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Tarski Geometry Axioms. Part III

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Summary. In the article, we continue the formalization of the work devoted to Tarski's geometry – the book "Metamathematische Methoden in der Geometrie" by W. Schwabhäuser, W. Szmielew, and A. Tarski. After we prepared some introductory formal framework in our two previous Mizar articles, we focus on the regular translation of underlying items faithfully following the abovementioned book (our encoding covers first seven chapters). Our development utilizes also other formalization efforts of the same topic, e.g. Isabelle/HOL by Makarios, Metamath or even proof objects obtained directly from Prover9.

In addition, using the native Mizar constructions (cluster registrations) the propositions ("Satz") are reformulated under weaker conditions, i.e. by using fewer axioms or by proposing an alternative version that uses just another axioms (ex. Satz 2.1 or Satz 2.2).

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0. Introduction

Some chapters of the book "Metamathematische Methoden in der Geometrie" by W. Schwabhäuser, W. Szmielew, and A. Tarski (SST) [12] have been formalized within the classical two-valued logic with proof checkers: Isabelle/HOL by Makarios [7, 8] (Chapter 2 and 3), Metamath (Chapters 2 to 6), Mizar ([11, 3], [5]) or by means of Coq [10, 2]. Some of the results were obtained with the help of other automatic proof assistants, either partially [4], or completely [1].

In the first part of this article, we use the Mizar system to systematically formalize Chapters 2 to 7 of the SST book.

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In addition, using the native Mizar constructions (cluster registrations) the propositions ("Satz") are reformulated with fewer hypotheses, i.e. by using fewer number of axioms or by proposing an alternative version that uses just another axioms (e.g., Satz 2.1 or Satz 2.2).

The proposition "6.28 Satz" introduced by Beeson (*"This is used in Satz 11.4, but is never proved in the book, and belongs in Chapter 6, so we give it the name "Satz 6.28""* following Beeson[1]) has been added.

The proof of the 2 lemmas: 5.12 Lemma 3 and 4 were directly inspired by Narboux Lemma (see Thm. 26 from [11]) and "endofsegidand" from Metamath. One of the theorems was taken from [6].

In the following section, the equivalence between the simplified axiomatic system of Makarios [9] is proved with axioms defined in [11] and [3]. This equivalence has already been shown (by means of GeoCoq).

To recall using the notations of Makarios:

- Reflexivity axiom for equidistance (RE)
$$\forall_{a,b}\; ab \equiv ba$$
- Transitivity axiom for equidistance (TE)
$$\forall_{a,b,p,q,r,s}\; ab \equiv pq \wedge ab \equiv rs \Rightarrow pq \equiv rs$$
- Identity axiom for equidistance (IE)
$$\forall_{a,b,c}\; ab \equiv cc \Rightarrow a = b$$
- Axiom of segment construction (SC)
$$\forall_{a,b,c,q}\exists_x\; \mathrm{B}qax \wedge ax \equiv bc$$
- Five-segments axiom (FS)
$$\forall_{a,b,c,d,a',b',c',d'}\; a \neq b \wedge \mathrm{B}abc \wedge \mathrm{B}a'b'c' \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge$$
$$\wedge\; bd \equiv b'd' \;\Rightarrow\; cd \equiv c'd'$$
- Identity axiom for betweenness (IB)
$$\forall_{a,b}\; \mathrm{B}aba \Rightarrow a = b$$

[1]Tarski Formalization Project Archives maintained by Michael Beeson are available at `http://www.michaelbeeson.com/research/FormalTarski/index.php?include=archive6.php`

- Axiom of Pasch (IP)

$$\forall_{a,b,c,p,q}\ \mathrm{B}apc \wedge \mathrm{B}bqc \Rightarrow \exists_x \mathrm{B}pxb \wedge \mathrm{B}qxa$$

- Lower 2-dimensional axiom (LO_2)

$$\exists_{a,b,c}\ \neg\mathrm{B}abc \wedge \neg\mathrm{B}bca \wedge \neg\mathrm{B}cab$$

- Upper 2-dimensional axiom (Up_2)

$$\forall_{a,b,c,p,q}\ p \neq q \wedge ap \equiv aq \wedge bp \equiv bq \wedge cp \equiv cq \Rightarrow (\mathrm{B}abc \vee \mathrm{B}bca \vee \mathrm{B}cab)$$

- Euclidean axiom (Eu)

$$\forall_{a,b,c,d,t}\ \mathrm{B}adt\ \wedge\ \mathrm{B}bdc\ \wedge\ a \neq d\ \Rightarrow\ \exists_{x,y}\ \mathrm{B}abx\ \wedge\ \mathrm{B}acy\ \wedge\ \mathrm{B}xty$$

- Axiom of continuity (Co)

$$\forall_{X,Y}(\exists_a\forall_{x,y}\ x \in X \wedge y \in Y \Rightarrow \mathrm{B}axy) \Rightarrow (\exists_b\forall_{x,y}\ x \in X \wedge y \in Y \Rightarrow \mathrm{B}xby)$$

- (FS')

$$\forall_{a,b,c,d,a',b',c',d'}\ a \neq b\ \wedge\ \mathrm{B}abc\ \wedge\ \mathrm{B}a'b'c'\ \wedge\ ab \equiv a'b'\ \wedge\ bc \equiv b'c'\ \wedge$$
$$\wedge\ ad \equiv a'd'\ \wedge\ bd \equiv b'd'\ \Rightarrow\ dc \equiv c'd'$$

We show that $\mathsf{CE}_2 = \{$(RE), (TE), (IE), (FS), (IB), (IP), (Lo_2), (Up_2), (Eu), (Co)$\}$ is equivalent to the system defined in [11] and [3].

Moreover, it can be shown that the real Euclidean plane is a model for the axiom system $\mathsf{CE}_2' = \{$(TE), (IE), (SC), (FS'), (IB), (IP), (Lo_2), (Up_2), (Eu), (Co)$\}$ of the system proposed by Makarios.

Like Makarios we show the equivalence between CE_2 and CE_2', but using less axioms, more particularly we show that

- $\{$(RE), (TE), (FS)$\} \vdash$ (FS')
- $\{$(TE), (IE), (SC), (FS')$\} \vdash$ (FS)

Additionally, we prove that

$$\{\text{(TE), (IE), (SC), (FS')}\} \vdash \text{(RE)}.$$

We don't use (IB) and (IP).

1. Congruence Properties

From now on S denotes Tarski plane satisfying the axiom of congruence symmetry and the axiom of congruence equivalence relation, and a, b, c, d, e, f denote points of S.

Now we state the propositions:

(1) 2.1 Satz:
$\overline{ab} \cong \overline{ab}$.

(2) 2.1 Satz bis:
Let us consider Tarski plane S satisfying the axiom of congruence equivalence relation and the axiom of segment construction, and points a, b of S. Then $\overline{ab} \cong \overline{ab}$.

(3) 2.2 Satz:
If $\overline{ab} \cong \overline{cd}$, then $\overline{cd} \cong \overline{ab}$. The theorem is a consequence of (1).

(4) 2.2 Satz bis:
Let us consider Tarski plane S satisfying the axiom of congruence equivalence relation and the axiom of segment construction, and points a, b, c, d of S. If $\overline{ab} \cong \overline{cd}$, then $\overline{cd} \cong \overline{ab}$. The theorem is a consequence of (2).

(5) 2.3 Satz:
If $\overline{ab} \cong \overline{cd}$ and $\overline{cd} \cong \overline{ef}$, then $\overline{ab} \cong \overline{ef}$. The theorem is a consequence of (3).

(6) 2.4 Satz:
If $\overline{ab} \cong \overline{cd}$, then $\overline{ba} \cong \overline{cd}$. The theorem is a consequence of (5).

(7) 2.5 Satz:
If $\overline{ab} \cong \overline{cd}$, then $\overline{ab} \cong \overline{dc}$. The theorem is a consequence of (5).

(8) 2.8 Satz:
Let us consider Tarski plane S satisfying the axiom of congruence identity and the axiom of segment construction, and points a, b of S. Then $\overline{aa} \cong \overline{bb}$.

Let S be a Tarski plane. We say that S satisfies (A5) from SST if and only if

(Def. 1) for every points a, b, c, d, a', b', c', d' of S such that $a \neq b$ and b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$ holds $\overline{cd} \cong \overline{c'd'}$.

Now we state the proposition:

(9) S satisfies the axiom of SAS if and only if S satisfies (A5) from SST. The theorem is a consequence of (6) and (7).

One can check that every Tarski plane satisfying the axiom of congruence symmetry and the axiom of congruence equivalence relation which satisfies (A5) from SST satisfies also the axiom of SAS and every Tarski plane satisfying the axiom of congruence symmetry and the axiom of congruence equivalence relation which satisfies the axiom of SAS satisfies also (A5) from SST.

Let S be a Tarski plane and a, b, c, d, a', b', c', d' be points of S. We say that AFS $\binom{a,\, b,\, c,\, d}{a',\, b',\, c',\, d'}$ if and only if

(Def. 2) b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$.

Now we state the proposition:

(10) Let us consider Tarski plane S satisfying the axiom of congruence symmetry, the axiom of congruence equivalence relation, and the axiom of SAS, and points a, b, c, d, a', b', c', d' of S. Suppose AFS $\binom{a,\, b,\, c,\, d}{a',\, b',\, c',\, d'}$ and $a \neq b$. Then $\overline{cd} \cong \overline{c'd'}$.

From now on S denotes Tarski plane satisfying the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of congruence identity, the axiom of segment construction, and the axiom of SAS and q, a, b, c, a', b', c', x_1, x_2 denote points of S. Now we state the propositions:

(11) 2.11 SATZ:
If b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$, then $\overline{ac} \cong \overline{a'c'}$. The theorem is a consequence of (6), (7), (8), and (3).

(12) 2.12 SATZ:
Suppose $q \neq a$. If a lies between q and x_1 and $\overline{ax_1} \cong \overline{bc}$ and a lies between q and x_2 and $\overline{ax_2} \cong \overline{bc}$, then $x_1 = x_2$. The theorem is a consequence of (3), (5), (1), and (11).

2. BETWEENNESS RELATION

Now we state the proposition:

(13) 3.1 SATZ:
Let us consider Tarski plane S satisfying the axiom of congruence identity and the axiom of segment construction, and points a, b of S. Then b lies between a and b.

From now on S denotes Tarski plane satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch and a, b, c, d denote points of S.

Now we state the propositions:

(14) 3.2 SATZ:
If b lies between a and c, then b lies between c and a. The theorem is

a consequence of (13).

(15) 3.3 Satz:
a lies between a and b.

(16) 3.4 Satz:
Let us consider Tarski plane S satisfying the axiom of betweenness identity and the axiom of Pasch, and points a, b, c of S. If b lies between a and c and a lies between b and c, then $a = b$.

From now on S denotes Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d denote points of S. Now we state the propositions:

(17) 3.5 Satz:
If b lies between a and d and c lies between b and d, then b lies between a and c and c lies between a and d.

(18) 3.6 Satz:
If b lies between a and c and c lies between a and d, then c lies between b and d and b lies between a and d.

(19) 3.7 Satz:
If b lies between a and c and c lies between b and d and $b \neq c$, then c lies between a and d and b lies between a and d.

Let S be a Tarski plane and a, b, c, d be points of S.

We say that between4(a, b, c, d) if and only if

(Def. 3) b lies between a and c and b lies between a and d and c lies between a and d and c lies between b and d.

Let S be a Tarski plane and a, b, c, d, e be points of S. We say that between5(a, b, c, d, e) if and only if

(Def. 4) b lies between a and c and b lies between a and d and b lies between a and e and c lies between a and d and c lies between a and e and d lies between a and e and c lies between b and d and c lies between b and e and d lies between b and e and d lies between c and e.

From now on S denotes Tarski plane satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch and a, b, c, d, e denote points of S.

Now we state the propositions:

(20) 3.9 Satz (n = 3):
If b lies between a and c, then b lies between c and a.

(21) 3.9 Satz (n = 4):
If between4(a, b, c, d), then between4(d, c, b, a).

(22) 3.9 Satz (n = 5):
If between5(a, b, c, d, e), then between5(e, d, c, b, a).

(23) 3.10 Satz (n = 4):

Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch, and points a, b, c, d of S. Suppose between4(a, b, c, d). Then

(i) b lies between a and c, and

(ii) b lies between a and d, and

(iii) c lies between a and d, and

(iv) c lies between b and d.

(24) 3.10 Satz (n = 5):

Suppose between5(a, b, c, d, e). Then

(i) b lies between a and c, and

(ii) b lies between a and d, and

(iii) b lies between a and e, and

(iv) c lies between a and d, and

(v) c lies between a and e, and

(vi) d lies between a and e, and

(vii) c lies between b and d, and

(viii) c lies between b and e, and

(ix) d lies between b and e, and

(x) d lies between c and e, and

(xi) between4(a, b, c, d), and

(xii) between4(a, b, c, e), and

(xiii) between4(a, c, d, e), and

(xiv) between4(b, c, d, e).

From now on S denotes Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, p denote points of S. Now we state the propositions:

(25) 3.11 Satz (n = 3, l = 1):

If b lies between a and c and p lies between a and b, then between4(a, p, b, c).

(26) 3.11 Satz (n = 3, l = 2):

If b lies between a and c and p lies between b and c, then between4(a, b, p, c).

(27) 3.11 Satz (n = 3, l = 1):

If between4(a, b, c, d) and p lies between a and b, then between5(a, p, b, c, d).

(28) 3.11 Satz (n = 3, l = 2):

If between4(a, b, c, d) and p lies between b and c, then between5(a, b, p, c, d).

(29) 3.11 Satz (n $=3$, l $=3$):
If between4(a,b,c,d) and p lies between c and d, then between5(a,b,c,p,d).

(30) 3.12 Satz (n $=3$, l $=1$):
If b lies between a and c and c lies between a and p, then between4(a,b,c,p) and if $a \neq c$, then between4(a,b,c,p).

(31) 3.12 Satz (n $=3$, l $=2$):
If b lies between a and c and c lies between b and p, then c lies between b and p and if $b \neq c$, then between4(a,b,c,p).

(32) 3.12 Satz (n $=4$, l $=1$):
If between4(a,b,c,d) and d lies between a and p, then between5(a,b,c,d,p) and if $a \neq d$, then between5(a,b,c,d,p).

(33) 3.12 Satz (n $=4$, l $=2$):
If between4(a,b,c,d) and d lies between b and p, then between4(b,c,d,p) and if $b \neq d$, then between5(a,b,c,d,p).

(34) 3.12 Satz (n $=4$, l $=3$):
If between4(a,b,c,d) and d lies between c and p, then d lies between c and p and if $c \neq d$, then between5(a,b,c,d,p).

Let us note that there exists Tarski plane satisfying seven Tarski's geometry axioms which satisfies Lower Dimension Axiom. Now we state the propositions:

(35) 3.13 Satz:
Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, and Lower Dimension Axiom. Then there exist points a, b, c of S such that

(i) b does not lie between a and c, and

(ii) c does not lie between b and a, and

(iii) a does not lie between c and b, and

(iv) $a \neq b$, and

(v) $b \neq c$, and

(vi) $c \neq a$.

The theorem is a consequence of (13).

(36) 3.14 Satz:
Let us consider Tarski plane S satisfying the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of congruence identity, the axiom of segment construction, and Lower Dimension Axiom, and points a, b of S. Then there exists a point c of S such that

(i) b lies between a and c, and

(ii) $b \neq c$.

The theorem is a consequence of (35) and (3).

(37) 3.15 Satz (n = 3):
Let us consider Tarski plane S satisfying the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and Lower Dimension Axiom, and points a_1, a_2 of S. Suppose $a_1 \neq a_2$. Then there exists a point a_3 of S such that

(i) a_2 lies between a_1 and a_3, and

(ii) a_1, a_2, a_3 are mutually different.

The theorem is a consequence of (36).

(38) 3.15 Satz (n = 4):
Let us consider Tarski plane S satisfying seven Tarski's geometry axioms and Lower Dimension Axiom, and points a_1, a_2 of S. Suppose $a_1 \neq a_2$. Then there exist points a_3, a_4 of S such that

(i) between4(a_1, a_2, a_3, a_4), and

(ii) a_1, a_2, a_3, a_4 are mutually different.

The theorem is a consequence of (37).

(39) 3.15 Satz (n = 5):
Let us consider Tarski plane S satisfying seven Tarski's geometry axioms and Lower Dimension Axiom, and points a_1, a_2 of S. Suppose $a_1 \neq a_2$. Then there exist points a_3, a_4, a_5 of S such that

(i) between5$(a_1, a_2, a_3, a_4, a_5)$, and

(ii) a_1,a_2,a_3,a_4,a_5 are mutually different.

The theorem is a consequence of (38) and (37).

(40) 3.17 Satz:
Let us consider Tarski plane S satisfying seven Tarski's geometry axioms, and points a, b, c, p, a', b', c' of S. Suppose b lies between a and c and b' lies between a' and c and p lies between a and a'. Then there exists a point q of S such that

(i) q lies between p and c, and

(ii) q lies between b and b'.

The theorem is a consequence of (14).

3. Collinearity

Let S be a Tarski plane and a, b, c, d, a', b', c', d' be points of S. We say that IFS $\binom{a,\,b,\,c,\,d}{a',\,b',\,c',\,d'}$ if and only if

(Def. 5) b lies between a and c and b' lies between a' and c' and $\overline{ac} \cong \overline{a'c'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{cd} \cong \overline{c'd'}$.

From now on S denotes Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, a', b', c', d' denote points of S.

Now we state the propositions:

(41) 4.2 Satz:
If IFS $\binom{a,\,b,\,c,\,d}{a',\,b',\,c',\,d'}$, then $\overline{bd} \cong \overline{b'd'}$. The theorem is a consequence of (3), (6), (7), and (14).

(42) 4.3 Satz:
If b lies between a and c and b' lies between a' and c' and $\overline{ac} \cong \overline{a'c'}$ and $\overline{bc} \cong \overline{b'c'}$, then $\overline{ab} \cong \overline{a'b'}$. The theorem is a consequence of (6), (8), (7), and (41).

(43) 4.5 Satz:
If b lies between a and c and $\overline{ac} \cong \overline{a'c'}$, then there exists b' such that b' lies between a' and c' and $\triangle abc \cong \triangle a'b'c'$. The theorem is a consequence of (3), (8), (13), (14), (11), and (12).

(44) 4.6 Satz:
If b lies between a and c and $\triangle abc \cong \triangle a'b'c'$, then b' lies between a' and c'. The theorem is a consequence of (43), (3), (5), (6), (1), (7), and (41).

(45) 4.11 Satz:
Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch, and points a, b, c of S. Suppose a, b and c are collinear. Then

(i) b, c and a are collinear, and

(ii) c, a and b are collinear, and

(iii) c, b and a are collinear, and

(iv) b, a and c are collinear, and

(v) a, c and b are collinear.

(46) 4.12 Satz:
Let us consider Tarski plane S satisfying the axiom of congruence identity and the axiom of segment construction, and points a, b of S. Then a, a and b are collinear.

(47) Let us consider Tarski plane S satisfying the axiom of congruence symmetry and the axiom of congruence equivalence relation, and points a, b, c, a', b', c' of S. Suppose $\triangle abc \cong \triangle a'b'c'$. Then $\triangle bca \cong \triangle b'c'a'$. The theorem is a consequence of (6) and (7).

(48) 4.13 Satz:
Let us consider Tarski plane S satisfying seven Tarski's geometry axioms, and points a, b, c, a', b', c' of S. Suppose a, b and c are collinear and $\triangle abc \cong \triangle a'b'c'$. Then a', b' and c' are collinear. The theorem is a consequence of (47) and (44).

Let us consider Tarski plane S satisfying the axiom of congruence symmetry and the axiom of congruence equivalence relation and points a, b, c, a', b', c' of S. Now we state the propositions:

(49) If $\triangle bac \cong \triangle b'a'c'$, then $\triangle abc \cong \triangle a'b'c'$. The theorem is a consequence of (6) and (7).

(50) If $\triangle acb \cong \triangle a'c'b'$, then $\triangle abc \cong \triangle a'b'c'$. The theorem is a consequence of (6) and (7).

From now on S denotes Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, a', b', c', d', p, q denote points of S.

Now we state the proposition:

(51) 4.14 Satz:
If a, b and c are collinear and $\overline{ab} \cong \overline{a'b'}$, then there exists a point c' of S such that $\triangle abc \cong \triangle a'b'c'$. The theorem is a consequence of (3), (11), (14), (6), (7), (49), (43), and (50).

Let S be a Tarski plane and a, b, c, d, a', b', c', d' be points of S. We say that FS $\binom{a,\, b,\, c,\, d}{a',\, b',\, c',\, d'}$ if and only if

(Def. 6) a, b and c are collinear and $\triangle abc \cong \triangle a'b'c'$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$.

Now we state the propositions:

(52) 4.16 Satz:
If FS $\binom{a,\, b,\, c,\, d}{a',\, b',\, c',\, d'}$ and $a \neq b$, then $\overline{cd} \cong \overline{c'd'}$. The theorem is a consequence of (44), (47), (41), (14), and (49).

(53) 4.17 Satz:
If $a \neq b$ and a, b and c are collinear and $\overline{ap} \cong \overline{aq}$ and $\overline{bp} \cong \overline{bq}$, then $\overline{cp} \cong \overline{cq}$. The theorem is a consequence of (1) and (52).

(54) 4.18 Satz:
If $a \neq b$ and a, b and c are collinear and $\overline{ac} \cong \overline{ac'}$ and $\overline{bc} \cong \overline{bc'}$, then $c = c'$. The theorem is a consequence of (53) and (3).

(55) 4.19 Satz:
If c lies between a and b and $\overline{ac} \cong \overline{ac'}$ and $\overline{bc} \cong \overline{bc'}$, then $c = c'$. The

theorem is a consequence of (3), (14), and (54).

4. Line Segments

From now on S denotes Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, e, f, a', b', c', d' denote points of S.

Now we state the propositions:

(56) 5.1 Satz:
If $a \neq b$ and b lies between a and c and b lies between a and d, then c lies between a and d or d lies between a and c.

(57) 5.2 Satz:
If $a \neq b$ and b lies between a and c and b lies between a and d, then c lies between b and d or d lies between b and c. The theorem is a consequence of (56).

(58) 5.3 Satz:
If b lies between a and d and c lies between a and d, then b lies between a and c or c lies between a and b. The theorem is a consequence of (13), (14), (3), and (57).

Let S be a Tarski plane and a, b, c, d be points of S. We say that $a, b \leqslant c, d$ if and only if

(Def. 7) there exists a point y of S such that y lies between c and d and $\overline{ab} \cong \overline{cy}$.

Now we state the propositions:

(59) 5.5 Satz:
$a, b \leqslant c, d$ if and only if there exists a point x of S such that b lies between a and x and $\overline{ax} \cong \overline{cd}$. The theorem is a consequence of (3), (51), (44), (6), and (7).

(60) 5.6 Satz:
If $a, b \leqslant c, d$ and $\overline{ab} \cong \overline{a'b'}$ and $\overline{cd} \cong \overline{c'd'}$, then $a', b' \leqslant c', d'$. The theorem is a consequence of (59), (51), (3), (5), and (44).

(61) 5.7 Satz:
$a, b \leqslant a, b$. The theorem is a consequence of (13) and (1).

(62) 5.8 Satz:
If $a, b \leqslant c, d$ and $c, d \leqslant e, f$, then $a, b \leqslant e, f$. The theorem is a consequence of (59), (3), (51), (44), and (5).

(63) 5.9 Satz:
If $a, b \leqslant c, d$ and $c, d \leqslant a, b$, then $\overline{ab} \cong \overline{cd}$. The theorem is a consequence of (59), (14), (3), (12), and (16).

(64) 5.10 Satz:

(i) $a, b \leqslant c, d$, or

(ii) $c, d \leqslant a, b$.

The theorem is a consequence of (3), (59), (14), and (56).

(65) 5.11 Satz:
$a, a \leqslant b, c$. The theorem is a consequence of (59).

(66) 5.12 Lemma 1:
Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, the axiom of Pasch, the axiom of congruence symmetry, and the axiom of congruence equivalence relation, and points a, b, c, d of S. If $a, b \leqslant c, d$, then $b, a \leqslant c, d$.

(67) 5.12 Lemma 2:
If $a, b \leqslant c, d$, then $a, b \leqslant d, c$. The theorem is a consequence of (59) and (7).

(68) 5.12 Lemma 3:
If b lies between a and c and $\overline{ac} \cong \overline{ab}$, then $c = b$. The theorem is a consequence of (14), (6), (3), (7), (44), and (16).

(69) 5.12 Lemma 4:
If c lies between a and b and $a, b \leqslant a, c$, then $b = c$. The theorem is a consequence of (59) and (68).

(70) 5.12 Satz:
If a, b and c are collinear, then b lies between a and c iff $a, b \leqslant a, c$ and $b, c \leqslant a, c$. The theorem is a consequence of (1), (14), (6), (67), (69), and (13).

5. Lines and Halflines

Let S be a Tarski plane and a, b, p be points of S. We say that $a \underset{p}{\simeq} b$ if and only if

(Def. 8) $p \neq a$ and $p \neq b$ and (a lies between p and b or b lies between p and a).

From now on p denotes a point of S. Now we state the proposition:

(71) 6.2 Satz:
If $a \neq p$ and $b \neq p$ and $c \neq p$ and p lies between a and c, then p lies between b and c iff $a \underset{p}{\simeq} b$. The theorem is a consequence of (14) and (57).

(72) 6.3 Satz:
$a \underset{p}{\simeq} b$ if and only if $a \neq p$ and $b \neq p$ and there exists c such that $c \neq p$

and p lies between a and c and p lies between b and c. The theorem is a consequence of (3) and (71).

(73) 6.4 Satz:
$a \underset{p}{\cong} b$ if and only if a, p and b are collinear and p does not lie between a and b. The theorem is a consequence of (14), (16), and (13).

(74) 6.5 Satz:
If $a \neq p$, then $a \underset{p}{\cong} a$.

(75) 6.6 Satz:
If $a \underset{p}{\cong} b$, then $b \underset{p}{\cong} a$.

(76) 6.7 Satz:
If $a \underset{p}{\cong} b$ and $b \underset{p}{\cong} c$, then $a \underset{p}{\cong} c$.

(77) METAMATH, segcon2:
There exists a point x of S such that

(i) a lies between p and x or x lies between p and a, and

(ii) $\overline{px} \cong \overline{bc}$.

The theorem is a consequence of (3), (14), and (57).

In the sequel r denotes a point of S. Now we state the proposition:

(78) 6.11 Satz a):
If $r \neq a$ and $b \neq c$, then there exists a point x of S such that $x \underset{a}{\cong} r$ and $\overline{ax} \cong \overline{bc}$. The theorem is a consequence of (77) and (3).

Let S be a Tarski plane and a, p be points of S. The functor HalfLine(p, a) yielding a set is defined by the term

(Def. 9) $\{x$, where x is a point of $S : x \underset{p}{\cong} a\}$.

From now on x, y denote points of S. Now we state the propositions:

(79) 6.11 Satz b):
If $r \neq a$ and $b \neq c$ and $x \underset{a}{\cong} r$ and $\overline{ax} \cong \overline{bc}$ and $y \underset{a}{\cong} r$ and $\overline{ay} \cong \overline{bc}$, then $x = y$. The theorem is a consequence of (72), (14), (12), and (57).

(80) 6.13 Satz:
If $a \underset{p}{\cong} b$, then $p, a \leqslant p, b$ iff a lies between p and b. The theorem is a consequence of (1), (79), and (70).

Let S be a non empty Tarski plane and p, q be points of S. The functor Line(p, q) yielding a subset of S is defined by the term

(Def. 10) $\{x$, where x is a point of $S : p$, q and x are collinear$\}$.

In the sequel S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms and p, q, r, s denote points of S.

Now we state the proposition:

(81) 6.15 SATZ:
If $p \neq q$ and $p \neq r$ and p lies between q and r, then $\mathrm{Line}(p, q) = (\mathrm{HalfLine}(p, q) \cup \{p\}) \cup \mathrm{HalfLine}(p, r)$. The theorem is a consequence of (14), (57), and (13).

Let S be a non empty Tarski plane and A be a subset of S. We say that A is a line if and only if

(Def. 11) there exist points p, q of S such that $p \neq q$ and $A = \mathrm{Line}(p, q)$.

Now we state the proposition:

(82) 6.16 SATZ:
If $p \neq q$ and $s \neq p$ and $s \in \mathrm{Line}(p, q)$, then $\mathrm{Line}(p, q) = \mathrm{Line}(p, s)$. The theorem is a consequence of (56), (14), (58), and (57).

In the sequel S denotes a non empty Tarski plane satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch and a, b, p, q denote points of S.

Now we state the proposition:

(83) 6.17 SATZ:
(i) $p, q \in \mathrm{Line}(p, q)$, and
(ii) $\mathrm{Line}(p, q) = \mathrm{Line}(q, p)$.

The theorem is a consequence of (13) and (14).

In the sequel S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms, A, B denote subsets of S, and a, b, c, p, q, r, s denote points of S.

Now we state the proposition:

(84) Let us consider Tarski plane S satisfying seven Tarski's geometry axioms, and elements a, b, c of S. Then $a \neq b$ and a, b and c are collinear if and only if c lies on the line passing through a and b.

Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms and points a, b, x, y of S. Now we state the propositions:

(85) If the line passing through a and b is equal to the line passing through x and y, then $\mathrm{Line}(a, b) = \mathrm{Line}(x, y)$. The theorem is a consequence of (84).

(86) If $a \neq b$ and $x \neq y$ and $\mathrm{Line}(a, b) = \mathrm{Line}(x, y)$, then the line passing through a and b is equal to the line passing through x and y.

(87) 6.18 SATZ:
If A is a line and $a \neq b$ and $a, b \in A$, then $A = \mathrm{Line}(a, b)$. The theorem is a consequence of (85).

(88) 6.19 SATZ:
If $a \neq b$ and A is a line and $a, b \in A$ and B is a line and $a, b \in B$, then $A = B$. The theorem is a consequence of (87).

(89) 6.21 Satz:
If A is a line and B is a line and $A \neq B$ and $a \in A$ and $a \in B$ and $b \in A$ and $b \in B$, then $a = b$.

(90) 6.23 Satz:
If there exists p and there exists q such that $p \neq q$, then a, b and c are collinear iff there exists A such that A is a line and a, b, $c \in A$. The theorem is a consequence of (87) and (13).

(91) 6.24 Satz:
Let us consider Tarski plane S satisfying (A8). Then there exist points a, b, c of S such that a, b and c are not collinear.

(92) 6.25 Satz:
Let us consider a non empty Tarski plane S satisfying seven Tarski's geometry axioms, and points a, b of S. Suppose S satisfies (A8) and $a \neq b$. Then there exists a point c of S such that a, b and c are not collinear. The theorem is a consequence of (91), (13), and (87).

(93) Let us consider Tarski plane S satisfying seven Tarski's geometry axioms, and points p, a, b of S. If $a \mathrel{\underset{p}{\cong}} b$ and $p, a \leqslant p, b$, then a lies between p and b.

(94) Let us consider Tarski plane S satisfying seven Tarski's geometry axioms, and elements a, b, c, d, e, f, g, h of S. Suppose $c, d \not\leqslant a, b$ and $\overline{ab} \cong \overline{ef}$ and $\overline{cd} \cong \overline{gh}$. Then $e, f \leqslant g, h$. The theorem is a consequence of (64) and (60).

(95) 6.28 Satz, introduced by Beeson:
Let us consider Tarski plane S satisfying seven Tarski's geometry axioms, and elements a, b, c, a_1, b_1, c_1 of S. Suppose $a \mathrel{\underset{b}{\cong}} c$ and $a_1 \mathrel{\underset{b_1}{\cong}} c_1$ and $\overline{ba} \cong \overline{b_1a_1}$ and $\overline{bc} \cong \overline{b_1c_1}$. Then $\overline{ac} \cong \overline{a_1c_1}$. The theorem is a consequence of (7), (6), (42), (94), (93), and (14).

6. Point Reflection

Let S be a Tarski plane and a, b, m be points of S. We say that Middle(a, m, b) if and only if

(Def. 12) m lies between a and b and $\overline{ma} \cong \overline{mb}$.

From now on S denotes Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch and a, b, m denote points of S.

Now we state the proposition:

(96) 7.2 SATZ:
If $\text{Middle}(a, m, b)$, then $\text{Middle}(b, m, a)$.

From now on S denotes Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, and the axiom of betweenness identity and a, b, m denote points of S.

Now we state the propositions:

(97) 7.3 SATZ:
$\text{Middle}(a, m, a)$ if and only if $m = a$.

(98) 7.4 EXISTENCE:
Let us consider a point p of S. Then there exists a point p' of S such that $\text{Middle}(p, a, p')$. The theorem is a consequence of (7), (3), and (97).

From now on S denotes Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, and the axiom of SAS and a denotes a point of S.

(99) 7.4 UNIQUENESS:
Let us consider points p, p_1, p_2 of S. If $\text{Middle}(p, a, p_1)$ and $\text{Middle}(p, a, p_2)$, then $p_1 = p_2$. The theorem is a consequence of (3) and (12).

Let S be Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, the axiom of betweenness identity, and the axiom of SAS and a, p be points of S. The functor $\mathrm{S}_a(p)$ yielding a point of S is defined by

(Def. 13) $\text{Middle}(p, a, it)$.

From now on S denotes Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, the axiom of betweenness identity, and the axiom of SAS and a, p, p' denote points of S.

Now we state the proposition:

(100) 7.6 SATZ:
$\mathrm{S}_a(p) = p'$ if and only if $\text{Middle}(p, a, p')$.

From now on S denotes Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, the axiom of betweenness identity, the axiom of SAS, and the axiom of Pasch and a, p, p' denote points of S.

Now we state the propositions:

(101) 7.7 Satz:
$\mathrm{S}_a((\mathrm{S}_a(p))) = p$. The theorem is a consequence of (14) and (3).

(102) 7.8 Satz:
There exists p such that $\mathrm{S}_a(p) = p'$. The theorem is a consequence of (101).

(103) 7.9 Satz:
If $\mathrm{S}_a(p) = \mathrm{S}_a(p')$, then $p = p'$. The theorem is a consequence of (101).

From now on S denotes Tarski plane satisfying the axiom of congruence identity, the axiom of congruence symmetry, the axiom of congruence equivalence relation, the axiom of segment construction, the axiom of betweenness identity, and the axiom of SAS and a, p denote points of S.

Now we state the proposition:

(104) 7.10 Satz:
$\mathrm{S}_a(p) = p$ if and only if $p = a$. The theorem is a consequence of (13) and (1).

From now on S denotes Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, m, p, p', q, r, s denote points of S.

Now we state the propositions:

(105) 7.13 Satz:
$\overline{pq} \cong \overline{\mathrm{S}_a(p)\mathrm{S}_a(q)}$. The theorem is a consequence of (104), (14), (26), (28), (3), (6), (7), (11), (5), (1), and (41).

(106) 7.15 Satz:
q lies between p and r if and only if $\mathrm{S}_a(q)$ lies between $\mathrm{S}_a(p)$ and $\mathrm{S}_a(r)$. The theorem is a consequence of (101).

(107) 7.16 Satz:
$\overline{pq} \cong \overline{rs}$ if and only if $\overline{\mathrm{S}_a(p)\mathrm{S}_a(q)} \cong \overline{\mathrm{S}_a(r)\mathrm{S}_a(s)}$. The theorem is a consequence of (101).

(108) 7.17 Satz:
If $\mathrm{Middle}(p, a, p')$ and $\mathrm{Middle}(p, b, p')$, then $a = b$. The theorem is a consequence of (105), (101), (5), (6), (7), (55), and (104).

(109) 7.18 Satz:
If $\mathrm{S}_a(p) = \mathrm{S}_b(p)$, then $a = b$. The theorem is a consequence of (108).

(110) 7.19 Satz:
$\mathrm{S}_b((\mathrm{S}_a(p))) = \mathrm{S}_a((\mathrm{S}_b(p)))$ if and only if $a = b$. The theorem is a consequence of (106), (107), (101), (108), and (104).

(111) 7.20 Satz:
If a, m and b are collinear and $\overline{ma} \cong \overline{mb}$, then $a = b$ or $\mathrm{Middle}(a, m, b)$. The theorem is a consequence of (14), (13), (7), (6), (1), (42), and (3).

From now on S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms and a, b, c, d, p denote points of S.

Now we state the proposition:

(112) 7.21 Satz:

Suppose a, b and c are not collinear and $b \neq d$ and $\overline{ab} \cong \overline{cd}$ and $\overline{bc} \cong \overline{da}$ and a, p and c are collinear and b, p and d are collinear. Then

(i) $\mathrm{Middle}(a, p, c)$, and

(ii) $\mathrm{Middle}(b, p, d)$.

The theorem is a consequence of (14), (51), (48), (7), (6), (3), (52), (13), (83), (88), and (111).

From now on a_1, a_2, b_1, b_2, m_1, m_2 denote points of S.

Now we state the propositions:

(113) 7.22 Satz, part 1:

Suppose c lies between a_1 and a_2 and c lies between b_1 and b_2 and $\overline{ca_1} \cong \overline{cb_1}$ and $\overline{ca_2} \cong \overline{cb_2}$ and $\mathrm{Middle}(a_1, m_1, b_1)$ and $\mathrm{Middle}(a_2, m_2, b_2)$ and $c, a_1 \leqslant c, a_2$. Then c lies between m_1 and m_2. The theorem is a consequence of (59), (3), (13), (1), (105), (104), (60), (14), (103), (56), (80), (106), (40), (107), (7), (6), (41), (53), and (108).

(114) 7.22 Satz, part 2:

Suppose c lies between a_1 and a_2 and c lies between b_1 and b_2 and $\overline{ca_1} \cong \overline{cb_1}$ and $\overline{ca_2} \cong \overline{cb_2}$ and $\mathrm{Middle}(a_1, m_1, b_1)$ and $\mathrm{Middle}(a_2, m_2, b_2)$ and $c, a_2 \leqslant c, a_1$. Then c lies between m_1 and m_2. The theorem is a consequence of (59), (3), (13), (14), (1), (105), (104), (60), (103), (56), (80), (106), (40), (107), (7), (6), (41), (53), and (108).

(115) 7.22 Satz, Krippenlemma, (Gupta 1965, 3.45 Theorem):

Suppose c lies between a_1 and a_2 and c lies between b_1 and b_2 and $\overline{ca_1} \cong \overline{cb_1}$ and $\overline{ca_2} \cong \overline{cb_2}$ and $\mathrm{Middle}(a_1, m_1, b_1)$ and $\mathrm{Middle}(a_2, m_2, b_2)$. Then c lies between m_1 and m_2. The theorem is a consequence of (64), (113), and (114).

Let S be a Tarski plane and a_1, a_2, b_1, b_2, c, m_1, m_2 be points of S. We say that $\mathrm{Krippenfigur}(a_1, m_1, b_1, c, b_2, m_2, a_2)$ if and only if

(Def. 14) c lies between a_1 and a_2 and c lies between b_1 and b_2 and $\overline{ca_1} \cong \overline{cb_1}$ and $\overline{ca_2} \cong \overline{cb_2}$ and $\mathrm{Middle}(a_1, m_1, b_1)$ and $\mathrm{Middle}(a_2, m_2, b_2)$.

Now we state the proposition:

(116) Krippenfigur:

If $\mathrm{Krippenfigur}(a_1, m_1, b_1, c, b_2, m_2, a_2)$, then c lies between m_1 and m_2.

Let us observe that there exists Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms which is non empty.

In the sequel S denotes a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and a, b, c, p, q, r denote points of S. Now we state the proposition:

(117) If $\overline{ca} \cong \overline{cb}$, then there exists a point x of S such that Middle(a, x, b). The theorem is a consequence of (14), (111), (13), (1), (36), (3), (7), (10), (6), (43), (41), (48), (88), (83), (87), and (53).

7. Note about Simplification of Tarski's Axioms of Geometry by Makarios

Let S be a Tarski plane. We say that S satisfies (RE) if and only if

(Def. 15) for every points a, b of S, $\overline{ab} \cong \overline{ba}$.

We say that S satisfies (TE) if and only if

(Def. 16) for every points a, b, p, q, r, s of S such that $\overline{ab} \cong \overline{pq}$ and $\overline{ab} \cong \overline{rs}$ holds $\overline{pq} \cong \overline{rs}$.

We say that S satisfies (IE) if and only if

(Def. 17) for every points a, b, c of S such that $\overline{ab} \cong \overline{cc}$ holds $a = b$.

We say that S satisfies (SC) if and only if

(Def. 18) for every points a, b, c, q of S, there exists a point x of S such that a lies between q and x and $\overline{ax} \cong \overline{bc}$.

We say that S satisfies (FS) if and only if

(Def. 19) for every points a, b, c, d, a', b', c', d' of S such that $a \neq b$ and b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$ holds $\overline{cd} \cong \overline{c'd'}$.

We say that S satisfies (IB) if and only if

(Def. 20) for every points a, b of S such that b lies between a and a holds $a = b$.

We say that S satisfies (IP) if and only if

(Def. 21) for every points a, b, c, p, q of S such that p lies between a and c and q lies between b and c there exists a point x of S such that x lies between p and b and x lies between q and a.

We say that S satisfies (Lo_2) if and only if

(Def. 22) there exist points a, b, c of S such that b does not lie between a and c and c does not lie between b and a and a does not lie between c and b.

We say that S satisfies (Up_2) if and only if

(Def. 23) for every points a, b, c, p, q of S such that $p \neq q$ and $\overline{ap} \cong \overline{aq}$ and $\overline{bp} \cong \overline{bq}$ and $\overline{cp} \cong \overline{cq}$ holds b lies between a and c or c lies between b and a or a lies between c and b.

We say that S satisfies (Eu) if and only if

(Def. 24) for every points a, b, c, d, t of S such that d lies between a and t and d lies between b and c and $a \neq d$ there exist points x, y of S such that b lies between a and x and c lies between a and y and t lies between x and y.

We say that S satisfies (Co) if and only if

(Def. 25) for every sets X, Y such that there exists a point a of S such that for every points x, y of S such that $x \in X$ and $y \in Y$ holds x lies between a and y there exists a point b of S such that for every points x, y of S such that $x \in X$ and $y \in Y$ holds b lies between x and y.

We say that S satisfies (FS') if and only if

(Def. 26) for every points a, b, c, d, a', b', c', d' of S such that $a \neq b$ and b lies between a and c and b' lies between a' and c' and $\overline{ab} \cong \overline{a'b'}$ and $\overline{bc} \cong \overline{b'c'}$ and $\overline{ad} \cong \overline{a'd'}$ and $\overline{bd} \cong \overline{b'd'}$ holds $\overline{dc} \cong \overline{c'd'}$.

In the sequel S denotes a Tarski plane. Now we state the propositions:

(118) S satisfies the axiom of congruence symmetry if and only if S satisfies (RE).

(119) S satisfies the axiom of congruence equivalence relation if and only if S satisfies (TE).

(120) S satisfies the axiom of congruence identity if and only if S satisfies (IE).

(121) S satisfies the axiom of segment construction if and only if S satisfies (SC).

(122) S satisfies the axiom of betweenness identity if and only if S satisfies (IB).

(123) S satisfies the axiom of Pasch if and only if S satisfies (IP).

(124) S satisfies Lower Dimension Axiom if and only if S satisfies (Lo_2).

(125) S satisfies Upper Dimension Axiom if and only if S satisfies (Up_2).

(126) S satisfies Euclid Axiom if and only if S satisfies (Eu).

(127) Let us consider Tarski plane S satisfying the axiom of congruence symmetry and the axiom of congruence equivalence relation. Then S satisfies the axiom of SAS if and only if S satisfies (FS).

(128) Let us consider a non empty Tarski plane S. Then S satisfies Continuity Axiom if and only if S satisfies (Co).

One can verify that every Tarski plane which satisfies (RE) satisfies also the axiom of congruence symmetry and every Tarski plane which satisfies (TE) satisfies also the axiom of congruence equivalence relation and every Tarski plane which satisfies (IE) satisfies also the axiom of congruence identity and every Tarski plane which satisfies (SC) satisfies also the axiom of segment construction.

Every Tarski plane which satisfies (IB) satisfies also the axiom of betweenness identity and every Tarski plane which satisfies (IP) satisfies also the axiom of Pasch and every Tarski plane which satisfies (Lo_2) satisfies also Lower Dimension Axiom and every Tarski plane which satisfies (Up_2) satisfies also Upper Dimension Axiom and every Tarski plane which satisfies (Eu) satisfies also Euclid Axiom.

Every Tarski plane which satisfies (Co) satisfies also Continuity Axiom and every Tarski plane which satisfies the axiom of congruence symmetry satisfies also (RE) and every Tarski plane which satisfies the axiom of congruence equivalence relation satisfies also (TE) and every Tarski plane which satisfies the axiom of congruence identity satisfies also (IE) and every Tarski plane which satisfies the axiom of segment construction satisfies also (SC) and every Tarski plane which satisfies the axiom of betweenness identity satisfies also (IB). Every Tarski plane which satisfies the axiom of Pasch satisfies also (IP) and every Tarski plane which satisfies Lower Dimension Axiom satisfies also (Lo_2) and every Tarski plane which satisfies Upper Dimension Axiom satisfies also (Up_2) and every Tarski plane which satisfies Euclid Axiom satisfies also (Eu) and every non empty Tarski plane which satisfies Continuity Axiom satisfies also (Co) and there exists a Tarski plane which satisfies (RE) and (TE).

(129) Let us consider Tarski plane S satisfying (RE) and (TE). Then S satisfies the axiom of SAS if and only if S satisfies (FS).

One can check that every Tarski plane satisfying (RE) and (TE) which satisfies (FS) satisfies also the axiom of SAS and there exists Tarski plane satisfying (RE) and (TE) which satisfies (FS).

From now on S denotes a Tarski plane. Now we state the propositions:

(130) Makarios, Lemma 6:
Let us consider a Tarski plane S. Suppose S satisfies (RE) and (TE). Then S satisfies (FS) if and only if S satisfies (FS').

(131) Let us consider Tarski plane S satisfying (RE) and (TE). Then S satisfies (FS) if and only if S satisfies (FS').

Let us note that every Tarski plane satisfying (RE) and (TE) which satisfies (FS') satisfies also (FS) and there exists a Tarski plane which satisfies (TE) and (SC) and there exists Tarski plane satisfying (RE) and (TE) which satisfies (FS') and there exists Tarski plane satisfying (RE), (TE), and (FS') which satisfies (SC). Now we state the propositions:

(132) Let us consider Tarski plane S satisfying (TE) and (SC), and points a, b of S. Then $\overline{ab} \cong \overline{ab}$.

(133) Let us consider Tarski plane S satisfying (IE) and (SC), and points a, b of S. Then b lies between a and b.

(134) Let us consider Tarski plane S satisfying (TE) and (SC), and points a, b, c, d of S. If $\overline{ab} \cong \overline{cd}$, then $\overline{cd} \cong \overline{ab}$.

(135) Let us consider Tarski plane S satisfying (TE), (SC), and (FS'), and points a, b, c, d, e, f of S. Suppose $a \neq b$ and a lies between b and c and a lies between d and e and $\overline{ba} \cong \overline{da}$ and $\overline{ac} \cong \overline{ae}$ and $\overline{bf} \cong \overline{df}$. Then $\overline{fc} \cong \overline{ef}$. The theorem is a consequence of (2).

Let S be a Tarski plane. We say that S satisfies (RE') if and only if

(Def. 27) for every points a, b, c, d of S such that $a \neq b$ and a lies between b and c holds $\overline{dc} \cong \overline{cd}$.

Now we state the proposition:

(136) Every Tarski plane satisfying (TE), (SC), and (FS') satisfies (RE'). The theorem is a consequence of (2) and (135).

Let us note that every Tarski plane which satisfies (TE), (SC), and (FS') satisfies also (RE') and there exists Tarski plane satisfying (IE) which satisfies (RE') and there exists Tarski plane satisfying (RE') and (IE) which satisfies (SC) and there exists a non empty Tarski plane satisfying (IE) which is trivial and there exists a non empty Tarski plane satisfying (IE) and (SC) which is trivial. Now we state the proposition:

(137) Every trivial, non empty Tarski plane satisfying (IE) and (SC) satisfies (RE). The theorem is a consequence of (8).

One can verify that there exists a non empty Tarski plane satisfying (TE), (IE), and (SC) which satisfies (RE'). Now we state the proposition:

(138) Every non empty Tarski plane satisfying (RE'), (TE), (IE), and (SC) satisfies (RE). The theorem is a consequence of (8), (13), and (4).

Note that there exists a non empty Tarski plane satisfying (TE), (IE), and (SC) which satisfies (FS'). Now we state the propositions:

(139) Every non empty Tarski plane satisfying (TE), (IE), (SC), and (FS') satisfies (RE).

(140) Every non empty Tarski plane satisfying (TE), (IE), (SC), and (FS') satisfies (FS). The theorem is a consequence of (138).

8. Main Results and Corollaries

Let us note that every Tarski plane which satisfies (RE), (TE), and (FS) satisfies also (FS') and every non empty Tarski plane which satisfies (TE), (IE), (SC), and (FS') satisfies also (FS) and every non empty Tarski plane which satisfies (TE), (IE), (SC), and (FS') satisfies also (RE) and every non empty Tarski plane which satisfies (TE), (IE), (SC), and (FS') satisfies also the axiom of SAS and there exists a non empty Tarski plane which satisfies (RE), (TE), (IE), (SC), (FS), (IB), (IP), (Lo_2), (Up_2), (Eu), and (Co).

An axiomatic system CE_2 is a non empty Tarski plane satisfying (RE), (TE), (IE), (SC), (FS), (IB), (IP), (Lo_2), (Up_2), (Eu), and (Co).

An axiomatic system CE_2' is a non empty Tarski plane satisfying (TE), (IE), (SC), (FS'), (IB), (IP), (Lo_2), (Up_2), (Eu), and (Co). Now we state the propositions:

(141) Every axiomatic system CE_2 is an axiomatic system CE_2'.

(142) Every axiomatic system CE_2' is an axiomatic system CE_2.

(143) Every axiomatic system CE_2 satisfies seven Tarski's geometry axioms, Lower Dimension Axiom, Upper Dimension Axiom, Euclid Axiom, and Continuity Axiom.

(144) Every axiomatic system CE_2' satisfies seven Tarski's geometry axioms, Lower Dimension Axiom, Upper Dimension Axiom, Euclid Axiom, and Continuity Axiom.

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