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On Subnomials

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Summary. While discussing the sum of consecutive powers as a result of division of two binomials W.W. Sawyer [12] observes

> "It is a curious fact that most algebra textbooks give our ast result twice. It appears in two different chapters and usually there is no mention in either of these that it also occurs in the other. The first chapter, of course, is that on factors. The second is that on geometrical progressions. Geometrical progressions are involved in nearly all financial questions involving compound interest – mortgages, annuities, etc."

It's worth noticing that the first issue involves a simple arithmetical division of 99...9 by 9. While the above notion seems not have changed over the last 50 years, it reflects only a special case of a broader class of problems involving two variables. It seems strange, that while binomial formula is discussed and studied widely [7], [8], little research is done on its counterpart with all coefficients equal to one, which we will call here the subnomial. The study focuses on its basic properties and applies it to some simple problems usually proven by induction [6].

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From now on a, b, i, j, k, l, m, n denote natural numbers.

Let a be a positive real number and n be a natural number. Let us note that a^n is positive.

Let a be a non negative real number. One can check that a^n is non negative.

Let us observe that $\sqrt{a^2}$ reduces to a.

Observe that there exists a complex which is real and there exists a complex which is non real.

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Let a be a non real complex. One can verify that $\Im(a)$ is non zero.

Let a be a real number. One can check that $\Re(a)$ reduces to a.

Now we state the proposition:

(1) Let us consider a non zero real number a, and a complex a_1. If $a \cdot a_1$ is a real number, then a_1 is a real number.

Note that every binary relation which is $\mathbb{R}$-valued is also $\mathbb{C}$-valued and every binary relation which is $\mathbb{Q}$-valued is also $\mathbb{R}$-valued and every binary relation which is $\mathbb{Q}$-valued is also $\mathbb{C}$-valued and every binary relation which is $\mathbb{Z}$-valued is also $\mathbb{Q}$-valued and every binary relation which is $\mathbb{Z}$-valued is also $\mathbb{R}$-valued and every binary relation which is $\mathbb{Z}$-valued is also $\mathbb{C}$-valued.

Every binary relation which is $\mathbb{N}$-valued is also $\mathbb{Z}$-valued and every binary relation which is $\mathbb{N}$-valued is also $\mathbb{Q}$-valued and every binary relation which is $\mathbb{N}$-valued is also $\mathbb{R}$-valued and every binary relation which is $\mathbb{N}$-valued is also $\mathbb{C}$-valued and every binary relation which is real-valued is also $\mathbb{R}$-valued and every binary relation which is complex-valued is also $\mathbb{C}$-valued.

Let a be an object. Observe that $1 \cdot \operatorname{len}\langle a\rangle$ reduces to 1.

Let f be a finite sequence. Let us note that $(\langle a\rangle \frown f)(1)$ reduces to a and $(f \frown \langle a\rangle)(1 + \operatorname{len} f)$ reduces to a.

Let x be a complex. Observe that $\sum\langle x\rangle$ reduces to x.

Let f, g be finite sequences. Let us note that $(f \frown g)\restriction \operatorname{dom} f$ reduces to f and $(f \frown g)\restriction \operatorname{len} f$ reduces to f.

Now we state the proposition:

(2) Let us consider a finite sequence f. Then there exists a non empty set D such that f is a finite sequence of elements of D.

Let f be a finite sequence. One can check that $f(0)$ reduces to 0 and $f\restriction \operatorname{len} f$ reduces to f. Note that $f_{\lfloor \operatorname{len} f}$ is empty.

Let n be a natural number. One can verify that $\overline{\overline{\operatorname{Seg} n}}$ reduces to n and $\overline{\overline{\mathbb{Z}_n}}$ reduces to n.

Note that $\operatorname{len} \mathrm{id}_{\operatorname{Seg} n}$ reduces to n and $\operatorname{len} \operatorname{idseq}(n)$ reduces to n.

Let m, n be natural numbers. One can check that $(\operatorname{idseq}(m+n))(m)$ reduces to m and $(\operatorname{Rev}(\operatorname{idseq}(m + (n + 1))))(m + 1)$ reduces to $n + 1$.

Let a, b be natural numbers. The functors: $\min(a, b)$ and $\max(a, b)$ yield natural numbers. Let f be a finite sequence and n be a natural number. One can check that $f\restriction(\operatorname{len} f + n)$ reduces to f and $f\restriction \operatorname{Seg} \max(\operatorname{len} f, n)$ reduces to f. One can verify that $f_{\lfloor \operatorname{len} f+n}$ is empty and $f_{\lfloor \operatorname{len} f}(n)$ is zero.

Let us consider an element n of $\mathbb{N}$, a set D, and a finite sequence f of elements of D. Now we state the propositions:

(3) If $n \in \operatorname{dom} f$, then $\operatorname{len}(f\restriction n) = n$.

(4) If $n \geqslant \operatorname{len} f$, then $\operatorname{len}(f\restriction n) = \operatorname{len} f$.

Let f be a finite sequence and n be a non zero natural number. One can verify that $f(\operatorname{len} f + n)$ is zero.

Let f be a finite sequence of elements of $\mathbb{R}$ and i, j be natural numbers. One can verify that $(f{\restriction}i){\restriction}(i + j)$ reduces to $f{\restriction}i$.

Let a be a natural number. Note that $\sum(a \mapsto 0)$ reduces to 0. Note that $\sum(a \mapsto 0)$ is zero.

Let b be an object. One can verify that $\operatorname{len}(a \mapsto b)$ reduces to a.

Let a be a non zero natural number. Observe that $a \mapsto b$ is non empty and $a \mapsto b$ is constant.

Let us observe that the value of $a \mapsto b$ reduces to b.

Let f be a constant finite sequence. Let us observe that $\operatorname{Rev}(f)$ reduces to f.

Let a be a natural number, b be a non zero natural number, and x be an object. One can check that $((a+b) \mapsto x)(b)$ reduces to x. Let us observe that $(a \mapsto x)(a + b)$ is zero.

Let a be an object and n be a natural number. Observe that $\operatorname{Rev}(n \mapsto a)$ reduces to $n \mapsto a$.

Note that $\binom{n}{0 \cdot 1}$ reduces to 1 and $\binom{n}{n \cdot 1}$ reduces to 1.

Let f be a non-negative yielding finite sequence of elements of $\mathbb{R}$ and i be a natural number. One can check that $f(i)$ is non negative and every finite sequence which is empty is also non-positive yielding.

Let f be a non-positive yielding finite sequence of elements of $\mathbb{R}$ and i be a natural number. Note that $f(i)$ is non positive.

Let f be a non-negative yielding finite sequence of elements of $\mathbb{R}$ and i, j be natural numbers. One can check that $(f{\restriction}j)(i)$ is non negative and $f_{\lfloor j}(i)$ is non negative.

Let f be an empty, real-valued finite sequence. One can verify that $\prod f$ is non negative.

Let f be a non-negative yielding finite sequence of elements of $\mathbb{R}$. One can verify that $\sum f$ is non negative and $\prod f$ is non negative.

Let f be a non-positive yielding finite sequence of elements of $\mathbb{R}$. Let us note that $\sum f$ is non positive.

Let a be an object and f be a non-negative yielding finite sequence of elements of $\mathbb{R}$. One can check that $f(a)$ is non negative.

Let f be a non-positive yielding finite sequence of elements of $\mathbb{R}$. One can verify that $f(a)$ is non positive.

Let D be a set and f, g be D-valued finite sequences. Let us note that $f \frown g$ is D-valued.

Let f be a finite sequence of elements of $\mathbb{R}$ and n be a natural number. One can verify that $(f{\restriction}n)_{\lfloor n}$ is empty and $f_{\lfloor \max(\operatorname{len} f, n)}$ is empty.

Let D be a set. One can verify that there exists a finite sequence of elements of D which is empty and every finite sequence of elements of D which is empty is also non-negative yielding and every finite sequence which is non-negative yielding and $\mathbb{Z}$-valued is also $\mathbb{N}$-valued and every finite sequence of elements of $\mathbb{Z}$ which is non-negative yielding is also $\mathbb{N}$-valued.

Let f be a $\mathbb{C}$-valued finite sequence. Note that $f+0$ reduces to f and $f-0$ reduces to f.

Let x be an object. One can check that $\langle x \rangle(1)$ reduces to x.

Now we state the propositions:

(5) Let us consider a finite sequence f. Then every permutation of $\operatorname{dom} f$ is a permutation of $\operatorname{dom} \operatorname{Rev}(f)$.

(6) $\operatorname{Rev}(\operatorname{idseq}(n))$ is a permutation of $\operatorname{Seg} n$.

Let us consider a finite sequence f. Now we state the propositions:

(7) $\operatorname{idseq}(\operatorname{len} f)$ is a permutation of $\operatorname{dom} f$.

(8) $\operatorname{Rev}(\operatorname{idseq}(\operatorname{len} f))$ is a permutation of $\operatorname{dom} \operatorname{Rev}(f)$. The theorem is a consequence of (6).

(9) Let us consider a function f, and a permutation h of $\operatorname{dom} f$. Then $\operatorname{dom}(f \cdot h) = \operatorname{dom} f$.

Let f be a finite sequence and h be a permutation of $\operatorname{dom} f$. Observe that $f \cdot h$ is finite sequence-like and $f \cdot h$ is $(\operatorname{dom} f)$-defined.

Let us consider a finite sequence f. Now we state the propositions:

(10) $f = \operatorname{Rev}(f) \cdot \operatorname{Rev}(\operatorname{idseq}(\operatorname{len} f))$.

PROOF: Reconsider $P = \operatorname{Rev}(\operatorname{idseq}(\operatorname{len} f))$ as a permutation of $\operatorname{dom} \operatorname{Rev}(f)$. Reconsider $g = \operatorname{Rev}(f) \cdot P$ as a finite sequence. For every object x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$ by [13, (25)], [1, (21)], [3, (57)], [4, (12)]. $\square$

(11) f and $\operatorname{Rev}(f)$ are fiberwise equipotent. The theorem is a consequence of (10) and (8).

(12) Let us consider a non empty set D, and a D-valued finite sequence r. Suppose $\operatorname{len} r = i + j$. Then there exist D-valued finite sequences p, q such that

 (i) $\operatorname{len} p = i$, and

 (ii) $\operatorname{len} q = j$, and

 (iii) $r = p \frown q$.

(13) Let us consider a non-negative yielding finite sequence f of elements of $\mathbb{R}$. Then $\sum f \geqslant \sum(f \restriction j)$.

(14) Let us consider a $\mathbb{C}$-valued finite sequence f, and complexes x_1, x_2. Then $(f + x_1) + x_2 = f + (x_1 + x_2)$.

Let f be a $\mathbb{C}$-valued finite sequence and x be a complex. One can check that $f + x - x$ reduces to f and $f - x + x$ reduces to f.

Let x, y be real numbers. One can check that $\max(\min(x, y), \max(x, y))$ reduces to $\max(x, y)$ and $\min(\min(x, y), \max(x, y))$ reduces to $\min(x, y)$.

Let z be a non negative real number. Let us observe that $\min(\min(x, y), y+z)$ reduces to $\min(x, y)$ and $\max(\max(x, y), y - z)$ reduces to $\max(x, y)$.

Let f be a finite sequence and i, j be natural numbers. Observe that $(f{\restriction}i){\restriction}(i+j)$ reduces to $f{\restriction}i$.

Let f be a non-negative yielding finite sequence of elements of $\mathbb{R}$ and n be a natural number. One can check that $f{\restriction}n$ is non-negative yielding and $f_{\lfloor n}$ is non-negative yielding.

Let f be a finite sequence of elements of $\mathbb{R}$. Note that $f - \min f$ is non-negative yielding and $f - \max f$ is non-positive yielding.

Let f be a finite sequence. Let us note that $\mathrm{Rev}(f)$ is $(\mathrm{len}\, f)$-element.

Let D be a non empty set and f be a D-valued finite sequence. Note that $\mathrm{Rev}(f)$ is D-valued.

Let a be a complex and f be a complex-valued finite sequence. Let us note that $a \cdot f$ is $(\mathrm{len}\, f)$-element.

Let a, b be real numbers and n be a natural number.

Note that $\mathrm{len}\langle\binom{n+1-1}{0}a^0b^{n+1-1}, \ldots, \binom{n+1-1}{n+1-1}a^{n+1-1}b^0\rangle$ reduces to $n + 1$.

Let us note that $\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle$ is $(n + 1)$-element.

Let us note that $\mathrm{len}\langle\binom{n+1-1}{0}, \ldots, \binom{n+1-1}{n+1-1}\rangle$ reduces to $n + 1$. One can verify that $\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle$ is non-negative yielding and $\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle$ is $(n + 1)$-element.

Let n be a non zero natural number. Let us note that $\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle(2)$ reduces to n and $\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle(n)$ reduces to n.

Now we state the propositions:

(15) Let us consider complex-valued functions f_1, f_2, f_3. Then $(f_1 \cdot f_2) \cdot f_3 = f_1 \cdot (f_2 \cdot f_3)$.

(16) Let us consider finite sequences f, g of elements of $\mathbb{C}$, and an object i. Then $(f \cdot g)(i) = f(i) \cdot g(i)$.

Let us consider real numbers x, y. Now we state the propositions:

(17) $\max(x, y) - \min(x, y) = |x - y|$.

(18) (i) $\min(x, y) \cdot \max(x, y) = x \cdot y$, and

(ii) $\min(x, y) + \max(x, y) = x + y$.

Let us consider a non-negative yielding finite sequence f of elements of $\mathbb{R}$. Now we state the propositions:

(19) $\sum f \geqslant \sum(f{\restriction}j)$.

(20) If $i \geqslant j$, then $\sum(f{\restriction}i) \geqslant \sum(f{\restriction}j)$. The theorem is a consequence of (19).

(21) $\sum f \geqslant f(n)$.

(22) Let us consider finite sequences f, g, h of elements of $\mathbb{C}$. Suppose $\operatorname{dom} h = \operatorname{dom} f \cap \operatorname{dom} g$. Then $\operatorname{len} h = \min(\operatorname{len} f, \operatorname{len} g)$.

Let us consider finite sequences f, g of elements of $\mathbb{C}$. Now we state the propositions:

(23) $\operatorname{len}(f + g) = \min(\operatorname{len} f, \operatorname{len} g)$. The theorem is a consequence of (22).

(24) $\operatorname{len}(f \cdot g) = \min(\operatorname{len} f, \operatorname{len} g)$. The theorem is a consequence of (22).

(25) $\operatorname{len}(f - g) = \min(\operatorname{len} f, \operatorname{len} g)$. The theorem is a consequence of (23).

(26) Let us consider non-negative yielding finite sequences f, g of elements of $\mathbb{R}$. Then $(f \cdot g)(n) \leqslant \sum f \cdot g(n)$. The theorem is a consequence of (21).

(27) Let us consider a real number r, and a non zero natural number n. Then there exists a finite sequence f of elements of $\mathbb{R}$ such that

 (i) $\operatorname{len} f = n$, and

 (ii) $\sum f = r$.

Let us consider a finite sequence f of elements of $\mathbb{C}$ and a complex x. Now we state the propositions:

(28) $f + x = f + \operatorname{len} f \mapsto x$.

PROOF: Reconsider $g = \operatorname{len} f \mapsto x$ as a finite sequence of elements of $\mathbb{C}$. $\operatorname{len}(f + g) = \min(\operatorname{len} f, \operatorname{len}(\operatorname{len} f \mapsto x))$. For every natural number i such that $i \in \operatorname{dom}(f + x)$ holds $(f + x)(i) = (f + g)(i)$ by [13, (25)], [1, (21)], [13, (29)]. $\square$

(29) $\sum(f + x) = \sum f + x \cdot \operatorname{len} f$. The theorem is a consequence of (28).

(30) Let us consider a complex-valued finite sequence f, and a complex x. Then $\sum(f - x) = \sum f - x \cdot \operatorname{len} f$. The theorem is a consequence of (29).

(31) Let us consider a finite sequence f of elements of $\mathbb{R}$, and a non-negative yielding finite sequence g of elements of $\mathbb{R}$. If for every natural number x, $f(x) \geqslant g(x)$, then f is non-negative yielding.

(32) Let us consider finite sequences f, g of elements of $\mathbb{R}$. If for every natural number x, $f(x) \geqslant g(x)$, then $\sum f \geqslant \sum g$.

(33) Let us consider a finite sequence f of elements of $\mathbb{C}$.
Then $\sum(f{\restriction}(1\ \textbf{qua}\ \text{natural number})) = f((1\ \textbf{qua}\ \text{natural number}))$.

(34) Let us consider a finite sequence f of elements of $\mathbb{C}$, and a natural number n. Then $\sum(f_{\lfloor n}{\restriction}1) = f(n + 1)$. The theorem is a consequence of (33).

(35) Let us consider a finite sequence f, and natural numbers a, b. Then $(f_{\lfloor a})_{\lfloor b} = f_{\lfloor a+b}$. The theorem is a consequence of (2).

Let us consider a finite sequence f of elements of $\mathbb{C}$. Now we state the propositions:

(36) $f = ((f \restriction i) \frown (f_{\downharpoonleft i} \restriction (1$ **qua** natural number$))) \frown f_{\downharpoonleft i+1}$. The theorem is a consequence of (35).

(37) $\sum f = \sum (f \restriction i) + f(i+1) + \sum f_{\downharpoonleft i+1}$. The theorem is a consequence of (35) and (34).

(38) Let us consider a finite sequence f, and a non zero natural number i. Then $f(n+i) = f_{\downharpoonleft n}(i)$. The theorem is a consequence of (2).

(39) Let us consider finite sequences f, g of elements of $\mathbb{R}$. Suppose for every natural number x, $f(x) \geqslant g(x)$ and there exists i such that $f(i+1) > g(i+1)$. Then $\sum f > \sum g$.

Proof: Consider i being a natural number such that $f(i+1) > g(i+1)$. $\sum f = \sum (f \restriction i) + f(i+1) + \sum f_{\downharpoonleft i+1}$. $\sum g = \sum (g \restriction i) + g(i+1) + \sum g_{\downharpoonleft i+1}$. For every natural number x, $(f \restriction i)(x) \geqslant (g \restriction i)(x)$ and $f_{\downharpoonleft i+1}(x) \geqslant g_{\downharpoonleft i+1}(x)$ by [13, (112)], [3, (17)], [13, (25)], (38). $\sum (f \restriction i) \geqslant \sum (g \restriction i)$ and $\sum f_{\downharpoonleft i+1} \geqslant \sum g_{\downharpoonleft i+1}$. $\square$

(40) Let us consider non-negative yielding finite sequences f, g of elements of $\mathbb{R}$. Then $\sum f \cdot \sum g \geqslant \sum (f \cdot g)$. The theorem is a consequence of (26) and (32).

(41) Let us consider a complex a, and a complex-valued finite sequence f. Then $\operatorname{len} f \mapsto a \cdot f = a \cdot f$.

Proof: For every object x such that $x \in \operatorname{dom}(\operatorname{len} f \mapsto a \cdot f)$ holds $(\operatorname{len} f \mapsto a \cdot f)(x) = (a \cdot f)(x)$ by [13, (25)], [1, (10)]. $\square$

(42) Let us consider complexes a, b. Then $a \cdot \langle b \rangle = \langle a \cdot b \rangle$.

Proof: For every object x such that $x \in \operatorname{Seg} 1$ holds $\langle a \cdot b \rangle(x) = a \cdot \langle b \rangle(x)$ by [2, (2)]. $\square$

Let us consider a complex a and complex-valued finite sequences f, g. Now we state the propositions:

(43) $a \cdot (f \frown g) = (a \cdot f) \frown (a \cdot g)$.

Proof: For every object x such that $x \in \operatorname{dom}(a \cdot (f \frown g))$ holds $(a \cdot (f \frown g))(x) = ((a \cdot f) \frown (a \cdot g))(x)$ by [2, (25)]. $\square$

(44) If $g = \operatorname{Rev}(f)$, then $\operatorname{Rev}(a \cdot f) = a \cdot g$.

Proof: Set $h = a \cdot f$. Set $h_1 = a \cdot g$. Set $h_2 = \operatorname{Rev}(h)$. For every object x such that $x \in \operatorname{dom} h_1$ holds $h_1(x) = h_2(x)$ by [13, (25)], [1, (21)], [13, (29)]. $\square$

Let a, b be real numbers and n be a natural number.

The functor $(a, b)\,\mathrm{Subnomial}\, n$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by the term

(Def. 1) $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle / \langle \binom{n}{0}, \ldots, \binom{n}{n} \rangle$.

Now we state the proposition:

(45) Let us consider real numbers a, b, and a natural number n. Then

(i) $\mathrm{len}\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle = \mathrm{len}\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$, and

(ii) $\mathrm{len}((a,b)\,\mathrm{Subnomial}\,n) = \mathrm{len}\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$, and

(iii) $\mathrm{len}\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle = \mathrm{len}((a,b)\,\mathrm{Subnomial}\,n)$, and

(iv) $\mathrm{dom}\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle = \mathrm{dom}\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$, and

(v) $\mathrm{dom}((a,b)\,\mathrm{Subnomial}\,n) = \mathrm{dom}\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$, and

(vi) $\mathrm{dom}\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle = \mathrm{dom}((a,b)\,\mathrm{Subnomial}\,n)$.

Let a, b be real numbers and n be a natural number.

Note that $\mathrm{len}((a,b)\,\mathrm{Subnomial}(n+1-1))$ reduces to $n+1$. Observe that $(a,b)\,\mathrm{Subnomial}\,n$ is $(n+1)$-element.

Let a, b be integers and n, m be natural numbers.

Observe that $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle(m)$ is integer.

Let n be a natural number. One can check that $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle$ is $\mathbb{Z}$-valued. Now we state the proposition:

(46) Let us consider integers a, b, and a natural number k. Suppose $k \in \mathrm{dom}\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$. Then $\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle(k) \mid \langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle(k)$.

Let l, k be natural numbers. Note that $\binom{l+k}{k}$ is positive.

Let l be a natural number and k be a non zero natural number. One can check that $\binom{l}{l+k}$ is zero and $\langle\binom{l}{0},\ldots,\binom{l}{l}\rangle(l+k+1)$ is zero.

Let k be a natural number. Observe that $\langle\binom{l+k}{0},\ldots,\binom{l+k}{l+k}\rangle(k+1)$ is positive.

Now we state the proposition:

(47) Let us consider natural numbers k, l. Suppose $k \in \mathrm{dom}\langle\binom{l}{0},\ldots,\binom{l}{l}\rangle$. Then $\langle\binom{l}{0},\ldots,\binom{l}{l}\rangle(k)$ is not zero.

Let a, b be integers and m, n be natural numbers. One can check that $((a,b)\,\mathrm{Subnomial}\,n)(m)$ is integer.

Let n be a natural number. Note that $(a,b)\,\mathrm{Subnomial}\,n$ is $\mathbb{Z}$-valued.

Let a, b be real numbers. One can verify that the functor $(a,b)\,\mathrm{Subnomial}\,n$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by

(Def. 2) $\mathrm{len}\,it = n+1$ and for every natural numbers i, l, m such that $i \in \mathrm{dom}\,it$ and $m = i-1$ and $l = n-m$ holds $it(i) = a^l \cdot b^m$.

Let a, b be positive real numbers and k, l be natural numbers. Note that $((a,b)\,\mathrm{Subnomial}(k+l))(k+1)$ is positive.

Let n be a natural number. Let us note that $\sum((a,b)\,\mathrm{Subnomial}\,n)$ is positive.

Let k be a natural number and n be a non zero natural number. One can verify that $\langle\binom{n}{0}0^00^n,\ldots,\binom{n}{n}0^n0^0\rangle(k)$ is zero and $((0,0)\,\mathrm{Subnomial}\,n)(k)$ is zero and $\langle\binom{n}{0}0^00^n,\ldots,\binom{n}{n}0^n0^0\rangle$ is empty yielding and $(0,0)\,\mathrm{Subnomial}\,n$ is empty yielding.

Let f be an empty yielding finite sequence of elements of $\mathbb{C}$. Let us observe that $\sum f$ is zero.

Let n be a natural number. One can check that $\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle(1)$ reduces to 1 and $\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle(n+1)$ reduces to 1.

Now we state the proposition:

(48) Let us consider real numbers a, b, and a natural number n. Then

(i) $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle(1) = ((a,b)\,\mathrm{Subnomial}\,n)(1)$, and

(ii) $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle(n+1) = ((a,b)\,\mathrm{Subnomial}\,n)(n+1)$.

Let us consider real numbers a, b. Now we state the propositions:

(49) $(a,b)\,\mathrm{Subnomial}(n+1) = a\cdot((a,b)\,\mathrm{Subnomial}\,n)\frown\langle b^{n+1}\rangle$.

Proof: For every natural number k such that $1 \leqslant k \leqslant \mathrm{len}((a,b)\,\mathrm{Subnomial}(n+1))$ holds $((a,b)\,\mathrm{Subnomial}(n+1))(k) = (a\cdot((a,b)\,\mathrm{Subnomial}\,n)\frown\langle b^{n+1}\rangle)(k)$ by [13, (25)], [1, (21)], [10, (6)], [5, (16)]. □

(50) $(a,b)\,\mathrm{Subnomial}(n+1) = \langle a^{n+1}\rangle\frown(b\cdot((a,b)\,\mathrm{Subnomial}\,n))$.

Proof: For every natural number k such that $1 \leqslant k \leqslant \mathrm{len}((a,b)\,\mathrm{Subnomial}(n+1))$ holds $((a,b)\,\mathrm{Subnomial}(n+1))(k) = (\langle a^{n+1}\rangle\frown(b\cdot((a,b)\,\mathrm{Subnomial}\,n)))(k)$ by [1, (13), (21)], [13, (25)], [10, (2), (6)]. □

(51) Let us consider real numbers a, b, and a natural number n. Then $a^{n+1} - b^{n+1} = (a-b)\cdot\sum((a,b)\,\mathrm{Subnomial}\,n)$. The theorem is a consequence of (49) and (50).

(52) Let us consider a real number a, and a non zero natural number n. Then $a^n = \sum((a,0)\,\mathrm{Subnomial}\,n)$. The theorem is a consequence of (51).

(53) Let us consider a real number a, and a natural number n. Then $a^{n+1} = \sum((a,1)\,\mathrm{Subnomial}\,n)\cdot(a-1)+1$. The theorem is a consequence of (51).

(54) Let us consider real numbers a, b, c, d, a natural number n, and an object x. Suppose $x \in \mathrm{dom}((a\cdot b, c\cdot d)\,\mathrm{Subnomial}\,n)$. Then $((a\cdot b, c\cdot d)\,\mathrm{Subnomial}\,n)(x) = ((a,d)\,\mathrm{Subnomial}\,n)(x)\cdot((b,c)\,\mathrm{Subnomial}\,n)(x)$.

(55) Let us consider real numbers a, b, c, d, and a natural number n. Then $(a\cdot b, c\cdot d)\,\mathrm{Subnomial}\,n = ((a,d)\,\mathrm{Subnomial}\,n)\cdot((b,c)\,\mathrm{Subnomial}\,n)$. The theorem is a consequence of (54).

Let us consider real numbers a, b and a natural number n. Now we state the propositions:

(56) $(a,b)\,\mathrm{Subnomial}\,n = ((a,1)\,\mathrm{Subnomial}\,n)\cdot((1,b)\,\mathrm{Subnomial}\,n)$. The theorem is a consequence of (55).

(57) $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle = \langle\binom{n}{0},\ldots,\binom{n}{n}\rangle\cdot((a,b)\,\mathrm{Subnomial}\,n)$.

Proof: $\mathrm{dom}\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle = \mathrm{dom}\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$. For every object c such that $c \in \mathrm{dom}\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle$ holds $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle(c) = \langle\binom{n}{0},\ldots,\binom{n}{n}\rangle(c)\cdot((a,b)\,\mathrm{Subnomial}\,n)(c)$ by [13, (25)], [1, (10)]. □

(58) (i) $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle = \langle\binom{n}{0}a^0 1^n,\ldots,\binom{n}{n}a^n 1^0\rangle\cdot((1,b)\operatorname{Subnomial} n)$, and

(ii) $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle = ((a,1)\operatorname{Subnomial} n)\cdot\langle\binom{n}{0}1^0b^n,\ldots,\binom{n}{n}1^nb^0\rangle$.

The theorem is a consequence of (57), (56), and (15).

(59) Let us consider real numbers a, b, c, d, and a natural number n. Then $\langle\binom{n}{0}a\cdot b^0c\cdot d^n,\ldots,\binom{n}{n}a\cdot b^nc\cdot d^0\rangle = (\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle\cdot((a,c)\operatorname{Subnomial} n))\cdot((b,d)\operatorname{Subnomial} n)$. The theorem is a consequence of (57) and (55).

(60) Let us consider a real number a, and natural numbers n, i. Suppose $i\in\operatorname{dom}((a,a)\operatorname{Subnomial} n)$. Then $((a,a)\operatorname{Subnomial} n)(i) = a^n$.

Let us consider a natural number n and a real number a. Now we state the propositions:

(61) $(a,a)\operatorname{Subnomial} n = (n+1)\mapsto a^n$.

PROOF: For every natural number j, $((a,a)\operatorname{Subnomial} n)(j) = ((n+1)\mapsto a^n)(j)$ by [13, (25)], (60), [1, (10)]. □

(62) $\prod((a,a)\operatorname{Subnomial} n) = a^{n\cdot(n+1)}$. The theorem is a consequence of (61).

(63) Let us consider a natural number n, and n-element, complex-valued finite sequences f, g. Then $\prod(f\cdot g) = \prod f\cdot\prod g$.

(64) Let us consider real numbers a, b, and a natural number n. Then $(a,b)\operatorname{Subnomial} n = \operatorname{Rev}((b,a)\operatorname{Subnomial} n)$.

PROOF: $\operatorname{dom}((a,b)\operatorname{Subnomial} n) = \operatorname{dom}\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$ and $\operatorname{dom}((b,a)\operatorname{Subnomial} n) = \operatorname{dom}\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$. For every object i such that $i\in\operatorname{dom}((a,b)\operatorname{Subnomial} n)$ holds $((a,b)\operatorname{Subnomial} n)(i) = (\operatorname{Rev}((b,a)\operatorname{Subnomial} n))(i)$ by [13, (26)], [3, (57), (59), (58)]. □

Let n be a natural number and i be a natural number. One can check that $((1,1)\operatorname{Subnomial}(n+i))(i+1)$ reduces to 1.

Let i be a non zero natural number. Observe that $((1,-1)\operatorname{Subnomial}(2\cdot i+n))(2\cdot i)$ reduces to -1.

Let i be an odd natural number. Let us observe that $((1,-1)\operatorname{Subnomial}(n+i))(i)$ reduces to 1.

Let a be a real number.

One can check that $n\mapsto a$ is constant and $(a,a)\operatorname{Subnomial} n$ is constant.

Let a, b be non negative real numbers and n, k be natural numbers. One can check that $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle(k)$ is non negative and $((a,b)\operatorname{Subnomial} n)(k)$ is non negative.

Now we state the propositions:

(65) Let us consider a real number a, and a natural number n. Then $\sum((a,a)\operatorname{Subnomial} n) = (n+1)\cdot a^n$. The theorem is a consequence of (60).

(66) Let us consider a real number a, and an even natural number n. Then $\sum((a,-a)\,\mathrm{Subnomial}\,n) = a^n$. The theorem is a consequence of (65) and (51).

Let n be an even natural number. Note that $\sum((1,-1)\,\mathrm{Subnomial}\,n)$ reduces to 1.

Let a be a real number and n be an odd natural number. One can verify that $\sum((a,-a)\,\mathrm{Subnomial}\,n)$ is zero.

Let n be a natural number. Let us observe that $\sum((1,1)\,\mathrm{Subnomial}(n+1-1))$ reduces to $n+1$.

One can verify that $\sum\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle$ is non zero.

Let a, b be non negative real numbers. Observe that $(a,b)\,\mathrm{Subnomial}\,n$ is non-negative yielding and $\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle$ is non-negative yielding and $\sum((a,b)\,\mathrm{Subnomial}\,n)$ is non negative and $\sum\langle\binom{n}{0}a^0b^n,\ldots,\binom{n}{n}a^nb^0\rangle$ is non negative.

Let us consider real numbers a, b. Now we state the propositions:

(67) $(a,b)\,\mathrm{Subnomial}\,n$ and $(b,a)\,\mathrm{Subnomial}\,n$ are fiberwise equipotent. The theorem is a consequence of (11) and (64).

(68) $\prod((a,b)\,\mathrm{Subnomial}\,n) = \prod((b,a)\,\mathrm{Subnomial}\,n)$.

(69) Let us consider a non negative real number a. Then $\prod((a,1)\,\mathrm{Subnomial}\,n) = a^{\binom{n+1}{2}}$. The theorem is a consequence of (62), (55), (63), and (67).

(70) $n!\cdot k! \leqslant (n+k)!$.

(71) $\binom{n+k}{k} = 1$ if and only if $n=0$ or $k=0$.

Proof: If $n \neq 0$ and $k \neq 0$, then $\binom{n+k}{k} \neq 1$ by [1, (14)], [10, (22)]. $\square$

(72) $n!\cdot k! = (n+k)!$ if and only if $n=0$ or $k=0$. The theorem is a consequence of (71).

Let n, k be non zero natural numbers. One can check that $(n+k)! - n!\cdot k!$ is positive. Now we state the propositions:

(73) Let us consider a real number a. Then $\sum((a,a)\,\mathrm{Subnomial}\,n) = \sum((1,1)\,\mathrm{Subnomial}\,n)\cdot\sum\langle\binom{n}{0}a^0 0^n,\ldots,\binom{n}{n}a^n 0^0\rangle$. The theorem is a consequence of (65).

(74) Let us consider real numbers a, b, c. Then $\sum\langle\binom{n}{0}a+b^0c^n,\ldots,\binom{n}{n}a+b^nc^0\rangle = \sum\langle\binom{n}{0}a^0b+c^n,\ldots,\binom{n}{n}a^nb+c^0\rangle$.

(75) $\langle\binom{n}{0},\ldots,\binom{n}{n}\rangle(i+1) = \binom{n}{i}$.

(76) $\binom{2\cdot n}{n} = \frac{(2\cdot n)!}{n!^2}$.

(77) $\langle\binom{2\cdot n+1}{0},\ldots,\binom{2\cdot n+1}{2\cdot n+1}\rangle(n+1) = \langle\binom{2\cdot n+1}{0},\ldots,\binom{2\cdot n+1}{2\cdot n+1}\rangle(n+2)$. The theorem is a consequence of (75).

(78) Let us consider a non zero integer a.
If $1 \leqslant k \leqslant n$, then $a \mid ((a, b) \operatorname{Subnomial} n)(k)$.

(79) Let us consider integers a, b. Then $a \cdot b \mid \langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(i) - ((a, b) \operatorname{Subnomial} n)(i)$. The theorem is a consequence of (45).

(80) Let us consider a $\mathbb{Z}$-valued finite sequence f, and an integer a. Suppose for every natural number n such that $n \in \operatorname{dom} f$ holds $a \mid f(n)$. Then $a \mid \sum f$.
PROOF: Reconsider $f_1 = f$ as a finite sequence of elements of $\mathbb{R}$. Reconsider $k = \min f_1$ as an integer. Reconsider $f_2 = f$ as a finite sequence of elements of $\mathbb{C}$. Reconsider $g = f_2 - k$ as a finite sequence of elements of $\mathbb{Z}$. Reconsider $l = |a|$ as a natural number. $a \mid k$ by [11, (12)]. If $m \in \operatorname{dom} g$, then $l \mid g(m)$ by [11, (10)], [9, (4), (13)]. $\sum(g + k) = \sum g + k \cdot \operatorname{len} g$. $\square$

(81) Let us consider integers a, b. Then $a \cdot b \cdot (a - b) \mid (a - b) \cdot (a + b)^n - (a^{n+1} - b^{n+1})$. The theorem is a consequence of (79), (80), and (51).

Let us consider non negative real numbers a, b.

(82) $\langle \binom{n}{0} a^0 b^n, \ldots, \binom{n}{n} a^n b^0 \rangle(i) \geqslant ((a, b) \operatorname{Subnomial} n)(i)$. The theorem is a consequence of (47) and (57).

(83) $(a + b)^n \geqslant \sum((a, b) \operatorname{Subnomial} n)$. The theorem is a consequence of (82) and (32).

Let us consider non negative real numbers a, b and a non zero natural number n. Now we state the propositions:

(84) $a^n + b^n \leqslant \sum((a, b) \operatorname{Subnomial} n)$. The theorem is a consequence of (48).

(85) $a \cdot (a + 2 \cdot b)^n + b^{n+1} \geqslant (a + b)^{n+1}$.

(86) $a \cdot (a + b)^n + (a + b) \cdot b^n \leqslant (a + b)^{n+1}$.

Let us consider positive real numbers a, b and a non zero natural number n. Now we state the propositions:

(87) $\sum((a, b) \operatorname{Subnomial}(n+1)) < \sum\langle \binom{n+1}{0} a^0 b^{n+1}, \ldots, \binom{n+1}{n+1} a^{n+1} b^0 \rangle$. The theorem is a consequence of (82), (57), and (39).

(88) $\sum((a + b, 0) \operatorname{Subnomial}(n + 1)) > \sum((a, b) \operatorname{Subnomial}(n + 1))$. The theorem is a consequence of (51) and (87).

(89) Let us consider real numbers a, b, and natural numbers n, i. Then $((a, b) \operatorname{Subnomial} n)(i) \leqslant ((|a|, |b|) \operatorname{Subnomial} n)(i)$. The theorem is a consequence of (45).

(90) Let us consider a real number a, a natural number n, and an odd natural number i. Then $((a, -a) \operatorname{Subnomial}(n + i))(i) = a^{n+i}$. The theorem is a consequence of (54) and (60).

(91) Let us consider a real number a, a natural number n, and a non zero natural number i. Then $((a, -a) \operatorname{Subnomial}(n + 2 \cdot i))(2 \cdot i) = -a^{n+2 \cdot i}$. The

theorem is a consequence of (54) and (60).

Let us consider real numbers a, b and a natural number n. Now we state the propositions:

(92) $(a, b)\,\mathrm{Subnomial}(n+1) = \langle a^{n+1} \rangle \frown (b \cdot ((a, b)\,\mathrm{Subnomial}\, n))$.

Proof: $\mathrm{dom}((a, b)\,\mathrm{Subnomial}(n+1)) = \mathrm{dom}(\langle a^{n+1} \rangle \frown (b \cdot ((a, b)\,\mathrm{Subnomial}\, n)))$ by [13, (29)], [2, (22)]. For every object i such that $i \in \mathrm{dom}((a, b)\,\mathrm{Subnomial}(n+1))$ holds $((a, b)\,\mathrm{Subnomial}(n+1))(i) = (\langle a^{n+1} \rangle \frown (b \cdot ((a, b)\,\mathrm{Subnomial}\, n)))(i)$ by [13, (25)], [1, (10), (13)], [2, (65)]. $\square$

(93) $(a, b)\,\mathrm{Subnomial}(n+2) = (\langle a^{n+2} \rangle \frown (a \cdot b \cdot ((a, b)\,\mathrm{Subnomial}\, n))) \frown \langle b^{n+2} \rangle$.

The theorem is a consequence of (92), (44), (64), (43), and (42).

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