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Multilinear Operator and Its Basic Properties

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Summary. In the first chapter, the notion of multilinear operator on real linear spaces is discussed. The algebraic structure [2] of multilinear operators is introduced here. In the second chapter, the results of the first chapter are extended to the case of the normed spaces. This chapter shows that bounded multilinear operators on normed linear spaces constitute the algebraic structure. We referred to [3], [7], [5], [6] in this formalization.

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1. Multilinear Operator on Real Linear Spaces

Let X be a non empty, non-empty finite sequence, i be an object, and x be an element of $\prod X$. The functor $\operatorname{reproj}(i, x)$ yielding a function from $X(i)$ into $\prod X$ is defined by

(Def. 1) for every object r such that $r \in X(i)$ holds $it(r) = x +\cdot (i, r)$.

Now we state the propositions:

(1) Let us consider a non empty, non-empty finite sequence X, an element x of $\prod X$, an element i of $\operatorname{dom} X$, and an object r. If $r \in X(i)$, then $(\operatorname{reproj}(i, x))(r)(i) = r$.

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(2) Let us consider a non empty, non-empty finite sequence X, an element x of $\prod X$, elements i, j of $\operatorname{dom} X$, and an object r. If $r \in X(i)$ and $i \neq j$, then $(\operatorname{reproj}(i,x))(r)(j) = x(j)$.

(3) Let us consider a non empty, non-empty finite sequence X, an element x of $\prod X$, and an element i of $\operatorname{dom} X$. Then $(\operatorname{reproj}(i,x))(x(i)) = x$.

Let X be a real linear space sequence, i be an element of $\operatorname{dom} X$, and x be an element of $\prod X$. The functor $\operatorname{reproj}(i,x)$ yielding a function from $X(i)$ into $\prod X$ is defined by

(Def. 2) there exists an element x_0 of $\prod \overline{X}$ such that $x_0 = x$ and $it = \operatorname{reproj}(i, x_0)$.

Now we state the propositions:

(4) Let us consider a real linear space sequence X, an element i of $\operatorname{dom} X$, an element x of $\prod X$, an element r of $X(i)$, and a function F. If $F = (\operatorname{reproj}(i,x))(r)$, then $F(i) = r$. The theorem is a consequence of (1).

(5) Let us consider a real linear space sequence X, elements i, j of $\operatorname{dom} X$, an element x of $\prod X$, an element r of $X(i)$, and functions F, s. If $F = (\operatorname{reproj}(i,x))(r)$ and $x = s$ and $i \neq j$, then $F(j) = s(j)$. The theorem is a consequence of (2).

(6) Let us consider a real linear space sequence X, an element i of $\operatorname{dom} X$, an element x of $\prod X$, and a function s. If $x = s$, then $(\operatorname{reproj}(i,x))(s(i)) = x$. The theorem is a consequence of (3).

Let X be a real linear space sequence, Y be a real linear space, and f be a function from $\prod X$ into Y. We say that f is multilinear if and only if

(Def. 3) for every element i of $\operatorname{dom} X$ and for every element x of $\prod X$, $f \cdot (\operatorname{reproj}(i,x))$ is a linear operator from $X(i)$ into Y.

One can verify that there exists a function from $\prod X$ into Y which is multilinear.

A multilinear operator from X into Y is a multilinear function from $\prod X$ into Y. Now we state the propositions:

(7) Let us consider real linear spaces X, Y, and a linear operator f from X into Y. Then $0_Y = f(0_X)$.

(8) Let us consider a real linear space sequence X, a real linear space Y, a multilinear operator g from X into Y, a point t of $\prod X$, and an element s of $\prod \overline{X}$. Suppose $s = t$ and there exists an element i of $\operatorname{dom} X$ such that $s(i) = 0_{X(i)}$. Then $g(t) = 0_Y$. The theorem is a consequence of (17) and (7).

(9) Let us consider a real linear space sequence X, a real linear space Y, a multilinear operator g from X into Y, and a finite sequence a of elements of $\mathbb{R}$. Suppose $\operatorname{dom} a = \operatorname{dom} X$. Let us consider points t, t_1 of $\prod X$, and

elements s, s_1 of $\prod \overline{X}$. Suppose $t = s$ and $t_1 = s_1$ and for every element i of $\operatorname{dom} X$, $s_1(i) = a_{/i} \cdot s(i)$. Then $g(t_1) = (\prod a) \cdot g(t)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every points t, t_1 of $\prod X$ for every elements s, s_1 of $\prod \overline{X}$ for every finite sequence b of elements of $\mathbb{R}$ such that $t = s$ and $t_1 = s_1$ and $b = a{\restriction}\$_1$ and $\$_1 \leqslant \operatorname{len} a$ and for every element i of $\operatorname{dom} X$, if $i \in \operatorname{Seg} \$_1$, then $s_1(i) = a_{/i} \cdot s(i)$ and if $i \notin \operatorname{Seg} \$_1$, then $s_1(i) = s(i)$ holds $g(t_1) = (\prod b) \cdot g(t)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. For every element i of $\operatorname{dom} X$, if $i \in \operatorname{Seg} \operatorname{len} a$, then $s_1(i) = a_{/i} \cdot s(i)$ and if $i \notin \operatorname{Seg} \operatorname{len} a$, then $s_1(i) = s(i)$. $\square$

Let X be a real linear space sequence and Y be a real linear space. The functor $\operatorname{MultOpers}(X, Y)$ yielding a subset of $\operatorname{RealVectSpace}((\text{the carrier of } \prod X), Y)$ is defined by

(Def. 4) for every set x, $x \in \mathit{it}$ iff x is a multilinear operator from X into Y.

One can check that $\operatorname{MultOpers}(X, Y)$ is non empty and functional and $\operatorname{MultOpers}(X, Y)$ is linearly closed.

The functor $\operatorname{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)$ yielding a strict RLS structure is defined by the term

(Def. 5) $\langle \operatorname{MultOpers}(X, Y), \operatorname{Zero}(\operatorname{MultOpers}(X, Y), \operatorname{RealVectSpace}((\text{the carrier of } \prod X), Y)), \operatorname{Add}(\operatorname{MultOpers}(X, Y), \operatorname{RealVectSpace}((\text{the carrier of } \prod X), Y)), \operatorname{Mult}(\operatorname{MultOpers}(X, Y), \operatorname{RealVectSpace}((\text{the carrier of } \prod X), Y)) \rangle$.

Now we state the proposition:

(10) Let us consider a real linear space sequence X, and a real linear space Y. Then $\langle \operatorname{MultOpers}(X, Y), \operatorname{Zero}(\operatorname{MultOpers}(X, Y), \operatorname{RealVectSpace}((\text{the carrier of } \prod X), Y)), \operatorname{Add}(\operatorname{MultOpers}(X, Y), \operatorname{RealVectSpace}((\text{the carrier of } \prod X), Y)), \operatorname{Mult}(\operatorname{MultOpers}(X, Y), \operatorname{RealVectSpace}((\text{the carrier of } \prod X), Y)) \rangle$ is a subspace of $\operatorname{RealVectSpace}((\text{the carrier of } \prod X), Y)$.

Let X be a real linear space sequence and Y be a real linear space. One can verify that $\operatorname{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)$ is non empty and $\operatorname{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital and $\operatorname{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)$ is constituted functions.

Let f be an element of $\operatorname{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)$ and v be a vector of $\prod X$. Let us note that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(11) Let us consider a real linear space sequence X, a real linear space Y, and vectors f, g, h of $\operatorname{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)$. Then $h = f + g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) + g(x)$.

(12) Let us consider a real linear space sequence X, a real linear space Y, vectors f, h of VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of $\prod X$, $h(x) = a \cdot f(x)$.

Let us consider a real linear space sequence X and a real linear space Y. Now we state the propositions:

(13) $0_{\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)} = (\text{the carrier of } \prod X) \longmapsto 0_Y$.

(14) (The carrier of $\prod X) \longmapsto 0_Y$ is a multilinear operator from X into Y.

2. Bounded Multilinear Operator on Normed Linear Spaces

Now we state the propositions:

(15) Let us consider a real norm space sequence X, an element i of $\operatorname{dom} X$, an element x of $\prod X$, an element r of $X(i)$, and a function F. If $F = (\operatorname{reproj}(i, x))(r)$, then $F(i) = r$. The theorem is a consequence of (1).

(16) Let us consider a real norm space sequence X, elements i, j of $\operatorname{dom} X$, an element x of $\prod X$, an element r of $X(i)$, and functions F, s. If $F = (\operatorname{reproj}(i, x))(r)$ and $x = s$ and $i \neq j$, then $F(j) = s(j)$. The theorem is a consequence of (2).

(17) Let us consider a real norm space sequence X, an element i of $\operatorname{dom} X$, an element x of $\prod X$, and a function s. If $x = s$, then $(\operatorname{reproj}(i, x))(s(i)) = x$. The theorem is a consequence of (3).

Let X be a real norm space sequence, Y be a real normed space, and f be a function from $\prod X$ into Y. We say that f is multilinear if and only if

(Def. 6) for every element i of $\operatorname{dom} X$ and for every element x of $\prod X$, $f \cdot (\operatorname{reproj}(i, x))$ is a linear operator from $X(i)$ into Y.

One can verify that there exists a function from $\prod X$ into Y which is multilinear.

A multilinear operator from X into Y is a multilinear function from $\prod X$ into Y. The functor MultOpers(X, Y) yielding a subset of RealVectSpace((the carrier of $\prod X), Y)$ is defined by

(Def. 7) for every set x, $x \in \mathit{it}$ iff x is a multilinear operator from X into Y.

Note that MultOpers(X, Y) is non empty and functional and MultOpers(X, Y) is linearly closed.

Now we state the proposition:

(18) Let us consider a real norm space sequence X, and a real normed space Y. Then $\langle$MultOpers(X, Y), Zero(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X), Y)$), Add(MultOpers(X, Y), RealVectSpace((the carrier of $\prod X)$,

$Y)), \mathrm{Mult}(\mathrm{MultOpers}(X,Y), \mathrm{RealVectSpace}((\text{the carrier of } \prod X), Y))\rangle$ is a subspace of $\mathrm{RealVectSpace}((\text{the carrier of } \prod X), Y)$.

Let X be a real norm space sequence and Y be a real normed space. Note that $\langle \mathrm{MultOpers}(X,Y), \mathrm{Zero}(\mathrm{MultOpers}(X,Y), \mathrm{RealVectSpace}((\text{the carrier of } \prod X), Y)), \mathrm{Add}(\mathrm{MultOpers}(X,Y), \mathrm{RealVectSpace}((\text{the carrier of } \prod X), Y)),$
$\mathrm{Mult}(\mathrm{MultOpers}(X,Y), \mathrm{RealVectSpace}((\text{the carrier of } \prod X), Y))\rangle$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The functor $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)$ yielding a strict real linear space is defined by the term

(Def. 8) $\langle \mathrm{MultOpers}(X,Y), \mathrm{Zero}(\mathrm{MultOpers}(X,Y), \mathrm{RealVectSpace}((\text{the carrier of } \prod X), Y)), \mathrm{Add}(\mathrm{MultOpers}(X,Y), \mathrm{RealVectSpace}((\text{the carrier of } \prod X), Y)), \mathrm{Mult}(\mathrm{MultOpers}(X,Y), \mathrm{RealVectSpace}((\text{the carrier of } \prod X), Y))\rangle$.

One can check that $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)$ is constituted functions.

Let f be an element of $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)$ and v be a vector of $\prod X$. One can check that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(19) Let us consider a real norm space sequence X, a real normed space Y, and vectors f, g, h of $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)$. Then $h = f + g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) + g(x)$.

(20) Let us consider a real norm space sequence X, a real normed space Y, vectors f, h of $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of $\prod X$, $h(x) = a \cdot f(x)$.

Let us consider a real norm space sequence X and a real normed space Y. Now we state the propositions:

(21) $0_{\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)} = (\text{the carrier of } \prod X) \longmapsto 0_Y$.

(22) (The carrier of $\prod X) \longmapsto 0_Y$ is a multilinear operator from X into Y.

Let X be a real norm space sequence, Y be a real normed space, I be a multilinear operator from X into Y, and x be a vector of $\prod X$. Let us observe that the functor $I(x)$ yields a point of Y. Note that $\prod X$ is constituted functions.

Let x be a point of $\prod X$ and i be an element of $\mathrm{dom}\, X$. One can check that the functor $x(i)$ yields a point of $X(i)$. Now we state the propositions:

(23) Let us consider a real norm space sequence G, and points p, q, r of $\prod G$. Then $p+q = r$ if and only if for every element i of $\mathrm{dom}\, G$, $r(i) = p(i)+q(i)$.

(24) Let us consider a real norm space sequence G, points p, r of $\prod G$, and a real number a. Then $a \cdot p = r$ if and only if for every element i of $\mathrm{dom}\, G$, $r(i) = a \cdot p(i)$.

(25) Let us consider a real norm space sequence G, and a point p of $\prod G$. Then $0_{\prod G} = p$ if and only if for every element i of $\operatorname{dom} G$, $p(i) = 0_{G(i)}$.

(26) Let us consider a real norm space sequence G, and points p, q, r of $\prod G$. Then $p - q = r$ if and only if for every element i of $\operatorname{dom} G$, $r(i) = p(i) - q(i)$. The theorem is a consequence of (23) and (24).

Let X be a real norm space sequence and x be a point of $\prod X$. The functor $\operatorname{NrProduct} x$ yielding a non negative real number is defined by

(Def. 9) there exists a finite sequence N of elements of $\mathbb{R}$ such that $\operatorname{dom} N = \operatorname{dom} X$ and for every element i of $\operatorname{dom} X$, $N(i) = \|x(i)\|$ and $it = \prod N$.

Now we state the proposition:

(27) Let us consider a real norm space sequence X, and a point x of $\prod X$. Then

(i) there exists an element i of $\operatorname{dom} X$ such that $x(i) = 0_{X(i)}$ iff $\operatorname{NrProduct} x = 0$, and

(ii) if there exists no element i of $\operatorname{dom} X$ such that $x(i) = 0_{X(i)}$, then $0 < \operatorname{NrProduct} x$.

Proof: Consider N being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} N = \operatorname{dom} X$ and for every element i of $\operatorname{dom} X$, $N(i) = \|x(i)\|$ and $\operatorname{NrProduct} x = \prod N$. There exists an element i of $\operatorname{dom} X$ such that $x(i) = 0_{X(i)}$ iff $\operatorname{NrProduct} x = 0$ by [1, (103)]. If there exists no element i of $\operatorname{dom} X$ such that $x(i) = 0_{X(i)}$, then $0 < \operatorname{NrProduct} x$ by [4, (42)]. $\square$

Let X be a real norm space sequence, Y be a real normed space, and I be a multilinear operator from X into Y. We say that I is Lipschitzian if and only if

(Def. 10) there exists a real number K such that $0 \leqslant K$ and for every point x of $\prod X$, $\|I(x)\| \leqslant K \cdot (\operatorname{NrProduct} x)$.

Now we state the proposition:

(28) Let us consider a real norm space sequence X, a real normed space Y, and a multilinear operator f from X into Y. If for every vector x of $\prod X$, $f(x) = 0_Y$, then f is Lipschitzian.

Let X be a real norm space sequence and Y be a real normed space. One can check that there exists a multilinear operator from X into Y which is Lipschitzian.

The functor $\operatorname{BoundedMultOpers}(X, Y)$ yielding a subset of $\operatorname{VectorSpaceOfMultOpers}_{\mathbb{R}}(X, Y)$ is defined by

(Def. 11) for every set x, $x \in it$ iff x is a Lipschitzian multilinear operator from X into Y.

Note that BoundedMultOpers(X, Y) is non empty and BoundedMultOpers(X, Y) is linearly closed.

Now we state the proposition:

(29) Let us consider a real norm space sequence X, and a real normed space Y. Then $\langle$BoundedMultOpers(X, Y), Zero(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Add(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Mult(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$)$\rangle$ is a subspace of VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$.

Let X be a real norm space sequence and Y be a real normed space. Observe that $\langle$BoundedMultOpers(X, Y), Zero(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Add(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Mult(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$)$\rangle$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The functor VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$ yielding a strict real linear space is defined by the term

(Def. 12) $\langle$BoundedMultOpers(X, Y), Zero(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Add(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$), Mult(BoundedMultOpers(X, Y), VectorSpaceOfMultOpers$_{\mathbb{R}}(X, Y)$)$\rangle$.

Let us note that every element of VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$ is function-like and relation-like.

Let f be an element of VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$ and v be a vector of $\prod X$. Note that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(30) Let us consider a real norm space sequence X, a real normed space Y, and vectors f, g, h of VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$. Then $h = f + g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) + g(x)$. The theorem is a consequence of (19).

(31) Let us consider a real norm space sequence X, a real normed space Y, vectors f, h of VectorSpaceOfBoundedMultOpers$_{\mathbb{R}}(X, Y)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of $\prod X$, $h(x) = a \cdot f(x)$. The theorem is a consequence of (20).

(32) Let us consider a real norm space sequence X, and a real normed space Y. Then $0_{\mathrm{VectorSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)} = (\text{the carrier of } \prod X) \longmapsto 0_Y$. The theorem is a consequence of (21).

Let X be a real norm space sequence, Y be a real normed space, and f be an object. Assume $f \in \mathrm{BoundedMultOpers}(X, Y)$. The functor $\mathrm{PartFuncs}(f, X, Y)$ yielding a Lipschitzian multilinear operator from X into Y is defined by the term

(Def. 13) f.

Let u be a multilinear operator from X into Y. The functor $\mathrm{PreNorms}(u)$ yielding a non empty subset of $\mathbb{R}$ is defined by the term

(Def. 14) $\{\|u(t)\|$, where t is a vector of $\prod X$: for every element i of $\mathrm{dom}\, X$, $\|t(i)\| \leqslant 1\}$.

Now we state the propositions:

(33) Let us consider a real norm space sequence X, and an element s of $\prod X$. Then there exists a finite sequence F of elements of $\mathbb{R}$ such that

(i) $\mathrm{dom}\, F = \mathrm{dom}\, X$, and

(ii) for every element i of $\mathrm{dom}\, X$, $F(i) = \|s(i)\|$.

PROOF: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an element i of $\mathrm{dom}\, X$ such that $\$_1 = i$ and $\$_2 = \|s(i)\|$. For every natural number n such that $n \in \mathrm{Seg}\,\mathrm{len}\, X$ there exists an element d of $\mathbb{R}$ such that $\mathcal{Q}[n, d]$. Consider F being a finite sequence of elements of $\mathbb{R}$ such that $\mathrm{len}\, F = \mathrm{len}\, X$ and for every natural number n such that $n \in \mathrm{Seg}\,\mathrm{len}\, X$ holds $\mathcal{Q}[n, F_{/n}]$. For every element i of $\mathrm{dom}\, X$, $F(i) = \|s(i)\|$. $\square$

(34) Let us consider a finite sequence F of elements of $\mathbb{R}$. Suppose for every element i of $\mathrm{dom}\, F$, $0 \leqslant F(i) \leqslant 1$. Then $0 \leqslant \prod F \leqslant 1$.

(35) Let us consider a real norm space sequence X, and a point x of $\prod X$. Suppose for every element i of $\mathrm{dom}\, X$, $\|x(i)\| \leqslant 1$. Then $0 \leqslant \mathrm{NrProduct}\, x \leqslant 1$. The theorem is a consequence of (34).

(36) Let us consider a real norm space sequence X, a real normed space Y, a multilinear operator g from X into Y, and a point t of $\prod X$. Suppose there exists an element i of $\mathrm{dom}\, X$ such that $t(i) = 0_{X(i)}$. Then $g(t) = 0_Y$. The theorem is a consequence of (17).

(37) Let us consider a real norm space sequence X, and a point x of $\prod X$. Then there exists a finite sequence d of elements of $\mathbb{R}$ such that

(i) $\mathrm{dom}\, d = \mathrm{dom}\, X$, and

(ii) for every element i of $\mathrm{dom}\, X$, $d(i) = \|x(i)\|^{-1}$.

PROOF: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an element i of $\mathrm{dom}\, X$ such that $\$_1 = i$ and $\$_2 = \|x(i)\|^{-1}$. For every natural number n such that $n \in \mathrm{Seg}\,\mathrm{len}\, X$ there exists an element d of $\mathbb{R}$ such that $\mathcal{Q}[n, d]$. Consider F being a finite sequence of elements of $\mathbb{R}$ such that $\mathrm{len}\, F = \mathrm{len}\, X$ and for every natural number n such that $n \in \mathrm{Seg}\,\mathrm{len}\, X$ holds $\mathcal{Q}[n, F_{/n}]$. For every element i of $\mathrm{dom}\, X$, $F(i) = \|x(i)\|^{-1}$. $\square$

(38) Let us consider a real norm space sequence X, a point s of $\prod X$, and a finite sequence a of elements of $\mathbb{R}$. Then there exists a point s_1 of $\prod X$ such that for every element i of $\operatorname{dom} X$, $s_1(i) = a_{/i} \cdot s(i)$.

Proof: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an element i of $\operatorname{dom} X$ such that $\$_1 = i$ and $\$_2 = a_{/i} \cdot x(i)$. For every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ there exists an object d such that $\mathcal{Q}[n, d]$. Consider F being a finite sequence such that $\operatorname{dom} F = \operatorname{Seg} \operatorname{len} X$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ holds $\mathcal{Q}[n, F(n)]$. For every object y such that $y \in \operatorname{dom} \overline{X}$ holds $F(y) \in \overline{X}(y)$. For every element i of $\operatorname{dom} X$, $F(i) = a_{/i} \cdot x(i)$. □

(39) Let us consider a real norm space sequence X, a real normed space Y, a multilinear operator g from X into Y, and a finite sequence a of elements of $\mathbb{R}$. Suppose $\operatorname{dom} a = \operatorname{dom} X$. Let us consider points t, t_1 of $\prod X$. Suppose for every element i of $\operatorname{dom} X$, $t_1(i) = a_{/i} \cdot t(i)$. Then $g(t_1) = (\prod a) \cdot g(t)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every points t, t_1 of $\prod X$ for every finite sequence b of elements of $\mathbb{R}$ such that $b = a \restriction \$_1$ and $\$_1 \leqslant \operatorname{len} a$ and for every element i of $\operatorname{dom} X$, if $i \in \operatorname{Seg} \$_1$, then $t_1(i) = a_{/i} \cdot t(i)$ and if $i \notin \operatorname{Seg} \$_1$, then $t_1(i) = t(i)$ holds $g(t_1) = (\prod b) \cdot g(t)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. For every element i of $\operatorname{dom} X$, if $i \in \operatorname{Seg} \operatorname{len} a$, then $t_1(i) = a_{/i} \cdot t(i)$ and if $i \notin \operatorname{Seg} \operatorname{len} a$, then $t_1(i) = t(i)$. □

(40) Let us consider finite sequences F, G of elements of $\mathbb{R}$. Suppose $\operatorname{dom} F = \operatorname{dom} G$ and for every element i of $\operatorname{dom} F$, $G(i) = F(i)^{-1}$. Then $\prod G = (\prod F)^{-1}$.

(41) Let us consider a real norm space sequence X, a real normed space Y, and a Lipschitzian multilinear operator g from X into Y. Then $\operatorname{PreNorms}(g)$ is upper bounded. The theorem is a consequence of (35).

(42) Let us consider a real norm space sequence X, a real normed space Y, and a multilinear operator g from X into Y. Then g is Lipschitzian if and only if $\operatorname{PreNorms}(g)$ is upper bounded. The theorem is a consequence of (36), (37), (38), (39), (40), and (41).

Let X be a real norm space sequence and Y be a real normed space. The functor $\operatorname{BoundedMultOpersNorm}(X, Y)$ yielding a function from $\operatorname{BoundedMultOpers}(X, Y)$ into $\mathbb{R}$ is defined by

(Def. 15) for every object x such that $x \in \operatorname{BoundedMultOpers}(X, Y)$ holds $it(x) = \sup \operatorname{PreNorms}(\operatorname{PartFuncs}(x, X, Y))$.

Let f be a Lipschitzian multilinear operator from X into Y. One can verify that $\operatorname{PartFuncs}(f, X, Y)$ reduces to f.

Now we state the proposition:

(43) Let us consider a real norm space sequence X, a real normed space Y, and a Lipschitzian multilinear operator f from X into Y. Then $(\mathrm{BoundedMultOpersNorm}(X,Y))(f) = \sup \mathrm{PreNorms}(f)$.

Let X be a real norm space sequence and Y be a real normed space. The functor $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)$ yielding a non empty, strict normed structure is defined by the term

(Def. 16) $\langle \mathrm{BoundedMultOpers}(X,Y), \mathrm{Zero}(\mathrm{BoundedMultOpers}(X,Y), \mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)), \mathrm{Add}(\mathrm{BoundedMultOpers}(X,Y), \mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)), \mathrm{Mult}(\mathrm{BoundedMultOpers}(X,Y), \mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(X,Y)), \mathrm{BoundedMultOpersNorm}(X,Y)\rangle$.

Now we state the propositions:

(44) Let us consider a real norm space sequence X, and a real normed space Y. Then (the carrier of $\prod X) \longmapsto 0_Y = 0_{\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)}$. The theorem is a consequence of (32).

(45) Let us consider a real norm space sequence X, a real normed space Y, a point f of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)$, and a Lipschitzian multilinear operator g from X into Y. Suppose $g = f$. Let us consider a vector t of $\prod X$. Then $\|g(t)\| \leqslant \|f\| \cdot (\mathrm{NrProduct}\, t)$. The theorem is a consequence of (41), (36), (37), (38), (39), (40), and (43).

Let us consider a real norm space sequence X, a real normed space Y, and a point f of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)$. Now we state the propositions:

(46) $0 \leqslant \|f\|$. The theorem is a consequence of (41) and (43).

(47) If $f = 0_{\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)}$, then $0 = \|f\|$. The theorem is a consequence of (41), (44), and (43).

Let X be a real norm space sequence and Y be a real normed space. Let us note that every element of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)$ is function-like and relation-like.

Let f be an element of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)$ and v be a vector of $\prod X$. Note that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(48) Let us consider a real norm space sequence X, a real normed space Y, and points f, g, h of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)$. Then $h = f + g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) + g(x)$. The theorem is a consequence of (30).

(49) Let us consider a real norm space sequence X, a real normed space Y, points f, h of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)$, and a real number a. Then $h = a \cdot f$ if and only if for every vector x of $\prod X$, $h(x) = a \cdot f(x)$.

The theorem is a consequence of (31).

(50) Let us consider a real norm space sequence X, a real normed space Y, points f, g of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$, and a real number a. Then

(i) $\|f\| = 0$ iff $f = 0_{\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X,Y)}$, and

(ii) $\|a \cdot f\| = |a| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

Proof: $\|f + g\| \leqslant \|f\| + \|g\|$. $\|a \cdot f\| = |a| \cdot \|f\|$. □

(51) Let us consider a real norm space sequence X, and a real normed space Y. Then $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ is a real normed space.

Let X be a real norm space sequence and Y be a real normed space. Let us note that $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ is reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Now we state the proposition:

(52) Let us consider a real norm space sequence X, a real normed space Y, and points f, g, h of $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$. Then $h = f - g$ if and only if for every vector x of $\prod X$, $h(x) = f(x) - g(x)$. The theorem is a consequence of (48).

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