac{n\pi}{d},\ n=1,2\ldots\end{aligned} \tag{35}$$

where E_0 and E_n^m are constants;
– upper fluid domain

$$\begin{aligned}\phi(r,\varphi,z) = \textstyle\sum_{m=0,1,\ldots} \cos m\varphi\cdot\Big[-A^m \frac{1}{\lambda w^*}\left(\exp(-\lambda z) - \exp\lambda(z-2h)\right)\cdot J_m(\lambda r) + \\ +B^m\frac{1}{\lambda}\left(\sin\lambda z - \sin\lambda h\right)\frac{I_m(\lambda r)}{I_m(\lambda a)} + \sum_{n=1,2,\ldots} D_n^m \cos k_n^* z\cdot\frac{I_m(k_n^* r)}{I_m(k_n^* a)}\Bigg], \\ w^* = 1+\exp(-2\lambda h),\qquad k_n^* = \frac{(2n-1)\pi}{2h},\ n=1,2\ldots,\end{aligned} \tag{36}$$

where D_n^m are constants;
– infinite layer of fluid

$$\phi(r,\varphi,z) = \sum_{m=0,1,\ldots} \cos m\varphi \sum_{j=1,2,\ldots} C_j^m \cdot K_m(k_j r)\cos k_j z, \quad k_j = \frac{(2j-1)\pi}{2H}, \quad j = 1,2\ldots, \tag{37}$$

where C_j^m are constants.

One can check that equations (36) and (37) satisfy boundary condition (32). A remark is needed. The multiplier of B^m in equation (35) contains the denominator $\lambda \cdot \sin \lambda d \cdot I_m(\lambda a)$. It may happen that $\lambda d = s \cdot \pi$, where s is an integer, and thus, the denominator goes to zero. This case corresponds to the trivial solution $W(r,\varphi) = 0$, and therefore, the associated components with A^m and B^m in the potential function (35) should be cancelled out. It means that only cases with $\lambda d \neq s \cdot \pi$ are taken into account in equation (35). Obviously, solutions (35), (36) and (37) must satisfy boundary conditions at the common boundaries of the fluid sub-domains at $r = a$. These conditions mean that the fluid pressure and the fluid velocity at these boundaries must be uniquely defined by the solutions mentioned. Accordingly, the constants E^0, E_n^m, D_n^m and C_j^m in equations (35), (36) and (37) are not independent. In order to find relations between them, let us consider, in the first step, the pressure condition at the common boundaries of the lower and infinite fluid domains. At $r = a$, the following relation holds:

$$\begin{gathered} E_0 + \\ + \sum_{m=0,1,\ldots} \cos m\varphi \cdot \left[A^m \frac{\cosh \lambda z}{\lambda \sinh \lambda d} \cdot J_m(\lambda a) - B^m \frac{\cos \lambda z}{\lambda \sin \lambda d} + \sum_{n=1,2,\ldots} E_n^m \cos k_n z \right] = \\ = \sum_{m=0,1,\ldots} \cos m\varphi \cdot \sum_{j=1,2,\ldots} C_j^m \cdot K_m(k_j a) \cos k_j z, \\ k_n = \frac{n\pi}{d}, \quad k_j = \frac{2j-1}{2H}\pi, \quad n, j = 1,2,\ldots \end{gathered} \tag{38}$$

In order to find the desired relations between constants of the solution, equation (38) is multiplied in succession by $1, \cos k_1 z, \ldots, \cos k_n z, \ldots$ and then integrated in the range $0 \le z \le d,\ 0 \le \varphi \le 2\pi$. The respective integrals of this procedure read

$$JA^n = \int_0^d \frac{\cosh \lambda z}{\lambda \sinh \lambda d} \cos k_n z \, dz = \frac{(-1)^n}{\lambda^2 + (k_n)^2}, \quad k_n = \frac{n\pi}{d}, \ n = 0,1,2,\ldots, \tag{39}$$

$$JB^n = \int_0^d \frac{\cos \lambda z}{\lambda \sin \lambda d} \cos k_n z \, dz = \begin{cases} \dfrac{(-1)^n}{\lambda^2 - (k_n)^2}, & \lambda \neq k_n \,, \\ \lambda = k_n \ \text{ excluded from solution} \,, \end{cases} \tag{40}$$

$$JC_j^n = \int\limits_0^d \cos k_j z \cdot \cos k_n z \, dz = \begin{cases} \dfrac{(-1)^n k_j}{(k_j)^2 - (k_n)^2} \sin k_j d \quad \text{for} \quad k_j \neq k_n \, , \\ \dfrac{d}{2} \quad \text{for} \quad k_j = k_n \, . \end{cases} \tag{41}$$

These formulae hold for arbitrary $\lambda = \lambda_1, \lambda_2, \ldots$. For $m = 0$ and $k_n = 0$, the above procedure gives

$$E_0 = \frac{1}{d} \left\{ \frac{1}{\lambda^2} \left[-A^0 J_0(\lambda a) + B^0 \right] + \sum_{j=1,2...} C_j^0 \cdot K_0(k_j a) \frac{\sin k_j d}{k_j} \right\}. \tag{42}$$

In a similar way, for $n = 1, 2, \ldots$, $m = 0, 1, \ldots$, the following relation is obtained:

$$E_n^m = \frac{2}{d} \left[-A^m J_m(\lambda a) \cdot JA^n + B^m \cdot JB^n + \sum_{j=1,2,...} C_j^m \cdot K_m(k_j a) \cdot JC_j^n \right]. \tag{43}$$

Equations (42) and (43) enable us to express all the constants E^0, E_n^m ($n = 1, 2, \ldots$, $m = 0, 1, \ldots$) in terms of the remaining constants, i.e. in terms of A^m, B^m and C_j^m. It should be stressed that all these relations depend on λ. For the upper fluid domains, the associated boundary conditions at the common boundaries lead to the following relation:

$$\begin{aligned} \phi(r = a) = \sum_{m=0,1,...} \cos m\varphi \cdot \Bigg[-A^m \frac{1}{\lambda w^*} \left(\exp(-\lambda z) - \exp \lambda(z - 2h) \right) \cdot J_m(\lambda a) + \\ + B^m \frac{1}{\lambda} (\sin \lambda z - \sin \lambda h) + \sum_{n=1,2,...} D_n^m \cos k_n^* z \Bigg] = \\ = \sum_{m=0,1,...} \cos m\varphi \cdot \sum_{j=1,2,...} C_j^m \cdot K_m(k_j a) \cdot \cos k_j (z + c), \end{aligned} \tag{44}$$

$$w^* = 1 + \exp(-2\lambda h), \;\; k_n^* = \frac{(2n-1)\pi}{2h}, \;\; n = 1, 2 \ldots,$$
$$k_j = \frac{(2j-1)\pi}{2H}, \quad j = 1, 2, \ldots$$

A consecutive multiplication of terms in this equation by $\cos k_1^* z$, $\cos k_2^* z$, ... and integration of results in the range $0 \le z \le h$, gives the following set of integrals:

$$\begin{aligned} KA^n = \int\limits_0^h \frac{[\exp(-\lambda z) - \exp \lambda(z - 2h)]}{\lambda w^*} \cos k_n^* z \, dz = \frac{1}{\lambda^2 + (k_n^*)^2}, \\ k_n^* = \frac{(2n-1)\pi}{2h}, \;\; n = 1, 2, \ldots, \end{aligned} \tag{45}$$

$$KB^n = \int_0^h \frac{(\sin\lambda z - \sin\lambda h)}{\lambda}\cos k_n^* z\, dz =$$
$$= \begin{cases} \dfrac{1}{\lambda^2 - (k_n^*)^2}\left[1 + \dfrac{(-1)^n \lambda \sin\lambda h}{k_n^*}\right] & \text{for } \lambda \neq k_n^*, \\ -\dfrac{1}{2}\lambda^2 \quad \text{for} \quad \lambda = k_n^*, \end{cases} \tag{46}$$

$$KC_j^n = \int_0^h \cos k_j(z+c)\cos k_n^* z\, dz =$$
$$= \begin{cases} \dfrac{k_j \sin k_j c}{(k_n^*)^2 - (k_j)^2} & \text{for } k_n^* \neq k_j, \\ \dfrac{1}{2k_j}\left(k_j h \cdot \cos k_j c - \sin k_j c\right) & \text{for } k_j = k_n^*. \end{cases} \tag{47}$$

With respect to these formulae and the procedure applied, equation (44) gives

$$D_n^m = \frac{2}{h}\left[A^m J_m(\lambda a)\cdot KA^n - B^m \cdot KB^n + \sum_{j=1,2,\ldots} C_j^m \cdot K_m(k_j a)\cdot KC_j^n\right]. \tag{48}$$

Equations (43) and (48) allow us to eliminate two sets of constants from the equations of the problem considered. In order to eliminate the set of constants C_j^m, the remaining boundary condition is applied that the velocity component normal to the fluid boundary between the finite (cylindrical) and infinite fluid domains (infinite layer of fluid) must be the same. To write this condition, it is necessary to calculate the normal fluid velocities. Following the potential functions described by equations (35), (36) and (37), the radial fluid velocities are:

– lower fluid domain

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \sum_{m=0,1,\ldots}\cos m\varphi\cdot\left[A^m \frac{\cosh\lambda z}{\sinh\lambda d}\cdot\frac{dJ_m(u)}{du} + B^m\frac{\cos\lambda z}{\sin\lambda d}\frac{dI_m(u)}{du}\frac{1}{I_m(\lambda a)} + \right.$$
$$\left. + \sum_{n=1,2,\ldots} E_n^m k_n \cos k_n z\cdot\left.\frac{dI_m(u_n)}{du_n}\right|_{u_n=k_n a}\frac{1}{I_m(k_n a)}\right], \tag{49}$$
$$u = \lambda r, \quad u_n = k_n r, \quad k_n = \frac{n\pi}{d}, \quad n = 1, 2\ldots$$

– upper fluid domain

$$
\begin{aligned}
\left.\frac{\partial\phi}{\partial r}\right|_{r=a} &= \sum_{m=0,1,\ldots} \cos m\varphi \cdot \left[-A^m \frac{1}{w^*}\left(\exp(-\lambda z) - \exp\lambda(z-2h)\right) \cdot \frac{dJ_m(u)}{du} + \right. \\
&\quad + B^m \left(\sin\lambda z - \sin\lambda h\right) \frac{dI_m(u)}{du}\frac{1}{I_m(\lambda a)} + \\
&\quad \left. + \sum_{n=1,2,\ldots} D_n^m k_n^* \cos k_n^* z \cdot \left.\frac{dI_m(u_n)}{du_n}\right|_{u_n=k_n^* a} \frac{1}{I_m(k_n^* a)}\right], \\
u = \lambda r, \quad u_n &= k_n^* r, \quad w^* = 1 + \exp(-2\lambda h), \quad k_n^* = \frac{(2n-1)\pi}{2h}, \quad n = 1, 2 \ldots ;
\end{aligned}
\tag{50}
$$

– infinite layer of fluid

$$
\begin{aligned}
\left.\frac{\partial\phi}{\partial r}\right|_{r=a} &= \sum_{m=0,1,\ldots} \cos m\varphi \sum_{j=1,2,\ldots} C_j^m \cdot k_j \left.\frac{dK_m(u_j)}{du_j}\right|_{u_j=k_j a} \cos k_j z, \\
u_j &= k_j r, \quad k_j = \frac{(2j-1)\pi}{2H}, \quad j = 1, 2 \ldots
\end{aligned}
\tag{51}
$$

Equation (51) describes the fluid velocity at the boundary of the infinite fluid domain at $r = a$, $0 \le z \le H$. In a similar way, equation (49) describes this velocity at the same boundary of the lower cylindrical finite domain at $r = a$, $0 \le z \le d$, and finally, equation (50) expresses the fluid velocity at this boundary of the upper cylindrical fluid domain at $r = a$, $0 \le z \le h$. In the above equations, the independent variable z is taken as a local coordinate. From these equations, it follows that, in matching solutions at $r = a$, one may neglect the series with respect to m in the above relations and write the boundary condition for arbitrary $m = \text{const}$ as

$$
\begin{aligned}
&\left.\sum_{j=1,2,\ldots} C_j^m \cdot k_j \cdot K_m'(k_j a) \cos k_j z\right|_{0\le z\le H} = \\
&= \left[A^m \frac{\cosh\lambda z}{\sinh\lambda d} J_m'(z^*) - B^m \frac{\cos\lambda z}{\sin\lambda d}\frac{I_m'(z^*)}{I_m(z^*)} + \right. \\
&\quad \left.\left. + \sum_{n=1,2\ldots} E_n^m k_n \cos k_n z \frac{I_m'(k_n a)}{I_m(k_n a)}\right]\right|_{0\le z\le d} + \\
&\quad + \left.\left[\begin{array}{l} -A^m \dfrac{1}{w^*}\left[\exp(-\lambda z) - \exp\lambda(z-2h)\right] J_m'(z^*) + \\ B^m\left(\sin\lambda z - \sin\lambda h\right)\dfrac{I_m'(z^*)}{I_m(z^*)} + \displaystyle\sum_{n=1,2\ldots} D_n^m k_n^* \cos k_n^* z \frac{I_m'(k_n^* a)}{I_m(k_n^* a)} \end{array}\right]\right|_{0\le z\le \sim h}.
\end{aligned}
\tag{52}
$$

In this relation, prime denotes derivatives of Bessel functions with respect to their argument and $z^* = \lambda a$. In order to express the constants C_j^n in terms of the remaining constants, equation (52) is multiplied in succession by the functions $\cos k_1 z$, $\cos k_2 z, \ldots$,

$\cos k_j z, \ldots$ with $k_j = (2j-1)\pi/(2H)$ and then integrated in the range $0 \le z \le H$. To carry out the above procedure, we need the following integrals:

$$
\begin{aligned}
RA_j &= \int_0^d \frac{\cosh \lambda z}{\sinh \lambda d} \cos k_j z \, dz + \\
&- \int_c^H \frac{1}{w^*} \left[\exp -\lambda(z-c) - \exp \lambda(z-c-2h)\right] \cos k_j z \, dz = \\
&= \frac{1}{\lambda^2 + (k_j)^2} \Big[\lambda \left(\cos k_j d - \cos k_j c \right) + \\
&+ k_j \left(\sin k_j d \frac{\cosh \lambda d}{\sinh \lambda d} + \sin k_j c \frac{\sinh \lambda h}{\cosh \lambda h d} \right) \Big] ,
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
RB_j &= - \int_0^d \frac{\cos \lambda z}{\sin \lambda d} \cos k_j z \, dz + \int_c^H \left[\sin \lambda(z-c) - \sin \lambda h\right] \cos k_j z \, dz = \\
&= \begin{cases}
\dfrac{1}{\lambda^2 - (k_j)^2} \Big[k_j \dfrac{\cos \lambda d}{\sin \lambda d} \sin k_j d - \lambda \cos k_j d + \\
\quad + \lambda \cos k_j c + (-1)^j \cdot k_j \cdot \sin \lambda h \Big] + \\
+ \dfrac{\sin \lambda h}{k_j} \left[\sin k_j c + (-1)^j \right] \quad \text{for} \quad \sin \lambda d \ne 0, \quad \lambda \ne k_j \ , \\
- \dfrac{1}{4\lambda} \dfrac{1}{\sin \lambda d} (2\lambda d + \sin 2\lambda d) + \dfrac{\sin \lambda h}{\lambda} (\sin \lambda c - \sin \lambda H) - \dfrac{h}{2} \sin \lambda c + \\
+ \dfrac{1}{4\lambda} (\sin \lambda h \cdot \sin \lambda H + \cos \lambda c) \quad \text{for} \quad \sin \lambda d \ne 0, \quad \lambda = k_j \ ,
\end{cases}
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
RC_j^n &= \int_0^d k_n \cos k_n z \cdot \cos k_j z \, dz = \\
&= \begin{cases}
\dfrac{(-1)^n k_n k_j}{(k_j)^2 - (k_n)^2} \sin k_j d \quad \text{for} \quad k_n \ne k_j, \\
\dfrac{k_n d}{2} \quad \text{for} \quad k_n = k_j, \quad k_n = \dfrac{n\pi}{d}, \quad n = 1, 2 \ldots \ ,
\end{cases}
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
RD_j^n &= \int_c^H k_n^* \cos k_n^* (z-c) \cdot \cos k_j z \, dz = \\
&= \begin{cases}
- \dfrac{k_j k_n^*}{(k_j)^2 - (k_n^*)^2} \sin k_j c \quad \text{for} \quad k_j \ne k_n^*, \\
\dfrac{1}{2} \left((k_j h) \cdot \cos k_j c - \sin k_j c \right) \quad \text{for} \quad k_j = k_n^*, \\
k_j = \dfrac{2j-1}{2H} \pi, \quad k_n^* = \dfrac{2n-1}{2h} \pi, \quad j, n = 1, 2, \ldots \ .
\end{cases}
\end{aligned}
\tag{56}
$$

In accordance with these formulae and equation (52), one obtains

$$
\begin{aligned}
C_j^m = \frac{2}{k_j H}\frac{1}{K_m'(k_j a)}\Bigg\{ & A^m J_m'(\lambda a) RA_j + B^m \frac{I_m'(\lambda a)}{I_m(\lambda a)} RB_j + \\
& + \sum_{n=1,2...} E_n^m \frac{I_m'(k_n a)}{I_m(k_n a)} RC_j^n + \sum_{n=1,2...} D_n^m \frac{I_m'(k_n^* a)}{I_m(k_n^* a)} RD_j^n \Bigg\} .
\end{aligned}
\tag{57}
$$

Substitution of this relation into equation (43) gives

$$
\begin{aligned}
& A^m J_m(\lambda a)\cdot JA^n - B^m \cdot JB^n + \frac{d}{2}E_n^m = \sum_{j=1,2,...} C_j^m \cdot K^m(k_j a)\cdot JC_j^n = \\
& = \sum_{j=1,2,...} \frac{2}{k_j H}\frac{K^m(k_j a)}{K_m'(k_j a)}\Bigg\{ A^m J_m'(\lambda a) RA_j + B^m \frac{I_m'(\lambda a)}{I_m(\lambda a)} RB_j + \\
& + \sum_{n=1,2...} E_n^m \frac{I_m'(k_n a)}{I_m(k_n a)} RC_j^n + \sum_{n=1,2...} D_n^m \frac{I_m'(k_n^* a)}{I_m(k_n^* a)} RD_j^n \Bigg\} JC_j^n, \\
& \quad m = 0,1,2,\ldots, \quad n = 1,2,\ldots\ .
\end{aligned}
\tag{58}
$$

At the same time, from substitution of equation (57) into equation (48), the following is obtained:

$$
\begin{aligned}
& -A^m J_m(\lambda a)\cdot KA^n + B^m \cdot KB^n + \frac{h}{2}D_n^m = \sum_{j=1,2,...} C_j^m \cdot K^m(k_j a)\cdot KC_j^n = \\
& = \sum_{j=1,2,...} \frac{2}{k_j H}\frac{K^m(k_j a)}{K_m'(k_j a)}\Bigg\{ A^m J_m'(\lambda a) RA_j + B^m \frac{I_m'(\lambda a)}{I_m(\lambda a)} RB_j + \\
& + \sum_{n=1,2...} E_n^m \frac{I_m'(k_n a)}{I_m(k_n a)} RC_j^n + \sum_{n=1,2...} D_n^m \frac{I_m'(k_n^* a)}{I_m(k_n^* a)} RD_j^n \Bigg\} KC_j^n, \\
& \quad m = 0,1,2,\ldots, \quad n = 1,2,\ldots\ .
\end{aligned}
\tag{59}
$$

Formally, equations (58) and (59) allow us to express all the constants E_n^m, D_n^m in terms of the constants A^m and B^m. It should be stressed, however, that these coupled equations correspond to an infinite number of these constants ($n = 1,2,...,\infty$), and therefore, in order to find the desired relations, it is necessary to resort to a finite number of terms in the infinite series entering these equations and to solve the resulting system of algebraic equations by means of a numerical procedure. It is perhaps important to add here that, in calculating quotients of the Bessel functions entering equations (58) and (59), for a relatively large arguments, it is necessary to resort to asymptotic expansions of these functions (for details see Antoniewicz 1969). Thus, in order to make our further discussion clear, let us denote by ne, nd and nj the numbers of terms taken into account in the series corresponding to E_n^m, D_n^m and C_j^m, respectively. For such a finite system, it is reasonable to resort to a matrix notation, which is more convenient in description of the phenomenon. Thus, the finite set of constants

C_j^m is described as the matrix vector $(\boldsymbol{C}^m)$ of nj components. Accordingly, equation (57) is written in the following form:

$$\begin{aligned}(\boldsymbol{C}^m) = [\boldsymbol{G}^m] \cdot \{A^m J_m'(z)\,(\boldsymbol{RA}) + B^m \frac{I_m'(z)}{I_m(z)}\,(\boldsymbol{RB}) + \\ + [\boldsymbol{RC}] \cdot [\boldsymbol{BTA}] \cdot (\boldsymbol{E}^m) + [\boldsymbol{RD}] \cdot [\boldsymbol{BTB}] \cdot (\boldsymbol{D}^m)\},\end{aligned} \tag{60}$$

where $\boldsymbol{G}^m$ is a square diagonal matrix with the elements

$$G_j^m = \frac{2}{k_j H} \frac{1}{K_m'(k_j a)}, \quad j = 1, 2, ..., nj \tag{61}$$

and $\boldsymbol{BTA}$ and $\boldsymbol{BTB}$ are also square diagonal matrices with the elements

$$BTA_n^m = \frac{I_m'(k_n a)}{I_m(k_n a)}, \quad n = 1, 2, \ldots, ne, \quad BTB_n^m = \frac{I_m'(k_n^* a)}{I_m(k_n^* a)}, \quad n = 1, 2, \ldots, nd. \tag{62}$$

The matrices $\boldsymbol{RC}$ and $\boldsymbol{RD}$ in equation (60) are the rectangular matrices $(nj \times ne)$ and $(nj \times nd)$ with elements defined by equations (55) and (56), respectively.

Denoting by $\boldsymbol{GA}$ a square diagonal matrix with the elements

$$GA_j^m = \frac{2}{k_j H} \frac{K_m(k_j a)}{K_m'(k_j a)}, \quad j = 1, 2, \ldots, nj \tag{63}$$

and substituting the above formulae into equation (58), we arrive at the following matrix relation:

$$\begin{aligned}&A^m J_m(z)\,(\boldsymbol{JA}) - B^m\,(\boldsymbol{JB}) + \frac{d}{2}\,(\boldsymbol{E}^m) = \\ &= [\boldsymbol{JC}] \cdot [\boldsymbol{GA}^m] \cdot \left\{ A^m J_m'(z)\,(\boldsymbol{RA}) + B^m \frac{I_m'(z)}{I_m(z)}\,(\boldsymbol{RB}) + \right. \\ &\qquad + [\boldsymbol{RC}] \cdot [\boldsymbol{BTA}] \cdot (\boldsymbol{E}^m) + [\boldsymbol{RD}] \cdot [\boldsymbol{BTB}] \cdot (\boldsymbol{D}^m)\},\end{aligned} \tag{64}$$

where $\boldsymbol{JA}$ and $\boldsymbol{JB}$ are vector matrices defined by equations (39) and (40), and $\boldsymbol{JC}$ is a square matrix with elements described by equation (41).

In a similar way, equation (59) leads to the matrix equation

$$\begin{aligned}&-A^m J_m(z)\,(\boldsymbol{KA}) + B^m\,(\boldsymbol{KB}) + \frac{h}{2}\,(\boldsymbol{D}^m) = \\ &= [\boldsymbol{KC}] \cdot [\boldsymbol{GA}^m] \cdot \left\{ A^m J_m'(z)\,(\boldsymbol{RA}) + B^m \frac{I_m'(z)}{I_m(z)}\,(\boldsymbol{RB}) + \right. \\ &\qquad + [\boldsymbol{RC}] \cdot [\boldsymbol{BTA}] \cdot (\boldsymbol{E}^m) + [\boldsymbol{RD}] \cdot [\boldsymbol{BTB}] \cdot (\boldsymbol{D}^m)\},\end{aligned} \tag{65}$$

where $\boldsymbol{KA}$ and $\boldsymbol{KB}$ are vector matrices and $\boldsymbol{KC}$ is a square matrix, with elements described by equations (45), (46) and (47), respectively. At the same time, matrix equation (64) consists of ne equations corresponding to E_n^m unknown parameters, and equation (65) corresponds to nd equations inherent for D_n^m constants. In order to simplify our further discussion, it is convenient to introduce the following substitutions:

$$(\boldsymbol{W1}) = J_m'(z) \cdot [\boldsymbol{GA}^m] \cdot (\boldsymbol{RA}), \tag{66}$$

$$(\boldsymbol{W2}) = \frac{I_m'(z)}{I_m(z)} \cdot [\boldsymbol{GA}^m] \cdot (\boldsymbol{RB}), \tag{67}$$

$$[\boldsymbol{W3}] = [\boldsymbol{GA}^m] \cdot [\boldsymbol{RC}] \cdot [\boldsymbol{BTA}], \tag{68}$$

$$[\boldsymbol{W4}] = [\boldsymbol{GA}^m] \cdot [\boldsymbol{RD}] \cdot [\boldsymbol{BTB}]. \tag{69}$$

In view of these relations, equations (64) and (65) are reduced to the following forms:

$$\begin{aligned} &\left\{\frac{d}{2}[\boldsymbol{I}] - [\boldsymbol{JC}] \cdot [\boldsymbol{W3}]\right\} \cdot (\boldsymbol{E}^m) - [\boldsymbol{JC}] \cdot [\boldsymbol{W4}] \cdot (\boldsymbol{D}^m) = \\ &= A^m \{[\boldsymbol{JC}] \cdot (\boldsymbol{W1}) - J_m(z)(\boldsymbol{JA})\} + B^m \{[\boldsymbol{JC}] \cdot (\boldsymbol{W2}) + (\boldsymbol{JB})\} \end{aligned} \tag{70}$$

and

$$\begin{aligned} &- [\boldsymbol{KC}] \cdot [\boldsymbol{W3}] \cdot (\boldsymbol{E}^m) + \left\{\frac{h}{2}[\boldsymbol{I}] - [\boldsymbol{KC}] \cdot [\boldsymbol{W4}]\right\} \cdot (\boldsymbol{D}^m) = \\ &= A^m \{[\boldsymbol{KC}] \cdot (\boldsymbol{W1}) + J_m(z)(\boldsymbol{KA})\} + B^m \{[\boldsymbol{KC}] \cdot (\boldsymbol{W2}) - (\boldsymbol{KB})\}. \end{aligned} \tag{71}$$

For known values of d, h, m and λ and for a finite number of constants taken into account in the description of the phenomenon, this system of algebraic equations may be solved numerically. The final result of computations may be expressed in the following form:

$$\begin{aligned} (\boldsymbol{E}^m) &= A^m (\boldsymbol{R1}) + B^m (\boldsymbol{R2}), \\ (\boldsymbol{D}^m) &= A^m (\boldsymbol{R3}) + B^m (\boldsymbol{R4}). \end{aligned} \tag{72}$$

From substitution of these solutions into equation (60), we obtain

$$\begin{aligned} (\boldsymbol{C}^m) = [\boldsymbol{G}^m] \cdot \Big\{ & A^m J_m'(z)(\boldsymbol{RA}) + B^m \frac{I_m'(z)}{I_m(z)} (\boldsymbol{RB}) + \\ & + [\boldsymbol{RC}] \cdot [\boldsymbol{BTA}] \cdot [A^m (\boldsymbol{R1}) + B^m (\boldsymbol{R2})] + \\ & + [\boldsymbol{RD}] \cdot [\boldsymbol{BTB}] \cdot [A^m (\boldsymbol{R3}) + B^m (\boldsymbol{R4})]\Big\} = \\ & = A^m (\boldsymbol{R5}) + B^m (\boldsymbol{R6}). \end{aligned} \tag{73}$$

In order to complete the solution, it is necessary to find the constant E^0. For $m > 0$, this constant equals zero. For $m = 0$, equation (42) gives

$$\begin{aligned} E^0 &= \frac{1}{d}\left\{\frac{1}{\lambda^2}\left(-A^0 J_0(\lambda a) + B^0\right) + \left(\boldsymbol{KS}^0\right)^T \cdot \left(A^0(\boldsymbol{R5}) + B^0(\boldsymbol{R6})\right)\right\} = \\ &= \varepsilon_1 A^0 + \varepsilon_2 B^0, \end{aligned} \tag{74}$$

where

$$\left(\boldsymbol{KS}^0\right)^T = \left[\cdots, K_0(k_j a)\frac{\sin k_j d}{k_j}, \cdots\right] \tag{75}$$

is a row matrix of nj elements. Knowing the constants $(\boldsymbol{C}^m)$, $(\boldsymbol{D}^m)$, $(\boldsymbol{E}^m)$ and E^0, it is a simple task to express the velocity potential functions in the finite and infinite domains in terms of A^m and B^m, which are the independent parameters (variables) of the problem description presented above. This description can be easily reduced to one parameter – in our case, A^m. With the results obtained, one may solve the problem of free vibrations of the plate submerged in fluid.

5. Added Mass of Fluid and Fundamental Frequencies of the Plate Vibrating in Fluid

Free vibrations of an elastic plate immersed in fluid are described by equations (6) and (31). The boundary condition at the common boundary between the plate and fluid means that the plate velocities, normal to the plate surface, are equal to fluid velocities. This boundary condition leads to equation (32), which defines the boundary condition in space variables (time factor does not enter this equation). Substitution of relations (31) into equation (6) gives

$$\nabla^2\nabla^2 W + \frac{m^*}{D^*}\left[-\omega^2 W + \frac{\rho}{m^*}\left(-\omega^2\phi\big|_{low.} + \omega^2\phi\big|_{upp.}\right)\right] = 0, \tag{76}$$

where $m^* = (\rho_{plate} - \rho_{fluid}) \cdot \delta$ and $\rho = \rho_{fluid}$. This equation is written in the following form:

$$\nabla^2\nabla^2 W - \alpha^4\left[W + \frac{\rho}{m^*}\left(\phi|_{low.} - \phi|_{upp.}\right)\right] = 0, \tag{77}$$

where $\alpha^4 = (\omega c)^2$ and $c^2 = m^*/D^*$.

Compared with equation (10), it contains additional terms corresponding to values of the potential functions at the lower and upper surfaces of the plate. These terms are responsible for the mass of the fluid vibrating together with the plate. The unknown displacement $W(r, \varphi)$ (amplitude of vibrations) is described by a linear combination of eigenfunctions of the plate vibrating in air, i.e.

$$\begin{aligned} W(r,\varphi) &= \sum_{m=0,1,2\ldots}\sum_{i=1,2,\ldots} \cos(m\varphi)\left[A_i^m J_m(\lambda_i r) + B_i^m \frac{I_m(\lambda^i r)}{I_m(\lambda^i a)}\right] = \\ &= \sum_{m=0,1,2\ldots}\sum_{i=1,2,\ldots} A_i^m \cos(m\varphi)\left[J_m(\lambda_i r) - \frac{J_m(\lambda^i a)}{I_m(\lambda^i a)} I_m(\lambda^i r)\right] = \\ &= \sum_{m=0,1,2\ldots}\sum_{i=1,2,\ldots} A_i^m F_i^m(r,\varphi), \end{aligned} \tag{78}$$

where $F_i^m(r, \varphi)$ are defined by equation (29).

Obviously, each of these functions satisfies equation (10), and thus, the following relation holds:

$$\nabla^2\nabla^2 F_i^m = (\lambda_i^m)^4 F_i^m. \tag{79}$$

Substitution of equations (78) and (79) into (77) gives

$$\sum_m \sum_i A_i^m \left(\lambda_i^m\right)^4 F_i^m(r,\varphi) - \alpha^4 \left[\sum_m \sum_i A_i^m F_i^m(r,\varphi) + \frac{\rho}{m^*} \left(\phi|_{low.} - \phi|_{upp.} \right) \right] = 0. \quad (80)$$

The solution procedure displayed above is based on the set of eigenfunctions $F_i^m(r,\varphi)$ of the plate vibrating in air. These functions are unique within multiplication by a real number. Since the present problem is linear, the potential functions entering this equation may be obtained by a linear combination of potential functions corresponding to each of these eigenfunctions. On the other hand, the plate deflection is governed by a bi-harmonic partial differential equation, while the potential function should satisfy the harmonic equation. With respect to that and because of boundary conditions at the common plate-fluid boundary, it is impossible to employ directly the plate functions $F_i^m(r,\varphi)$ in the description of the potential functions. And thus, the term with the potentials in equation (84) is written in the following form:

$$\begin{aligned} \phi|_{low.} - \phi|_{upp.} = E_0 + \sum_{m=0,1,\ldots} \cos m\varphi \cdot \Bigg[A^m \left(\frac{\cosh \lambda d}{\lambda \sinh \lambda d} + \frac{\sinh \lambda h}{\lambda \cosh \lambda h} \right) J_m(\lambda r) + \\ - B^m \left(\frac{\cos \lambda d}{\lambda \sin \lambda d} - \frac{\sin \lambda h}{\lambda} \right) \frac{I_m(\lambda r)}{I_m(\lambda a)} + \sum_{n=1}^{ne} E_n^m (-1)^n \frac{I_m(k_n r)}{I_m(k_n a)} - \sum_{n=1}^{nd} D_n^m \frac{I_m(k_n^* r)}{I_m(k_n^* a)} \Bigg], \end{aligned} \quad (81)$$

where $B^m = -A^m J_m(\lambda a)$.

From substitution of equation (81) into equation (80), one obtains

$$\begin{aligned} \sum_m \sum_i A_i^m F_i^m (\lambda_i^m)^4 - \alpha^4 \Bigg\langle \sum_m \sum_i A_i^m F_i^m + \\ + \frac{\rho}{m^*} \Bigg\{ \sum_i E_i^0 + \sum_m \sum_i \cos m\varphi \Bigg[A_i^m \left(\frac{\cosh \lambda_i^m d}{\lambda_i^m \sinh \lambda_i^m d} + \frac{\sinh \lambda_i^m h}{\lambda_i^m \cosh \lambda_i^m h} \right) \times \\ \times J_m(\lambda_i^m r) + + B_i^m \left(\frac{\cos \lambda_i^m d}{\lambda_i^m \sin \lambda_i^m d} - \frac{\sin \lambda_i^m h}{\lambda_i^m} \right) \frac{I_m(\lambda_i^m r)}{I_m(\lambda_i^m a)} + \\ + \sum_{n=1}^{ne} E_n^m (-1)^n \frac{I_m(k_n r)}{I_m(k_n a)} - \sum_{n=1}^{nd} D_n^m \frac{I_m(k_n^* r)}{I_m(k_n^* a)} \Bigg] \Bigg\} \Bigg\rangle = 0. \end{aligned} \quad (82)$$

Each constant E_i^0 in this equation depends solely on A_i^0 and B_i^0. The constants E_n^m and D_n^m also depend on i. The expression in the curly brackets describes the added (co-vibrating) mass of fluid. Equation (82) is our fundamental equation for the problem of free vibrations of the plate in fluid. In order to find a standard set of algebraic equations for the eigenvalue problem considered, equation (82) is multiplied in succession by $F_i^m(r,\varphi)$ and then integrated in the range $(0 \le r \le a,\ 0 \le \varphi \le 2\pi)$. Obviously, these functions are orthogonal within the range of this integration. Since all constants in this equation, corresponding to chosen m, depend solely on m, the final system of equations of the problem may be decoupled into a set of matrix equations, each of which corresponds solely to m. In this way, instead of a relatively large system of equations, we can consider a set of individual subsets containing a smaller number of equations. This feature of equations (82) simplifies numerical computations. With respect to the above, for the assumed m = const, the fundamental equation of the problem is reduced to the following one:

$$\begin{aligned}
&\sum_i A_i^m F_i^m (\lambda_i^m)^4 - \alpha^4 \Bigg\langle \sum_i A_i^m F_i^m + \\
&\quad + \frac{\rho}{m^*} \Bigg\{ \sum_i E_i^0 + \sum_i \cos m\varphi \Bigg[A_i^m \left(\frac{\cosh \lambda_i^m d}{\lambda_i^m \sinh \lambda_i^m d} + \frac{\sinh \lambda_i^m h}{\lambda_i^m \cosh \lambda_i^m h} \right) J_m(\lambda_i^m r) + \\
&\qquad + B_i^m \left(\frac{\cos \lambda_i^m d}{\lambda_i^m \sin \lambda_i^m d} - \frac{\sin \lambda_i^m h}{\lambda_i^m} \right) \frac{I_m(\lambda_i^m r)}{I_m(\lambda_i^m a)} + \\
&\qquad + \sum_{n=1}^{ne} E_n^m (-1)^n \frac{I_m(k_n r)}{I_m(k_n a)} - \sum_{n=1}^{nd} D_n^m \frac{I_m(k_n^* r)}{I_m(k_n^* a)} \Bigg] \Bigg\} \Bigg\rangle = 0.
\end{aligned} \tag{83}$$

For the case $m > 0$, the term with the constants E_i^0 in this equation should be cancelled out. As mentioned above, in deriving algebraic equations, it is necessary to perform integrations of the products of the functions $F_i^m(r, \varphi)$ with functions entering equation (83), i.e. we have to calculate the following integrals:

$$\int_0^a f(r) \cdot F_i^m(r, \varphi = 0) \cdot r\, dr, \tag{84}$$

where $f(r)$ denotes functions, mainly Bessel functions, of the following form:

$$\begin{aligned}
&f(r) = \\
&= \left[F_i^{m(r)}, 1, J_0(\lambda_1 r), \ldots, J_0(\lambda_3 r), I_0(\lambda_1 r), \ldots, I_0(\lambda_3 r), I_0(k_n r), \ldots, I_0(k_n^* r) \right].
\end{aligned} \tag{85}$$

The system of equations obtained in this way contains all the constants entering equation (83) for $i = 1, 2, \ldots, i_{\max}$. Since the constants E^0, E_n^m, D_n^m and B^m depend on A^m, the final system of equations of the problem corresponds solely to the constants A_i^m $(i = 1, 2, \ldots, i_{\max})$. Each equation of this system is then multiplied by

$$\gamma_i^m = \frac{1}{(\lambda_i^m)^4} \cdot \frac{1}{\left\| F_i^m \right\|}, \quad \text{where} \quad \left\| F_i^m \right\| = \int_0^a (F_i^m)^2 \cdot r\, dr. \tag{86}$$

By the manipulations described above, the following homogeneous system of equations is obtained:

$$[\boldsymbol{AM}] \cdot (\boldsymbol{A}) = \boldsymbol{0}, \tag{87}$$

where $(\boldsymbol{A})$ is the vector of constants $A_1, A_2, \ldots, A_{i_{\max}}$, and $[\boldsymbol{AM}]$ is the fundamental matrix of the problem.

Accordingly, with this equation, the problem considered is reduced to the eigenvalues problem of the matrix $[\boldsymbol{AM}]$. For example, for $i_{\max} = 3$, the fundamental matrix may be expressed in the following form:

$$[\boldsymbol{AM}] = \begin{bmatrix} \left\|F_1^m\right\| + \dfrac{\rho}{m^*}(\ldots) & \dfrac{\rho}{m^*}(\ldots) & \dfrac{\rho}{m^*}(\ldots) \\ \dfrac{\rho}{m^*}(\ldots) & \left\|F_2^m\right\| + \dfrac{\rho}{m^*}(\ldots) & \dfrac{\rho}{m^*}(\ldots) \\ \dfrac{\rho}{m^*}(\ldots) & \dfrac{\rho}{m^*}(\ldots) & \left\|F_3^m\right\| + \dfrac{\rho}{m^*}(\ldots) \end{bmatrix} \times \tag{88}$$

$$\times \begin{bmatrix} \gamma_1^m & & \\ & \gamma_2^m & \\ & & \gamma_3^m \end{bmatrix}.$$

The eigenvalues β_i of this matrix relate to α_i in equation (81) through the formula

$$(\alpha_i)^4 = \frac{1}{\beta_i}. \tag{89}$$

6. Numerical Examples

The solution of the problem presented above is illustrated by numerical examples. Two steel plates of radii $a = 0.5$ m and $a = 1.0$ m, and thickness $\delta = 4$ mm are considered. These plates are installed in a layer of fluid of depth $H = 0.6$ m at a certain distance from the fluid bottom. Since the fluid pressure load on the plate depends on the width of the gap between the plate and the fluid bottom, a set of gap widths is considered. For each assumed gap, a set of the fundamental frequencies of the plate submerged in fluid is calculated. For comparison, a set of eigenfrequencies of the plate vibrating in air is also calculated. Some of the results obtained in calculations are drawn up in Table 2. The changes in eigenfrequencies associated with changing gap widths are illustrated in Fig. 4, where the plots show the distribution of fundamental frequencies versus the gap width. From the data collected in this table and from the plots in this figure, it may be seen that the maximum reduction in the eigenfrequency takes place for the lowest eigenfrequencies of the plate. From the practical point of view, the most important is the reduction of these lowest frequencies.

7. Concluding Remarks

The formulation developed in this paper makes it possible to calculate the co-vibrating mass of fluid and the set of eigenfrequencies of a circular horizontal thin elastic plate submerged in fluid of constant depth. As compared to vibrations of the plate in air, the most important result of these investigations is an assessment of the reduction in

Table 2. Eigenfrequencies ω_i^m of the plate vibrating in air and water

			Plate radius [cm]					
			a = 50			a = 100		
		m \ i	1	2	3	1	2	3
air		0	124.406	749.192	1869.356	31.101	187.298	467.338
		1	350.348	1222.077	2590.755	87.586	305.519	647.688
		2	645.671	1767.539	3385.436	161.417	441.884	846.359
water gap width [cm]	d = 1	0	2.461	89.541	174.188	1.291	17.431	66.673
		1	2.638	99.967	386.231	5.440	35.421	107.368
		2	6.487	196.218	547.888	13.668	61.155	158.814
	d = 2	0	3.334	110.536	188.620	1.808	23.887	89.594
		1	3.709	132.594	392.238	7.560	48.244	143.397
		2	9.079	252.568	558.060	18.835	82.568	210.21
	d = 3	0	3.924	120.008	201.709	2.192	28.375	104.581
		1	4.516	153.074	395.734	9.100	56.972	166.407
		2	11.016	284.581	565.951	22.487	96.737	242.074
	d = 4	0	4.366	125.177	212.265	2.504	31.807	115.394
		1	5.186	167.336	398.277	10.329	63.507	182.615
		2	12.611	304.983	572.305	25.322	107.059	263.901
	d = 5	0	4.715	128.423	220.447	2.771	34.551	123.570
		1	5.768	177.792	400.243	11.353	68.622	194.574
		2	13.988	318.784	577.423	27.624	114.928	279.579
	d = 6	0	5.001	130.667	226.696	3.003	36.804	129.928
		1	6.285	185.709	401.803	12.230	72.733	203.644
		2	15.208	328.478	581.543	29.541	121.088	291.159
	d = 7	0	5.242	132.319	231.425	3.210	38.688	134.965
		1	6.754	191.840	403.061	12.993	76.093	210.646
		2	16.310	335.463	584.869	31.165	125.996	299.870
	d = 8	0	5.448	133.591	234.969	3.397	40.284	139.009
		1	7.184	196.671	404.092	13.666	78.875	216.122
		2	17.317	340.584	587.574	32.558	129.957	306.507
	d = 9	0	5.629	134.600	237.581	3.566	41.650	142.290
		1	7.582	200.531	404.953	14.264	81.200	220.444
		2	18.247	344.383	589.800	33.765	133.182	311.610
	d = 10	0	5.789	135.418	239.456	3.722	42.828	42.828
		1	7.953	203.651	405.688	14.801	83.156	223.879
		2	19.111	347.220	591.664	34.817	135.828	315.560

the plate eigenfrequencies due to the co-vibrating mass of fluid. At the same time, the theory developed here makes it possible to assess the influence of the gap width on this reduction. The maximum reduction in the eigenfrequency of the plate takes place for the smallest gap width. With growing gap width, the corresponding eigenfrequencies go asymptotically to a constant value.

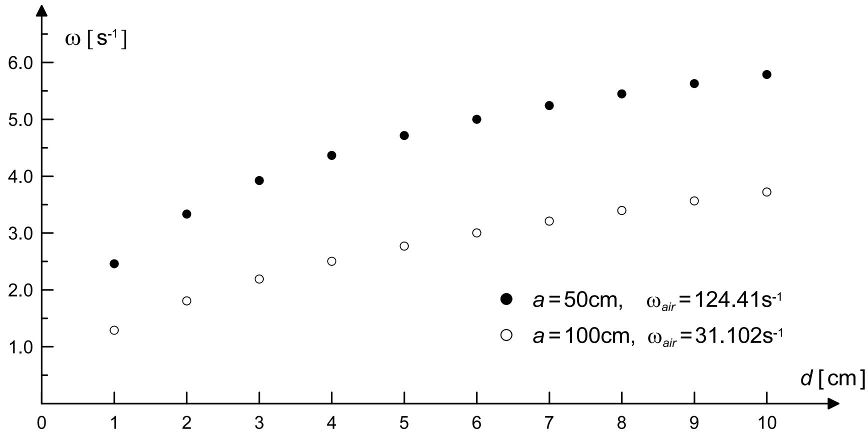

Fig. 4. Distribution of the fundamental frequencies versus the gap width

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