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Zariski Topology

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Summary. We formalize in the Mizar system [3], [4] basic definitions of commutative ring theory such as prime spectrum, nilradical, Jacobson radical, local ring, and semi-local ring [5], [6], then formalize proofs of some related theorems along with the first chapter of [1].

The article introduces the so-called Zariski topology. The set of all prime ideals of a commutative ring A is called the prime spectrum of A denoted by Spectrum A. A new functor Spec generates Zariski topology to make Spectrum A a topological space. A different role is given to Spec as a map from a ring morphism of commutative rings to that of topological spaces by the following manner: for a ring homomorphism $h : A \longrightarrow B$, we defined (Spec h) : Spec $B \longrightarrow$ Spec A by (Spec h)($\mathfrak{p}$) $= h^{-1}(\mathfrak{p})$ where $\mathfrak{p} \in$ Spec B.

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1. Preliminaries: Some Properties of Ideals

From now on R denotes a commutative ring, A, B denote non degenerated, commutative rings, h denotes a function from A into B, I, I_1, I_2 denote ideals of A, J, J_1, J_2 denote proper ideals of A, p denotes a prime ideal of A.

S denotes non empty subset of A, E, E_1, E_2 denote subsets of A, a, b, f denote elements of A, n denotes a natural number, and x denotes object.

Let us consider A and S. The functor Ideals(A, S) yielding a subset of Ideals A is defined by the term

(Def. 1) $\{I$, where I is an ideal of $A : S \subseteq I\}$.

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Let us observe that $\mathrm{Ideals}(A, S)$ is non empty.

Now we state the proposition:

(1) $\mathrm{Ideals}(A, S) = \mathrm{Ideals}(A, S\text{–ideal})$.

PROOF: $\mathrm{Ideals}(A, S) \subseteq \mathrm{Ideals}(A, S\text{–ideal})$. Consider y being an ideal of A such that $x = y$ and S–ideal $\subseteq y$. $\square$

Let A be a unital, non empty multiplicative loop with zero structure and a be an element of A. We say that a is nilpotent if and only if

(Def. 2) there exists a non zero natural number k such that $a^k = 0_A$.

Let us note that 0_A is nilpotent and there exists an element of A which is nilpotent.

Let us consider A. Observe that 1_A is non nilpotent.

Let us consider f. The functor $\mathrm{MultClSet}(f)$ yielding a subset of A is defined by the term

(Def. 3) the set of all f^i where i is a natural number.

Let us observe that $\mathrm{MultClSet}(f)$ is multiplicatively closed.

Now we state the propositions:

(2) Let us consider a natural number n. Then $(1_A)^n = 1_A$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (1_A)^{\$_1} = 1_A$. For every natural number n, $\mathcal{P}[n]$. $\square$

(3) $1_A \notin \sqrt{J}$. The theorem is a consequence of (2).

(4) $\mathrm{MultClSet}(1_A) = \{1_A\}$. The theorem is a consequence of (2).

Let us consider A, J, and f. The functor $\mathrm{Ideals}(A, J, f)$ yielding a subset of $\mathrm{Ideals}\, A$ is defined by the term

(Def. 4) $\{I$, where I is a subset of A : I is a proper ideal of A and $J \subseteq I$ and $I \cap \mathrm{MultClSet}(f) = \emptyset\}$.

Let us consider A, J, and f. Now we state the propositions:

(5) If $f \notin \sqrt{J}$, then $J \in \mathrm{Ideals}(A, J, f)$.

(6) If $f \notin \sqrt{J}$, then $\mathrm{Ideals}(A, J, f)$ has the upper Zorn property w.r.t. $\subseteq_{\mathrm{Ideals}(A,J,f)}$.

PROOF: Set $S = \mathrm{Ideals}(A, J, f)$. Set $P = \subseteq_S$. For every set Y such that $Y \subseteq S$ and $P\,|^2\, Y$ is a linear order there exists a set x such that $x \in S$ and for every set y such that $y \in Y$ holds $\langle y, x\rangle \in P$. $\square$

(7) If $f \notin \sqrt{J}$, then there exists a prime ideal m of A such that $f \notin m$ and $J \subseteq m$.

PROOF: Set $S = \mathrm{Ideals}(A, J, f)$. Set $P = \subseteq_S$. Consider I being a set such that I is maximal in P. Consider p being a subset of A such that $p = I$ and p is a proper ideal of A and $J \subseteq p$ and $p \cap \mathrm{MultClSet}(f) = \emptyset$. p is a quasi-prime ideal of A. $\square$

(8) There exists a maximal ideal m of A such that $J \subseteq m$.
Proof: $1_A \notin \sqrt{J}$. Set $S = \mathrm{Ideals}(A, J, 1_A)$. Set $P = \subseteq_S$. Consider I being a set such that I is maximal in P. Consider p being a subset of A such that $p = I$ and p is a proper ideal of A and $J \subseteq p$ and $p \cap \mathrm{MultClSet}(1_A) = \emptyset$. For every ideal q of A such that $p \subseteq q$ holds $q = p$ or q is not proper. □

(9) There exists a prime ideal m of A such that $J \subseteq m$. The theorem is a consequence of (8).

(10) If a is a non-unit of A, then there exists a maximal ideal m of A such that $a \in m$. The theorem is a consequence of (8).

2. Spectrum of Prime Ideals (Spectrum) and Maximal Ideals (m-Spectrum)

Let R be a commutative ring. The spectrum of R yielding a family of subsets of R is defined by the term

(Def. 5) $\begin{cases} \{I, \text{ where } I \text{ is an ideal of } R : I \text{ is quasi-prime and } I \neq \Omega_R\}, \\ \quad \textbf{if } R \text{ is not degenerated}, \\ \emptyset, \ \textbf{otherwise}. \end{cases}$

Let us consider A. Observe that the spectrum of A yields a family of subsets of A and is defined by the term

(Def. 6) the set of all I where I is a prime ideal of A.

Observe that the spectrum of A is non empty.

Let us consider R. The functor m-Spectrum(R) yielding a family of subsets of R is defined by the term

(Def. 7) $\begin{cases} \{I, \text{ where } I \text{ is an ideal of } R : I \text{ is quasi-maximal and } I \neq \Omega_R\}, \\ \quad \textbf{if } R \text{ is not degenerated}, \\ \emptyset, \ \textbf{otherwise}. \end{cases}$

Let us consider A. Observe that the functor m-Spectrum(A) yields a family of subsets of the carrier of A and is defined by the term

(Def. 8) the set of all I where I is a maximal ideal of A.

Observe that m-Spectrum(A) is non empty.

3. Local and Semi-Local Ring

Let us consider A. We say that A is local if and only if

(Def. 9) there exists an ideal m of A such that m-Spectrum(A) $= \{m\}$.

We say that A is semi-local if and only if

(Def. 10) m-Spectrum(A) is finite.

Now we state the propositions:

(11) If $x \in I$ and I is a proper ideal of A, then x is a non-unit of A.

(12) If for every objects m_1, m_2 such that m_1, $m_2 \in$ m-Spectrum(A) holds $m_1 = m_2$, then A is local.

(13) If for every x such that $x \in \Omega_A \setminus J$ holds x is a unit of A, then A is local. The theorem is a consequence of (8), (11), and (12).

In the sequel m denotes a maximal ideal of A. Now we state the propositions:

(14) If $a \in \Omega_A \setminus m$, then $\{a\}$–ideal $+ m = \Omega_A$.

(15) If for every a such that $a \in m$ holds $1_A + a$ is a unit of A, then A is local.
PROOF: For every x such that $x \in \Omega_A \setminus m$ holds x is a unit of A. □

Let us consider R. Let E be a subset of R. The functor PrimeIdeals(R, E) yielding a subset of the spectrum of R is defined by the term

(Def. 11) $\begin{cases} \{p, \text{ where } p \text{ is an ideal of } R : p \text{ is quasi-prime and } p \neq \Omega_R \text{ and } E \subseteq p\}, \\ \quad \textbf{if } R \text{ is not degenerated}, \\ \emptyset, \ \textbf{otherwise}. \end{cases}$

Let us consider A. Let E be a subset of A. Let us note that the functor PrimeIdeals(A, E) yields a subset of the spectrum of A and is defined by the term

(Def. 12) $\{p$, where p is a prime ideal of $A : E \subseteq p\}$.

Let us consider J. Observe that PrimeIdeals(A, J) is non empty.

From now on p denotes a prime ideal of A and k denotes a non zero natural number. Now we state the proposition:

(16) If $a \notin p$, then $a^k \notin p$.

4. Nilradical and Jacobson Radical

Let us consider A. The functor nilrad(A) yielding a subset of A is defined by the term

(Def. 13) the set of all a where a is a nilpotent element of A.

Now we state the proposition:

(17) nilrad(A) $= \sqrt{\{0_A\}}$.

Let us consider A. One can verify that nilrad(A) is non empty and nilrad(A) is closed under addition as a subset of A and nilrad(A) is left and right ideal as a subset of A.

Now we state the propositions:

(18) $\sqrt{J} = \bigcap \mathrm{PrimeIdeals}(A, J)$. The theorem is a consequence of (16), (7), and (9).

(19) $\mathrm{nilrad}(A) = \bigcap(\text{the spectrum of } A)$. The theorem is a consequence of (17) and (18).

(20) $I \subseteq \sqrt{I}$.

(21) If $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$.

PROOF: Consider s_1 being an element of A such that $s_1 = s$ and there exists an element n of $\mathbb{N}$ such that ${s_1}^n \in I$. Consider n_1 being an element of $\mathbb{N}$ such that ${s_1}^{n_1} \in I$. $n_1 \neq 0$ by [7, (8)], [2, (19)]. □

Let us consider A. The functor J-Rad(A) yielding a subset of A is defined by the term

(Def. 14) $\bigcap$ m-Spectrum(A).

5. Construction of Zariski Topology of the Prime Spectrum of A

Now we state the propositions:

(22) $\mathrm{PrimeIdeals}(A, S) \subseteq \mathrm{Ideals}(A, S)$.

(23) $\mathrm{PrimeIdeals}(A, S) = \mathrm{Ideals}(A, S) \cap (\text{the spectrum of } A)$. The theorem is a consequence of (22).

(24) $\mathrm{PrimeIdeals}(A, S) = \mathrm{PrimeIdeals}(A, S\text{–ideal})$. The theorem is a consequence of (23) and (1).

(25) If $I \subseteq p$, then $\sqrt{I} \subseteq p$.

PROOF: Consider s_1 being an element of A such that $s_1 = s$ and there exists an element n of $\mathbb{N}$ such that ${s_1}^n \in I$. Consider n_1 being an element of $\mathbb{N}$ such that ${s_1}^{n_1} \in I$. $n_1 \neq 0$. □

(26) If $\sqrt{I} \subseteq p$, then $I \subseteq p$. The theorem is a consequence of (20).

(27) $\mathrm{PrimeIdeals}(A, \sqrt{S\text{–ideal}}) = \mathrm{PrimeIdeals}(A, S\text{–ideal})$. The theorem is a consequence of (26) and (25).

(28) If $E_2 \subseteq E_1$, then $\mathrm{PrimeIdeals}(A, E_1) \subseteq \mathrm{PrimeIdeals}(A, E_2)$.

(29) $\mathrm{PrimeIdeals}(A, J_1) = \mathrm{PrimeIdeals}(A, J_2)$ if and only if $\sqrt{J_1} = \sqrt{J_2}$. The theorem is a consequence of (18) and (27).

(30) If $I_1 * I_2 \subseteq p$, then $I_1 \subseteq p$ or $I_2 \subseteq p$.

PROOF: If it is not true that $I_1 \subseteq p$ or $I_2 \subseteq p$, then $I_1 * I_2 \not\subseteq p$. □

(31) $\mathrm{PrimeIdeals}(A, \{1_A\}) = \emptyset$.

(32) The spectrum of $A = \mathrm{PrimeIdeals}(A, \{0_A\})$.

(33) Let us consider non empty subsets E_1, E_2 of A. Then there exists a non empty subset E_3 of A such that $\mathrm{PrimeIdeals}(A, E_1) \cup \mathrm{PrimeIdeals}(A, E_2) = \mathrm{PrimeIdeals}(A, E_3)$.

Proof: Set $I_1 = E_1$–ideal. Set $I_2 = E_2$–ideal. Reconsider $I_3 = I_1 * I_2$ as an ideal of A. PrimeIdeals(A, E_1) = PrimeIdeals(A, I_1). PrimeIdeals(A, I_3) $\subseteq$ PrimeIdeals$(A, I_1)\cup$PrimeIdeals(A, I_2). PrimeIdeals$(A, I_1)\cup$PrimeIdeals (A, I_2) $\subseteq$ PrimeIdeals(A, I_3). PrimeIdeals(A, I_3) = PrimeIdeals(A, E_1) $\cup$ PrimeIdeals(A, E_2). □

(34) Let us consider a family G of subsets of the spectrum of A. Suppose for every set S such that $S \in G$ there exists a non empty subset E of A such that S = PrimeIdeals(A, E). Then there exists a non empty subset F of A such that Intersect(G) = PrimeIdeals(A, F). The theorem is a consequence of (28).

Let us consider A. The functor Spec(A) yielding a strict topological space is defined by

(Def. 15) the carrier of *it* = the spectrum of A and for every subset F of *it*, F is closed iff there exists a non empty subset E of A such that F = PrimeIdeals(A, E).

Note that Spec(A) is non empty. Now we state the proposition:

(35) Let us consider points P, Q of Spec(A). Suppose $P \neq Q$. Then there exists a subset V of Spec(A) such that

(i) V is open, and

(ii) $P \in V$ and $Q \notin V$ or $Q \in V$ and $P \notin V$.

Note that there exists a commutative ring which is degenerated. Let R be a degenerated, commutative ring. Let us observe that ADTS(the spectrum of R) is T_0. Let us consider A. Observe that Spec(A) is T_0.

6. Continous Map of Zariski Topology Associated with a Ring Homomorphism

From now on M_0 denotes an ideal of B. Now we state the proposition:

(36) If h inherits ring homomorphism, then $h^{-1}(M_0)$ is an ideal of A.

In the sequel M_0 denotes a prime ideal of B.

(37) If h inherits ring homomorphism, then $h^{-1}(M_0)$ is a prime ideal of A.
Proof: For every elements x, y of A such that $x \cdot y \in h^{-1}(M_0)$ holds $x \in h^{-1}(M_0)$ or $y \in h^{-1}(M_0)$. $h^{-1}(M_0) \neq$ the carrier of A. □

Let us consider A, B, and h. Assume h inherits ring homomorphism. The functor Spec(h) yielding a function from Spec(B) into Spec(A) is defined by

(Def. 16) for every point x of Spec(B), *it*$(x) = h^{-1}(x)$.

Now we state the propositions:

(38) If h inherits ring homomorphism, then $\mathrm{Spec}(h)^{-1}\,\mathrm{PrimeIdeals}(A,E) = \mathrm{PrimeIdeals}(B,h^{\circ}E)$.
PROOF: $\mathrm{Spec}(h)^{-1}\,\mathrm{PrimeIdeals}(A,E) \subseteq \mathrm{PrimeIdeals}(B,h^{\circ}E)$. Consider q being a prime ideal of B such that $x = q$ and $h^{\circ}E \subseteq q$. $h^{-1}(q)$ is a prime ideal of A. □

(39) If h inherits ring homomorphism, then $\mathrm{Spec}(h)$ is continuous. The theorem is a consequence of (38).

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