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Algebraic Numbers

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Summary. This article provides definitions and examples upon an integral element of unital commutative rings. An algebraic number is also treated as consequence of a concept of "integral". Definitions for an integral closure, an algebraic integer and a transcendental numbers [14], [1], [10] and [7] are included as well. As an application of an algebraic number, this article includes a formal proof of a ring extension of rational number field $\mathbb{Q}$ induced by substitution of an algebraic number to the polynomial ring of $\mathbb{Q}[x]$ turns to be a field.

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1. Preliminaries

From now on i, j denote natural numbers and A, B denote rings.

Now we state the propositions:

(1) Let us consider rings L_1, L_2, L_3. Suppose L_1 is a subring of L_2 and L_2 is a subring of L_3. Then L_1 is a subring of L_3.

(2) $\mathbb{F}_{\mathbb{Q}}$ is a subfield of $\mathbb{C}_{\mathrm{F}}$.

(3) $\mathbb{F}_{\mathbb{Q}}$ is a subring of $\mathbb{C}_{\mathrm{F}}$.

(4) $\mathbb{Z}^{\mathrm{R}}$ is a subring of $\mathbb{C}_{\mathrm{F}}$.

Let us consider elements x, y of B and elements x_1, y_1 of A. Now we state the propositions:

(5) If A is a subring of B and $x = x_1$ and $y = y_1$, then $x + y = x_1 + y_1$.

(6) If A is a subring of B and $x = x_1$ and $y = y_1$, then $x \cdot y = x_1 \cdot y_1$.

Let c be a complex. Observe that $c(\in \mathbb{C}_{\mathrm{F}})$ reduces to c.

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2. Extended Evaluation Function

Let A, B be rings, p be a polynomial over A, and x be an element of B. The functor $\mathrm{ExtEval}(p, x)$ yielding an element of B is defined by

(Def. 1) there exists a finite sequence F of elements of B such that $it = \sum F$ and $\mathrm{len}\, F = \mathrm{len}\, p$ and for every element n of $\mathbb{N}$ such that $n \in \mathrm{dom}\, F$ holds $F(n) = p(n -' 1)(\in B) \cdot \mathrm{power}_B(x, n -' 1)$.

Now we state the proposition:

(7) Let us consider an element n of $\mathbb{N}$, rings A, B, and an element z of A. Suppose A is a subring of B. Then $\mathrm{power}_B(z(\in B), n) = \mathrm{power}_A(z, n)(\in B)$. The theorem is a consequence of (6).

Let us consider elements x_1, x_2 of A. Now we state the propositions:

(8) If A is a subring of B, then $x_1(\in B) + x_2(\in B) = (x_1 + x_2)(\in B)$. The theorem is a consequence of (5).

(9) If A is a subring of B, then $x_1(\in B) \cdot x_2(\in B) = (x_1 \cdot x_2)(\in B)$. The theorem is a consequence of (6).

(10) Let us consider a finite sequence F of elements of A, and a finite sequence G of elements of B. If A is a subring of B and $F = G$, then $(\sum F)(\in B) = \sum G$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence F of elements of A for every finite sequence G of elements of B such that $\mathrm{len}\, F = \$_1$ and $F = G$ holds $(\sum F)(\in B) = \sum G$. $\mathcal{P}[0]$ by [13, (43)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [4, (4)], [5, (3)], [4, (59)], [3, (11)]. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. $\square$

(11) Let us consider a natural number n, an element x of A, and a polynomial p over A. Suppose A is a subring of B. Then $p(n -' 1)(\in B) \cdot \mathrm{power}_B(x(\in B), n -' 1) = (p(n -' 1) \cdot \mathrm{power}_A(x, n -' 1))(\in B)$. The theorem is a consequence of (9) and (7).

(12) Let us consider an element x of A, and a polynomial p over A. Suppose A is a subring of B. Then $\mathrm{ExtEval}(p, x(\in B)) = (\mathrm{eval}(p, x))(\in B)$.

Proof: Consider F_1 being a finite sequence of elements of B such that $\mathrm{ExtEval}(p, x(\in B)) = \sum F_1$ and $\mathrm{len}\, F_1 = \mathrm{len}\, p$ and for every element n of $\mathbb{N}$ such that $n \in \mathrm{dom}\, F_1$ holds $F_1(n) = p(n -' 1)(\in B) \cdot \mathrm{power}_B(x(\in B), n -' 1)$. Consider F_2 being a finite sequence of elements of A such that $\mathrm{eval}(p, x) = \sum F_2$ and $\mathrm{len}\, F_2 = \mathrm{len}\, p$ and for every element n of $\mathbb{N}$ such that $n \in \mathrm{dom}\, F_2$ holds $F_2(n) = p(n -' 1) \cdot \mathrm{power}_A(x, n -' 1)$. $F_1 = F_2$ by [12, (29)], [5, (3)], (19). $\square$

(13) Let us consider an element x of B. Then $\mathrm{ExtEval}(\mathbf{0}.\, A, x) = 0_B$.

(14) Let us consider non degenerated rings A, B, and an element x of B. If A is a subring of B, then $\mathrm{ExtEval}(\mathbf{1}.\,A, x) = 1_B$.

(15) Let us consider an element x of B, and polynomials p, q over A. Suppose A is a subring of B. Then $\mathrm{ExtEval}(p+q, x) = \mathrm{ExtEval}(p, x)+\mathrm{ExtEval}(q, x)$. The theorem is a consequence of (8).

(16) Let us consider polynomials p, q over A. Suppose A is a subring of B and $\mathrm{len}\, p > 0$ and $\mathrm{len}\, q > 0$. Let us consider an element x of B. Then $\mathrm{ExtEval}(\text{Leading-Monomial}\, p * \text{Leading-Monomial}\, q, x) = (p(\mathrm{len}\, p -' 1) \cdot q(\mathrm{len}\, q -' 1))(\in B) \cdot \mathrm{power}_B(x, \mathrm{len}\, p + \mathrm{len}\, q -' 2)$. The theorem is a consequence of (13).

(17) Let us consider a polynomial p over A, and an element x of B. Suppose A is a subring of B. Then $\mathrm{ExtEval}(\text{Leading-Monomial}\, p, x) = p(\mathrm{len}\, p -' 1)(\in B) \cdot \mathrm{power}_B(x, \mathrm{len}\, p -' 1)$. The theorem is a consequence of (13).

Let us consider a commutative ring B, polynomials p, q over A, and an element x of B. Now we state the propositions:

(18) Suppose A is a subring of B. Then $\mathrm{ExtEval}(\text{Leading-Monomial}\, p*\text{Leading-Monomial}\, q, x) = \mathrm{ExtEval}(\text{Leading-Monomial}\, p, x)\cdot\mathrm{ExtEval}(\text{Leading-Monomial}\, q, x)$. The theorem is a consequence of (16), (9), (17), and (13).

(19) Suppose A is a subring of B. Then $\mathrm{ExtEval}(\text{Leading-Monomial}\, p*q, x) = \mathrm{ExtEval}(\text{Leading-Monomial}\, p, x) \cdot \mathrm{ExtEval}(q, x)$.

Proof: Set $p = \text{Leading-Monomial}\, p_1$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every polynomial q over A such that $\mathrm{len}\, q = \$_1$ holds $\mathrm{ExtEval}(p * q, x) = \mathrm{ExtEval}(p, x) \cdot \mathrm{ExtEval}(q, x)$. For every natural number k such that for every natural number n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$ by [9, (16)], [8, (31)], (15), (18). For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 4]. □

(20) If A is a subring of B, then $\mathrm{ExtEval}(p*q, x) = \mathrm{ExtEval}(p, x)\cdot\mathrm{ExtEval}(q, x)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every polynomial p over A such that $\mathrm{len}\, p = \$_1$ holds $\mathrm{ExtEval}(p * q, x) = \mathrm{ExtEval}(p, x) \cdot \mathrm{ExtEval}(q, x)$. For every natural number k such that for every natural number n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$ by [9, (16)], [8, (32)], (15), (19). For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 4]. □

(21) Let us consider an element x of B, and an element z_0 of A. Suppose A is a subring of B. Then $\mathrm{ExtEval}(\langle z_0 \rangle, x) = z_0(\in B)$. The theorem is a consequence of (13).

(22) Let us consider an element x of B, and elements z_0, z_1 of A. Suppose A is a subring of B. Then $\mathrm{ExtEval}(\langle z_0, z_1 \rangle, x) = z_0(\in B) + z_1(\in B) \cdot x$. The theorem is a consequence of (13).

3. Integral Element and Algebraic Numbers

Let A, B be rings and x be an element of B. We say that x is integral over A if and only if

(Def. 2) there exists a polynomial f over A such that $\mathrm{LC}\, f = 1_A$ and $\mathrm{ExtEval}(f, x) = 0_B$.

Now we state the proposition:

(23) Let us consider a non degenerated ring A, and an element a of A. If A is a subring of B, then $a(\in B)$ is integral over A. The theorem is a consequence of (12).

Let A be a non degenerated ring and B be a ring. Assume A is a subring of B. The integral closure over A in B yielding a non empty subset of B is defined by the term

(Def. 3) $\{z$, where z is an element of $B : z$ is integral over $A\}$.

Let c be a complex. We say that c is algebraic if and only if

(Def. 4) there exists an element x of $\mathbb{C}_{\mathrm{F}}$ such that $x = c$ and x is integral over $\mathbb{F}_{\mathbb{Q}}$.

Let x be an element of $\mathbb{C}_{\mathrm{F}}$. Note that x is algebraic if and only if the condition (Def. 5) is satisfied.

(Def. 5) x is integral over $\mathbb{F}_{\mathbb{Q}}$.

Let c be a complex. We say that c is algebraic integer if and only if

(Def. 6) there exists an element x of $\mathbb{C}_{\mathrm{F}}$ such that $x = c$ and x is integral over $\mathbb{Z}^{\mathrm{R}}$.

Let x be an element of $\mathbb{C}_{\mathrm{F}}$. Observe that x is algebraic integer if and only if the condition (Def. 7) is satisfied.

(Def. 7) x is integral over $\mathbb{Z}^{\mathrm{R}}$.

Let x be a complex. We introduce the notation x is transcendental as an antonym for x is algebraic.

Note that every complex which is rational is also algebraic and there exists a complex which is algebraic and there exists an element of $\mathbb{C}_{\mathrm{F}}$ which is algebraic and every complex which is integer is also algebraic integer and there exists a complex which is algebraic integer and there exists an element of $\mathbb{C}_{\mathrm{F}}$ which is algebraic integer.

Let A, B be rings and x be an element of B. The functor $\mathrm{AnnPoly}(x, A)$ yielding a non empty subset of $\mathrm{PolyRing}(A)$ is defined by the term

(Def. 8) $\{p$, where p is a polynomial over $A : \mathrm{ExtEval}(p, x) = 0_B\}$.

Now we state the propositions:

(24) Let us consider rings A, B, an element w of B, and elements x, y of PolyRing(A). Suppose A is a subring of B and x, $y \in$ AnnPoly(w, A). Then $x + y \in$ AnnPoly(w, A). The theorem is a consequence of (15).

(25) Let us consider a commutative ring B, an element z of B, and elements p, x of PolyRing(A). Suppose A is a subring of B and $x \in$ AnnPoly(z, A). Then $p \cdot x \in$ AnnPoly(z, A). The theorem is a consequence of (20).

(26) Let us consider a commutative ring B, an element w of B, and elements p, x of PolyRing(A). Suppose A is a subring of B and $x \in$ AnnPoly(w, A). Then $x \cdot p \in$ AnnPoly(w, A). The theorem is a consequence of (20).

(27) Let us consider a non degenerated ring A, a non degenerated commutative ring B, and an element w of B. Suppose A is a subring of B. Then AnnPoly(w, A) is a proper ideal of PolyRing(A).

Proof: AnnPoly(w, A) is closed under addition. AnnPoly(w, A) is left ideal. AnnPoly(w, A) is right ideal. AnnPoly(w, A) is proper by [8, (37)], (14). □

4. Properties of Polynomial Ring over Principal Ideal Domain

From now on K, L denote fields.

Now we state the propositions:

(28) Let us consider fields K, L, and an element w of L. Suppose K is a subring of L. Then there exists an element g of PolyRing(K) such that $\{g\}$–ideal = AnnPoly(w, K). The theorem is a consequence of (27).

(29) Let us consider fields K, L, and an element z of L. Suppose z is integral over K. Then AnnPoly(z, K) $\neq \{0_{\mathrm{PolyRing}(K)}\}$.

Proof: Consider f being a polynomial over K such that LC $f = 1_K$ and ExtEval(f, z) $= 0_L$. $f \notin \{0_{\mathrm{PolyRing}(K)}\}$ by [2, (47), (64)], [11, (7)]. □

(30) Let us consider a field K, and an element p of PolyRing(K). Suppose $p \neq \mathbf{0}.K$. Then p is a non zero element of the carrier of PolyRing(K).

Let us consider fields K, L and an element w of L. Now we state the propositions:

(31) If K is a subring of L, then AnnPoly(w, K) is quasi-prime. The theorem is a consequence of (20).

(32) If K is a subring of L and w is integral over K, then AnnPoly(w, K) is prime. The theorem is a consequence of (31) and (27).

(33) Let us consider fields K, L, and an element z of L. Suppose K is a subring of L and z is integral over K. Then there exists an element f of PolyRing(K) such that

(i) $f \neq \mathbf{0}.K$, and

(ii) $\{f\}$–ideal $=$ AnnPoly(z, K), and

(iii) $f =$ NormPoly f.

The theorem is a consequence of (28), (29), and (30).

(34) Let us consider fields K, L, an element z of L, and elements f, g of PolyRing(K). Suppose z is integral over K and $\{f\}$–ideal $=$ AnnPoly(z, K) and $f =$ NormPoly f and $\{g\}$–ideal $=$ AnnPoly(z, K) and $g =$ NormPoly g. Then $f = g$. The theorem is a consequence of (29) and (30).

Let K, L be fields and z be an element of L. Assume K is a subring of L and z is integral over K. The minimal polynomial of z over K yielding an element of the carrier of PolyRing(K) is defined by

(Def. 9) $it \neq \mathbf{0}.K$ and $\{it\}$–ideal $=$ AnnPoly(z, K) and $it =$ NormPoly it.

Assume K is a subring of L and z is integral over K. The degree of algebraic number z over K yielding an element of $\mathbb{N}$ is defined by the term

(Def. 10) deg(the minimal polynomial of z over K).

Let A, B be rings and x be an element of B. The functor HomExtEval(x, A) yielding a function from PolyRing(A) into B is defined by

(Def. 11) for every polynomial p over A, $it(p) =$ ExtEval(p, x).

Let x be an element of $\mathbb{C}_{\mathrm{F}}$. Note that HomExtEval$(x, \mathbb{F}_{\mathbb{Q}})$ is unity-preserving, additive, and multiplicative.

Now we state the propositions:

(35) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$.
Then $\mathbb{C}_{\mathrm{F}}$ is (PolyRing$(\mathbb{F}_{\mathbb{Q}})$)-homomorphic.

(36) Let us consider an element x of B, and an object z.
If $z \in$ rng HomExtEval(x, A), then $z \in B$.

Let x be an element of $\mathbb{C}_{\mathrm{F}}$. The functor FQ(x) yielding a subset of $\mathbb{C}_{\mathrm{F}}$ is defined by the term

(Def. 12) rng HomExtEval$(x, \mathbb{F}_{\mathbb{Q}})$.

Let us note that FQ(x) is non empty.

Let us consider elements x, z_1, z_2 of $\mathbb{C}_{\mathrm{F}}$. Now we state the propositions:

(37) If z_1, $z_2 \in$ FQ(x), then $z_1 + z_2 \in$ FQ(x). The theorem is a consequence of (3) and (15).

(38) If z_1, $z_2 \in$ FQ(x), then $z_1 \cdot z_2 \in$ FQ(x). The theorem is a consequence of (3) and (20).

(39) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$, and an element a of $\mathbb{F}_{\mathbb{Q}}$. Then $a \in$ FQ(x). The theorem is a consequence of (3) and (21).

Let x be an element of $\mathbb{C}_{\mathrm{F}}$. The functor FQ-add(x) yielding a binary operation on FQ(x) is defined by the term

(Def. 13) $+_{\mathbb{C}} \restriction \mathrm{FQ}(x)$.

The functor FQ-mult(x) yielding a binary operation on FQ(x) is defined by the term

(Def. 14) $\cdot_{\mathbb{C}} \restriction \mathrm{FQ}(x)$.

Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$ and elements z, w of FQ(x). Now we state the propositions:

(40) $(\mathrm{FQ\text{-}add}(x))(z, w) = z + w$.

(41) $(\mathrm{FQ\text{-}mult}(x))(z, w) = z \cdot w$.

(42) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$. Then $1_{\mathbb{C}_{\mathrm{F}}}(\in \mathrm{FQ}(x)) = 1_{\mathbb{C}_{\mathrm{F}}}$. The theorem is a consequence of (3) and (39).

(43) $(-1_{\mathbb{F}_{\mathbb{Q}}})(\in \mathbb{C}_{\mathrm{F}}) = -1_{\mathbb{C}_{\mathrm{F}}}$. The theorem is a consequence of (3).

Let x be an element of $\mathbb{C}_{\mathrm{F}}$. The functor $\mathbb{Q}[x]$ yielding a strict, non empty double loop structure is defined by the term

(Def. 15) $\langle \mathrm{FQ}(x), \mathrm{FQ\text{-}add}(x), \mathrm{FQ\text{-}mult}(x), 1_{\mathbb{C}_{\mathrm{F}}}(\in \mathrm{FQ}(x)), 0_{\mathbb{C}_{\mathrm{F}}}(\in \mathrm{FQ}(x))\rangle$.

Now we state the proposition:

(44) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$. Then $\mathbb{Q}[x]$ is a ring.
Proof: Reconsider $Z = \langle \mathrm{FQ}(x), \mathrm{FQ\text{-}add}(x), \mathrm{FQ\text{-}mult}(x), 1_{\mathbb{C}_{\mathrm{F}}}(\in \mathrm{FQ}(x)), 0_{\mathbb{C}_{\mathrm{F}}}(\in \mathrm{FQ}(x))\rangle$ as a non empty double loop structure. For every elements v, w of Z, $v + w = w + v$. For every elements u, v, w of Z, $(u + v) + w = u + (v + w)$. For every element v of Z, $v + 0_Z = v$. Every element of Z is right complementable by (36), [6, (9)], (39), (43). For every elements a, b, v of Z, $(a + b) \cdot v = a \cdot v + b \cdot v$. For every elements a, v, w of Z, $a \cdot (v + w) = a \cdot v + a \cdot w$ and $(v + w) \cdot a = v \cdot a + w \cdot a$. For every elements a, b, v of Z, $(a \cdot b) \cdot v = a \cdot (b \cdot v)$. For every element v of Z, $v \cdot 1_Z = v$ and $1_Z \cdot v = v$. $\square$

Let x be an element of $\mathbb{C}_{\mathrm{F}}$. One can verify that $\mathbb{Q}[x]$ is Abelian, add-associative, right zeroed, right complementable, associative, well unital, and distributive.

Let z be an element of $\mathbb{C}_{\mathrm{F}}$. One can verify that $\mathbb{Q}[z]$ is integral domain-like, commutative, and non degenerated.

Now we state the proposition:

(45) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$. Then $\mathbb{Q} \times \mathbb{Q} \subseteq \mathrm{FQ}(x) \times \mathrm{FQ}(x) \subseteq \mathbb{C} \times \mathbb{C}$. The theorem is a consequence of (39).

Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$. Now we state the propositions:

(46) The addition of $\mathbb{F}_{\mathbb{Q}} =$ (the addition of $\mathbb{Q}[x]) \restriction \mathbb{Q}$. The theorem is a consequence of (45).

(47) The multiplication of $\mathbb{F}_{\mathbb{Q}}$ = (the multiplication of $\mathbb{Q}[x]$) $\restriction \mathbb{Q}$. The theorem is a consequence of (45).

(48) $\mathbb{F}_{\mathbb{Q}}$ is a subring of $\mathbb{Q}[x]$. The theorem is a consequence of (46), (47), (42), (3), and (39).

Let us consider elements f, g of PolyRing(K). Now we state the propositions:

(49) Suppose $f \neq 0_{\mathrm{PolyRing}(K)}$ and $\{f\}$–ideal is prime and $g \notin \{f\}$–ideal. Then $\{f, g\}$–ideal = the carrier of PolyRing(K).

(50) Suppose $f \neq 0_{\mathrm{PolyRing}(K)}$ and $\{f\}$–ideal is prime and $g \notin \{f\}$–ideal. Then $\{f\}$–ideal and $\{g\}$–ideal are co-prime. The theorem is a consequence of (49).

(51) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$, and an element a of $\mathbb{Q}[x]$. Then there exists an element g of PolyRing($\mathbb{F}_{\mathbb{Q}}$) such that $a = (\mathrm{HomExtEval}(x, \mathbb{F}_{\mathbb{Q}}))(g)$.

Let us consider elements x, a of $\mathbb{C}_{\mathrm{F}}$. Now we state the propositions:

(52) Suppose $a \neq 0_{\mathbb{C}_{\mathrm{F}}}$ and $a \in$ the carrier of $\mathbb{Q}[x]$. Then there exists an element g of PolyRing($\mathbb{F}_{\mathbb{Q}}$) such that

(i) $g \notin \mathrm{AnnPoly}(x, \mathbb{F}_{\mathbb{Q}})$, and

(ii) $a = (\mathrm{HomExtEval}(x, \mathbb{F}_{\mathbb{Q}}))(g)$.

The theorem is a consequence of (51).

(53) Suppose x is algebraic and $a \neq 0_{\mathbb{C}_{\mathrm{F}}}$ and $a \in$ the carrier of $\mathbb{Q}[x]$. Then there exist elements f, g of PolyRing($\mathbb{F}_{\mathbb{Q}}$) such that

(i) $\{f\}$–ideal = $\mathrm{AnnPoly}(x, \mathbb{F}_{\mathbb{Q}})$, and

(ii) $g \notin \mathrm{AnnPoly}(x, \mathbb{F}_{\mathbb{Q}})$, and

(iii) $a = (\mathrm{HomExtEval}(x, \mathbb{F}_{\mathbb{Q}}))(g)$, and

(iv) $\{f\}$–ideal and $\{g\}$–ideal are co-prime.

The theorem is a consequence of (28), (3), (52), (32), (29), and (50).

(54) Suppose x is algebraic and $a \neq 0_{\mathbb{C}_{\mathrm{F}}}$ and $a \in$ the carrier of $\mathbb{Q}[x]$. Then there exists an element b of $\mathbb{C}_{\mathrm{F}}$ such that

(i) $b \in$ the carrier of $\mathbb{Q}[x]$, and

(ii) $a \cdot b = 1_{\mathbb{C}_{\mathrm{F}}}$.

The theorem is a consequence of (53), (3), (14), (15), and (20).

(55) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$. If x is algebraic, then $\mathbb{Q}[x]$ is a field. The theorem is a consequence of (54), (41), and (42).

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