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Modeling of Waves Propagating in Water with a Crushed Ice Layer on the Free Surface

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Abstract

A transformation of gravitational waves in fluid of constant depth with a crushed ice layer floating on the free fluid surface is considered. The propagating waves undergo a slight damping along their path of propagation. The main goal of the study is to construct an approximate descriptive model of this phenomenon. With regard to small displacements of the free surface, a viscous type model of damping is considered, which corresponds to a continuous distribution of dash-pots at the free surface of the fluid. A constant parameter of the dampers is assumed in advance as known parameter of damping. This parameter may be obtained by means of experiments in a laboratory flume.

Key words: gravitational wave, crushed ice, wave damping

1. Introduction

Under natural winter conditions, crushed ice layers may emerge in some costal sea zones. The sea surface in such zones is covered with floating nubbles of ice. Water waves arriving from the open sea undergo changes associated with these nubbles. These changes result from a small dissipation of the wave energy due to collisions between individual nubbles and to differences in the velocities of the ice nubbles and fluid. With respect to the size, shape and space distributions of the nubbles, where all these parameters are random, it is impossible to follow individual elements in a description of the phenomenon. Therefore, for practical reasons, we resort to a macroscopic description of this phenomenon, in which the wave damping is modelled by means of viscous dampers continuously distributed over the sea surface. An unknown parameter of the dampers may be obtained from laboratory experiments in a flume or by measurements of the sea elevation carried out under natural conditions. It may be changed in a certain range, dependent on local conditions, inherent for the coastal zone under consideration. Such a solution of the problem is similar to the one used by Lysmer & Kuhlemeyer (1969) in constructing absorbing boundary conditions for a finite dynamic model, used in description of waves propagating in infinite media. With respect to the above, in what follows we focus our attention on a continuous distribution of simple, one-parameter viscous dampers on the free surface of the fluid.

2. Formulation of the Problem

In order to describe a transformation of gravitational waves propagating in water with a crushed ice layer floating on its free surface, a time-dependent problem is considered in which the fluid, initially at rest, starts to move at a certain moment in time. To this end, an initial generation of gravitational waves in a semi-infinite layer of fluid by a piston – type generator is considered. The generator – fluid system is shown schematically in Fig. 1.

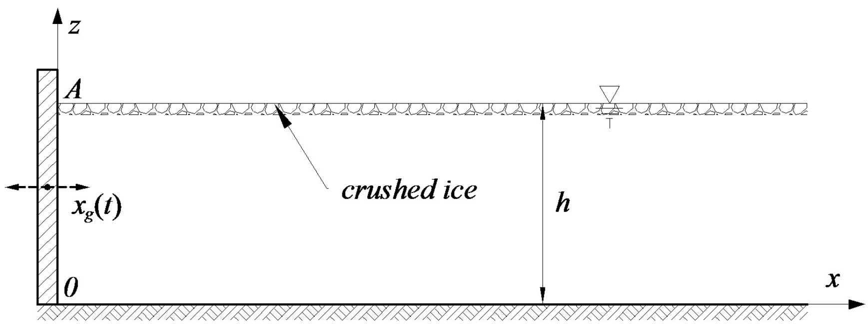

Fig. 1. Semi-infinite fluid domain with a layer of crushed ice

The problem considered here is similar to a well-known classical problem of forced vibrations of a damped system with one degree of freedom. In the case of a steady harmonic forcing of such a system, the oscillations amplitude is largest in the vicinity of a resonance frequency (eigenfrequency) of the same system when damping is neglected. Outside this range, when the difference between the forcing frequency and eigenfrequency of the system is relatively big, the damping of the system amplitude is rather small. It means that in order to asses the influence of damping on the system response, it would be necessary to force a system oscillation with a frequency that is closed to the resonance frequency of the system. In the case of a system with an infinite number of degrees of freedom, the associated resonance frequencies of the system are not known in advance, and therefore, in investigation of such a damped system, it is better to examine the associated impulse response function of the system at hand. With respect to this, for the initial value problem discussed in this paper, we confine our attention to a solution describing the free-surface elevation induced by an unit impulse of the generator motion (unit impulse of its velocity). The impulse response function of the generator – fluid system contains all information important in describing the behaviour of the system for arbitrary generations.

Thus, let us assume that, until the starting point $t = 0$, the generator – fluid system is at rest, i.e. the displacement, velocity, and acceleration of the generator plate and fluid are all equal to zero. In formulating this problem, it is assumed that the incompressible, non-viscous fluid motion is a potential motion with the potential function $\phi(x, z, t)$ satisfying Laplace's equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{1}$$

and appropriate boundary and initial conditions.

For the potential motion, the fluid pressure is obtained by linearization of the Bernoulli equation

$$p(x, z, t) = \rho \left[g(h - z) - \frac{\partial \phi}{\partial t} \right], \tag{2}$$

where ρ is the fluid density, and g is the gravitational acceleration.

The initial conditions for the velocity potential read:

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial z} = 0 \ \text{ for } \ 0 \le x \le \infty, \ 0 \le z \le h. \tag{3}$$

At the same time, for the fluid domain considered, the following boundary conditions hold:

$$\begin{aligned} &\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = \frac{\partial x_g(t)}{\partial t}, \quad \left.\frac{\partial \phi}{\partial z}\right|_{z=0} = 0, \quad \left.\frac{\partial \phi}{\partial z}\right|_{z=h} \cong \frac{\partial \eta}{\partial t}, \\ &\left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial z} \right) \to 0 \text{ as } x \to \infty, \end{aligned} \tag{4}$$

where $\partial x_g / \partial t$ denotes the horizontal velocity of the generator face (rigid wall OA in the figure).

Relations (4) are supplemented by the Bernoulli equation, written for points of the free surface, i.e.

$$\left. \frac{\partial \phi}{\partial t} + \frac{1}{\rho} p_{ice} + gz \right|_{z=\eta(x,t)} = 0, \tag{5}$$

where p_{ice} means the additional pressure at the surface, and $\eta(x, t)$ denotes its elevation.

In accordance with the above assumption of viscous damping, the second term in this relation is assumed in the following form:

$$p_{ice} = 2\rho g \mu \left. \frac{\partial \phi}{\partial z} \right|_{z=h}, \tag{6}$$

where μ is a parameter of the damping.
With respect to this formula, the damping parameter is measured in seconds.

By differentiating equations (5) and (6) with respect to time, and employing the third relation in (4), we arrive at the boundary condition for the free surface

$$\left.\frac{\partial^2\phi}{\partial t^2} + 2\mu g\frac{\partial^2\phi}{\partial z\partial t} + g\frac{\partial\phi}{\partial z}\right|_{z=h} = 0. \tag{7}$$

In constructing a general solution to Laplace's equation that satisfies the prescribed conditions, we resort to Fourier cosine transforms (Nowacki 1972)

$$\begin{aligned}\phi^*(s,z,t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \phi(x,z,t)\cos sx\, dx,\\ \phi(x,z,t) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \phi^*(s,z,t)\cos sx\, ds.\end{aligned} \tag{8}$$

To solve the boundary value problem, we multiply equation (1) by $\cos sx$ and integrate with respect to x over the range $(0, \infty)$. This procedure leads to the following result:

$$\frac{\partial^2\phi^*}{\partial z^2} - s^2\phi^* = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial t}x_g(t). \tag{9}$$

In a similar way, the Fourier transform of equation (7) gives

$$\left.\frac{\partial^2\phi^*}{\partial t^2} + 2\mu g\frac{\partial^2\phi^*}{\partial z\partial t} + g\frac{\partial\phi^*}{\partial z}\right|_{z=h} = 0. \tag{10}$$

Knowing that $\partial\phi^*/\partial z|_{z=0} = 0$, it is a simple task to find the solution of the Fourier transform of Laplace's equation (non-homogeneous linear differential equation 9)

$$\phi^*(s,z,t) = A(s,t)\cosh sz - \sqrt{\frac{2}{\pi}}\frac{1}{s^2}\frac{\partial x_g}{\partial t}. \tag{11}$$

From substitution of this solution into equation (10), the following relation is obtained:

$$\frac{\partial^2 A}{\partial t^2} + 2\mu gs\tanh sh\frac{\partial A}{\partial t} + gs\tanh sh\cdot A = \sqrt{\frac{2}{\pi}}\frac{1}{s^2\cosh sh}\frac{\partial^3}{\partial t^3}x_g(t) = F(s,t). \tag{12}$$

In turn, substitution of

$$r^2 = gs\tanh sh \tag{13}$$

into equation (12) leads to the result

$$\frac{\partial^2 A}{\partial t^2} + 2\mu r^2\frac{\partial A}{\partial t} + r^2 A = F(s,t). \tag{14}$$

To solve this equation, the standard Lagrange method of variation of parameters is employed, which gives the following general solution:

$$A(s,t) = C_1(s)\exp(w_1 t) + C_2(s)\exp(w_2 t) + \\ + \frac{1}{(w_2 - w_1)} \int_0^t F(s,\xi)\left[\exp w_2(t-\xi) - \exp w_1(t-\xi)\right]\, d\xi, \tag{15}$$

where C_1 and C_2 are constants of the solution, and

$$\begin{aligned} w_1 &= -\mu r^2 - r\sqrt{(\mu r)^2 - 1} = -\mu r^2 - r\beta, \\ w_2 &= -\mu r^2 + r\sqrt{(\mu r)^2 - 1} = -\mu r^2 + r\beta. \end{aligned} \tag{16}$$

From the initial conditions (3), i.e. at $t = 0$, it follows that both C_1 and C_2 are equal to zero, and thus, equation (15) is reduced to the following one:

$$A(s,t) = \frac{1}{2r\beta} \int_0^t F(s,\xi)\left[\exp w_2(t-\xi) - \exp w_1(t-\xi)\right]\, d\xi. \tag{17}$$

From substitution of this solution into equation (11), one obtains

$$\phi^*(s,z,t) = \\ = \frac{\cosh sz}{2r\beta} \int_0^t F(s,\xi)\left[\exp w_2(t-\xi) - \exp w_1(t-\xi)\right]\, d\xi - \sqrt{\frac{2}{\pi}}\frac{1}{s^2}\frac{\partial x_g}{\partial t}, \tag{18}$$

and, finally

$$\phi^*(s,z,t) = \sqrt{\frac{2}{\pi}} \left\{ \frac{1}{s^2}\frac{\cosh sz}{\cosh sh}\frac{1}{2r\beta} \times \right. \\ \left. \times \int_0^t \frac{\partial^2 x_g}{\partial \xi^2}\left[w_2 \exp w_2(t-\xi) - w_1 \exp w_1(t-\xi)\right] d\xi - \frac{1}{s^2}\frac{\partial x_g}{\partial t} \right\}. \tag{19}$$

The inverse transform of this equation gives the potential function

$$\phi(s,z,t) = \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{s^2} \left\{ \frac{1}{2r\beta}\frac{\cosh sz}{\cosh sh} \times \right. \\ \left. \times \int_0^t \frac{\partial^2 x_g}{\partial \xi^2}\left[w_2 \exp w_2(t-\xi) - w_1 \exp w_1(t-\xi)\right]\, d\xi - \frac{\partial x_g}{\partial t} \right\} ds. \tag{20}$$

Knowing the velocity potential, it is a simple task to calculate the free-surface elevation. From substitution of equation (20) into the dynamic boundary condition at the free surface, one obtains

$$\eta(x,t) = -\frac{1}{g}\left(\frac{\partial\phi}{\partial t} + 2\mu g\frac{\partial\phi}{\partial z}\right)\bigg|_{z=h} = \frac{2}{\pi}\int_0^\infty \frac{\tanh sh}{s}\cos sx \times \\ \times \int_0^t \frac{\partial x_g(t-\xi)}{\partial\xi}\exp(-\mu r^2\xi)\left[\cosh r\beta\xi - \mu r\frac{\sinh r\beta\xi}{\beta}\right] d\xi ds. \tag{21}$$

This free-surface elevation depends on the damping parameter μ. With respect to the range of this parameter, different shapes of the elevation may be obtained. For our further needs, it is convenient to write this equation in the following form:

$$\eta(x,t) = \int_0^t \frac{\partial x_g(t-\xi)}{\partial\xi} h(x,\xi)d\xi, \tag{22}$$

where $h(x,t)$ is the impulse response function of the problem considered. This function denotes the free-surface elevation at point $x > 0$, induced by the unit impulse of the generator velocity. With respect to equation (21), the impulse response function reads

$$h(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\tanh sh}{s}\cos sx \cdot \exp(-\mu r^2 t)\left[\cosh r\beta t - \mu r\frac{\sinh r\beta t}{\beta}\right] ds. \tag{23}$$

From the solution of the problem, described by equation (22), it follows that all information about the model description is contained in the impulse response function. Knowing this function, one may calculate the free-surface elevation for an assumed generator motion. It is worth adding here that, for the forced periodic generation of fluid motion, a change in the free-surface elevation due to a crushed ice layer floating on the surface is a very small quantity. In contrast, the damping of the fluid flow, due to the crushed ice, is important in the description of time-dependent free fluid motion. As described above, representative of the latter case is the fluid motion induced by the impulse generation described by the impulse response function. For this case, the magnitude of the damping parameter ($\mu > 0$) is expected to be important in the description of the phenomenon. Therefore, in our further investigations we will concentrate mainly on the examination of the response function properties. To this end, for an assumed set of space points ($x > 0$) and damping parameters ($\mu \geq 0$), distributions of this function with respect to an assumed range of time ($0 \leq t \leq t_{\max}$) will be calculated.

Thus, in the first step, let us consider the case $\mu = 0$, which corresponds to the free-fluid surface without an ice layer on it. For this case, $\beta = i$ (imaginary unit), and the solution is reduced to the following one:

$$h(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{\tanh sh}{s} \cos sx \cdot \cos rt \, ds. \tag{24}$$

With respect to the nomenclature of classical mechanics, in analysing vibrations of a damped system with one degree of freedom, three characteristic cases, dependent on the values of roots of the associated characteristic equation of the system are distinguished, i.e. under-damped, critically damped and over-damped. Periodic motion of the system may exist only for weak damping (the under-damped case). In the over-damped and critically damped cases, the motion is said to be aperiodic.

In the present problem of water waves, we have a system with an infinite number of degrees of freedom. In this case, in accordance with equations (16), there are also three formal cases dependent on values of the product μr in these equations. Since μ and r are non-negative numbers, these cases uniquely correspond to the ranges $\mu r > 1$, $\mu r = 1$ and $\mu r < 1$. It may be seen that for the first two cases the corresponding roots w_1 and w_2 are real numbers, and thus, the solutions of the problem are similar in their essential character. The third case ($\mu r < 1$) leads to complex roots w_1 and w_2. It may be important to note here that these cases do not depend on the damping parameter μ itself, but on the product of this parameter with r. It is easy to see that r corresponds directly to the wave frequency (wave length), which, in the Fourier method of solving the problem, belongs to the range $(0 \leq r \leq \infty)$. It means that the solution presented above depends on the continuous distribution of wave lengths varying from zero to infinity. In other words, in calculating the improper integrals all three cases necessarily occur. The final result of integration will obviously depend on the frequency of wave generation, and thus, for a given parameter $\mu > 0$, one should expect a type of solution inherent for this damping parameter, i.e. over-damped or under-damped waves. In order to illustrate the solution presented above, numerical examples are attached below for a chosen set of generation frequencies.

3. Numerical Experiments

In accordance with the solution presented in the preceding section, it is reasonable to consider three separate cases corresponding to the range of the damping parameter, i.e. $\mu = 0$ (fluid surface is free of damping), $0 < \mu \ll 0.01$ s (weak damping) and $\mu > 0.1$ s (a relatively strong damping). Thus, let us consider the first case, in which no damping occurs. In order to calculate the free-surface elevation, it is convenient to write equation (24) in the form

$$h(x,t) = J_1(x,t) + J_2(x,t), \tag{25}$$

where:

$$
\begin{aligned}
J_1(x,t) &= \frac{1}{\pi}\int\limits_0^\infty \frac{\tanh sh}{s}\cos(sx-rt)ds,\\
J_2(x,t) &= \frac{1}{\pi}\int\limits_0^\infty \frac{\tanh sh}{s}\cos(sx+rt)ds.
\end{aligned}
\tag{26}
$$

Careful examination of the second formula in (26) shows that $J_2(x,t) \approx 0$ for $t > 1$ s (Achenbach 1973, Szmidt 1999). In such a case, equation (25) gives

$$
h(x, t>1) \approx \frac{1}{\pi}\int\limits_0^\infty \frac{\tanh sh}{s}\cos(sx-rt)ds. \tag{27}
$$

Improper integrals in equations (26) and (27) may be estimated by discrete numerical integration. For the case of viscous damping, the range of integration in (23) is divided into two intervals dependent on the value of $\beta = \sqrt{(\mu r)^2 - 1}$. A common point of the intervals, say s_0, is defined by the formula

$$
g s_0 \tanh s_0 h = \frac{1}{\mu^2}. \tag{28}
$$

For weak damping, this formula gives

$$
s_0 \approx \frac{1}{g\mu^2}. \tag{29}
$$

The integrand of improper integral (23) at this isolated point has a finite value, and thus, the impulse response function is written in the following form:

$$
\begin{aligned}
h(x,t) = \frac{2}{\pi}\Bigg[&\int\limits_0^{s_0} \frac{\tanh sh\cdot\cos sx}{s}\exp(-\mu r^2 t)\left(\cos r\beta^* t - \mu r\frac{\sin r\beta^* t}{\beta^*}\right)\ ds\ +\\
&+\int\limits_{s^0}^{\infty} \frac{\tanh sh\cdot\cos sx}{s}\exp(-\mu r^2 t)\left(\cosh r\beta t - \mu r\frac{\sinh r\beta t}{\beta}\right)ds\Bigg],
\end{aligned}
\tag{30}
$$

where

$$
\beta^* = \sqrt{1-(\mu r)^2}. \tag{31}
$$

For weak damping ($0 < \mu << 0.01$ s), equation (29) gives $s_0 > 1000$ s, and the impulse response function may be described by the formula

$$
h(x,t) \approx \frac{2}{\pi}\left[\int\limits_0^{s_0} \frac{\tanh sh\cdot\cos sx}{s}\exp(-\mu r^2 t)\left(\cos r\beta^* t - \mu r\frac{\sin r\beta^* t}{\beta^*}\right)\ \right] ds. \tag{32}
$$

Solutions (30) and (32) are illustrated in the subsequent figures (2–5), in which the plots show the distribution of the impulse response functions in time for chosen set of the damping parameter (μ = 0, 0.01 s, 0.02 s, 0.1 s).

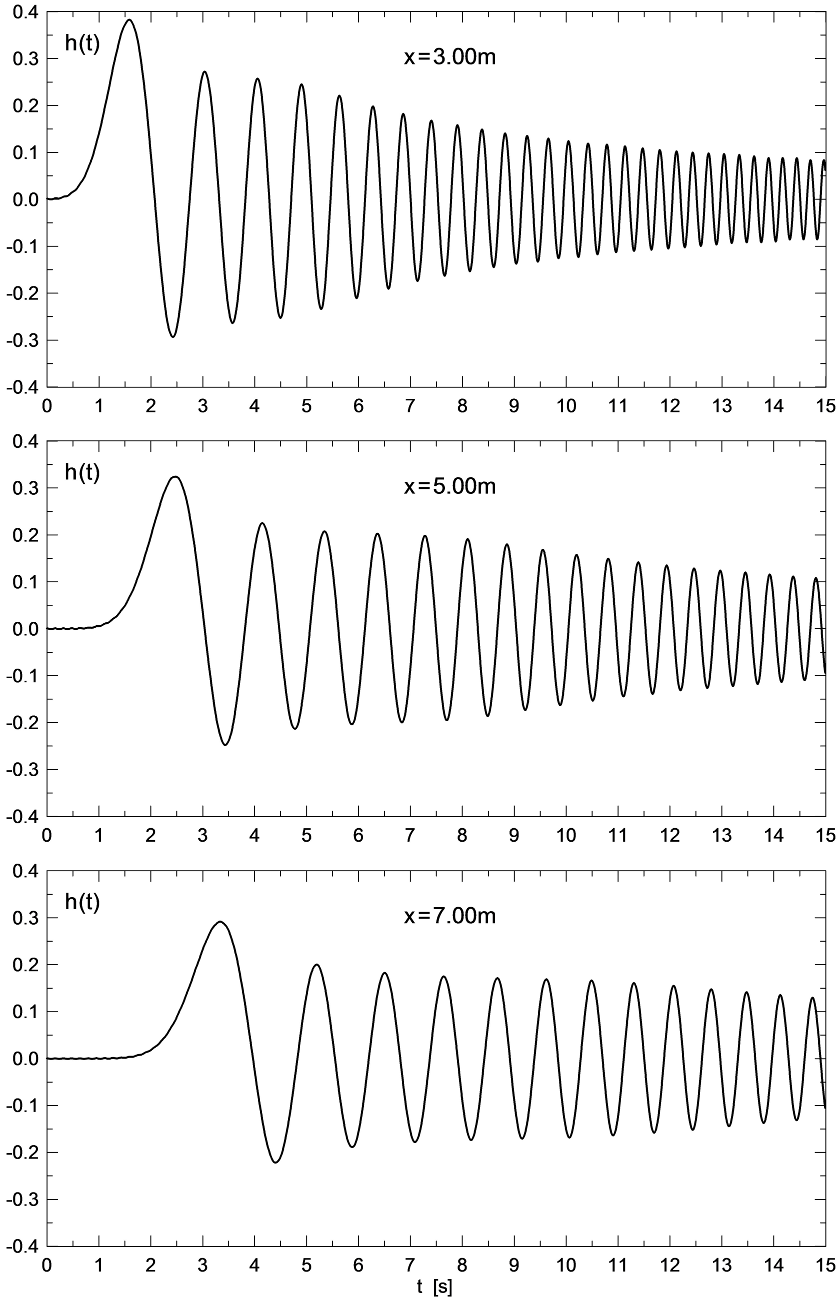

Fig. 2. Impulse response function at chosen points in space for $\mu = 0$

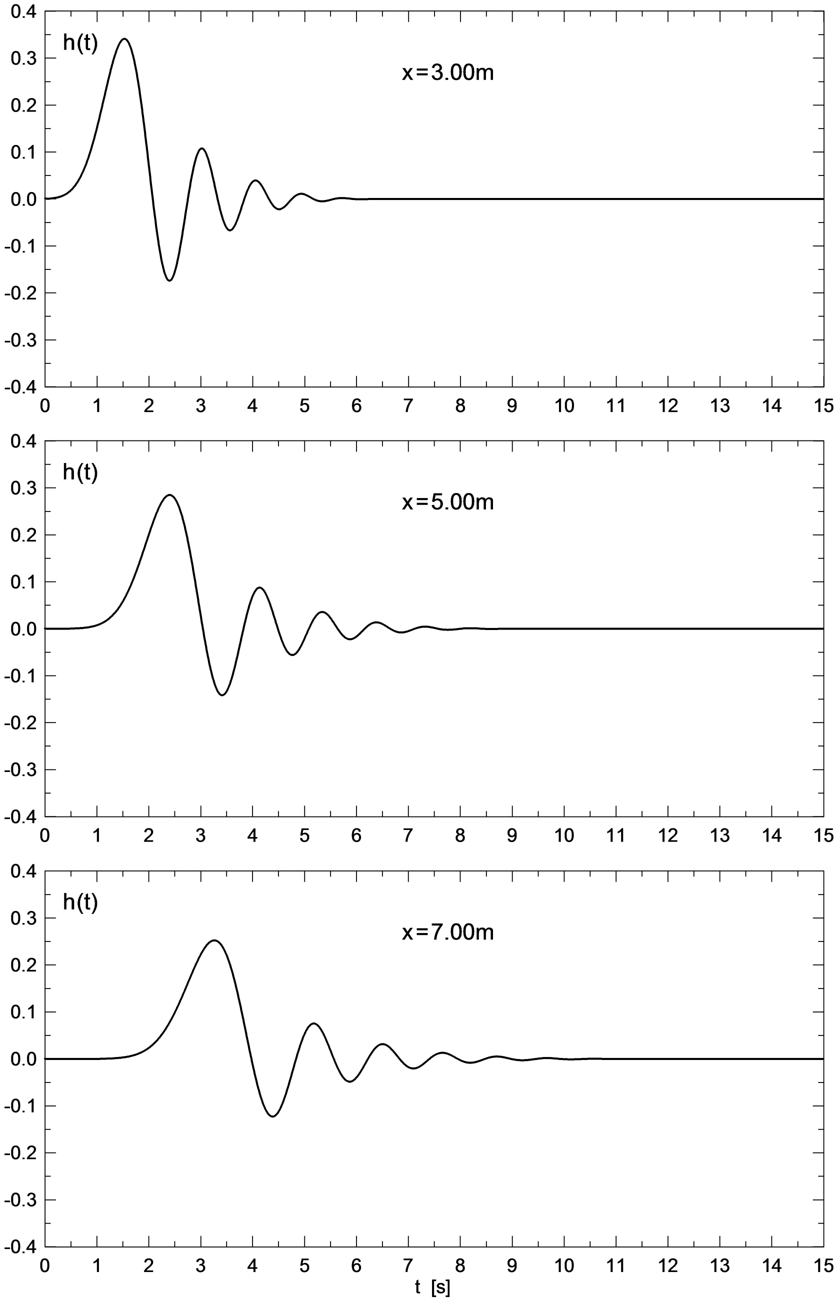

Fig. 3. Impulse response function at chosen points in space for $\mu = 0.01$

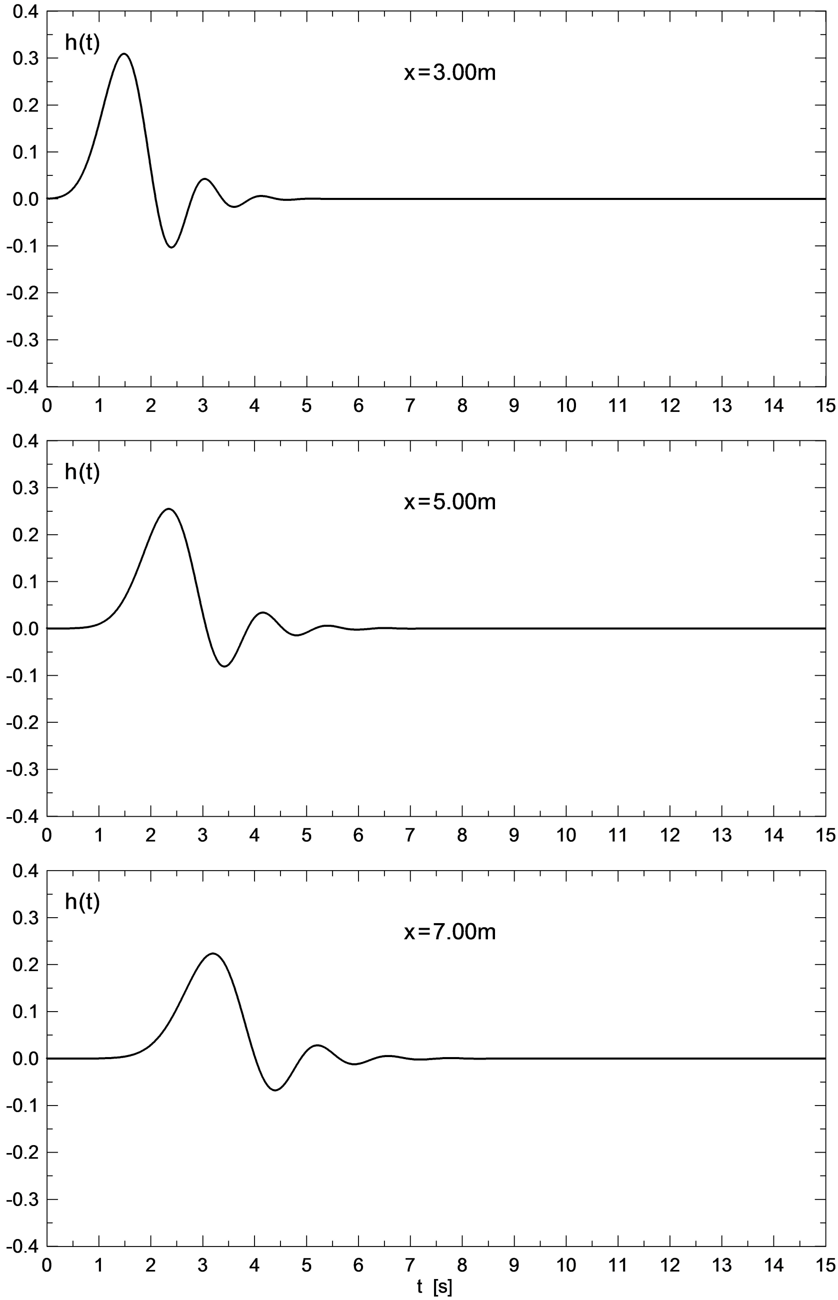

Fig. 4. Impulse response function at chosen points in space for $\mu = 0.02$

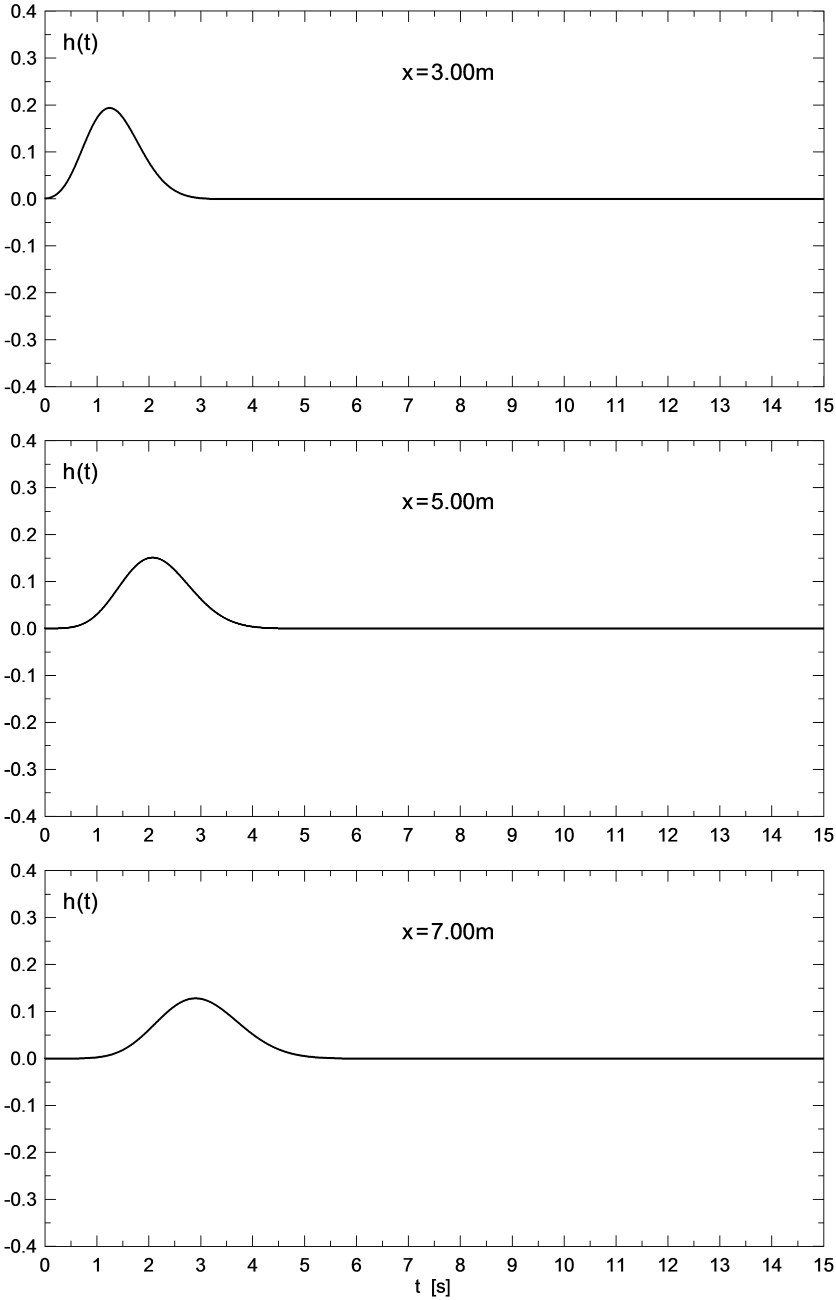

Fig. 5. Impulse response function at chosen points in space for $\mu = 0.1$

4. Concluding Remarks

The formulation developed in this paper makes it possible to calculate the free-surface elevation induced by the impulse motion of a piston-type generator placed at the beginning of a semi-infinite layer of fluid covered with crushed ice. The solution obtained is based on the assumption that the damping of surface waves caused by this ice layer may be properly described by one-parameter dash-pots distributed over the free surface of the fluid (μ in our formulation). With respect to natural conditions, in which sea waves are periodic in time, it would be desirable to develop a solution corresponding to a steady harmonic motion of the fluid. It should be stressed, however, that in such a case we deal with the problem of forced fluid motion with a continuous supply of energy, which is partially dissipated by the floating ice cover. In that case, it would be difficult to evaluate the share of the ice cover in the phenomenon of energy dissipation. Nevertheless, following the impulse response function, we may develop a solution for periodic motion by means of the convolution integral, as described by equation (22). In order to find a solution for periodic fluid motion, it is reasonable to consider the generator motion (vertical wall OA in Fig. 1)

$$x_g(t) = C\,[A(\tau)\cos\omega t + D(\tau)\sin\omega t]\,, \tag{33}$$

where $x_g(t)$ describes the generator displacement, C is a constant, and

$$\begin{aligned} A(\tau) &= \frac{\tau^3}{3!}\exp(-\tau), \\ D(\tau) &= 1 - \left(1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!}\right)\exp(-\tau), \quad \tau = \eta t. \end{aligned} \tag{34}$$

One can check that in the limiting case $t \to \infty$ the generation approaches harmonic generation. In practice, for a time exceeding one or two periods of generation ($T = 2\pi/\omega$) one may assume that we have a problem of harmonic generation with a constant amplitude. From equations (22) and plots in Fig. 5, it follows that for a relatively strong damping ($\mu \geq 0.1$ s) the range of integration in the convolution integral may be reduced to a few seconds ($t_{\max} < 10$ s).

References

Achenbach J. D. (1973) *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam.

Lysmer J., Kuhlemeyer R. L. (1969) Finite dynamic model for infinite media, *J. Eng. Mech. Division, Proc. of ASCE*, 859–877.

Nowacki W. (1972) *Dynamics of Structures* (in Polish), Arkady, Warszawa.

Szmidt J. K. (1999) *An introduction to Hydroelasticity of Sea Structures* (in Polish), Wyd. IBW-PAN, Gdańsk.