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# The Basic Existence Theorem of Riemann-Stieltjes Integral

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**Summary.** In this article, the basic existence theorem of Riemann-Stieltjes integral is formalized. This theorem states that if $f$ is a continuous function and $\rho$ is a function of bounded variation in a closed interval of real line, $f$ is Riemann-Stieltjes integrable with respect to $\rho$. In the first section, basic properties of real finite sequences are formalized as preliminaries. In the second section, we formalized the existence theorem of the Riemann-Stieltjes integral. These formalizations are based on [15], [12], [10], and [11].

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## 1. Preliminaries

Now we state the propositions:

(1) Let us consider a real number $E$, a finite sequence $q$ of elements of $\mathbb{R}$, and a finite sequence $S$ of elements of $\mathbb{R}$. Suppose $\operatorname{len} S = \operatorname{len} q$ and for every natural number $i$ such that $i \in \operatorname{dom} S$ there exists a real number $r$ such that $r = q(i)$ and $S(i) = r \cdot E$. Then $\sum S = \sum q \cdot E$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence $q$ of elements of $\mathbb{R}$ for every finite sequence $S$ of elements of $\mathbb{R}$ such that $\$_1 = \operatorname{len} S$




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and $\operatorname{len} S = \operatorname{len} q$ and for every natural number $i$ such that $i \in \operatorname{dom} S$ there exists a real number $r$ such that $r = q(i)$ and $S(i) = r \cdot E$ holds $\sum S = \sum q \cdot E$. $\mathcal{P}[0]$ by [7, (72)]. For every natural number $i$, $\mathcal{P}[i]$ from [1, Sch. 2]. $\square$

(2) Let us consider finite sequences $x$, $y$ of elements of $\mathbb{R}$. Suppose $\operatorname{len} x = \operatorname{len} y$ and for every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{dom} x$ there exists a real number $v$ such that $v = x(i)$ and $y(i) = |v|$. Then $|\sum x| \leqslant \sum y$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequences $x$, $y$ of elements of $\mathbb{R}$ such that $\$_1 = \operatorname{len} x$ and $\operatorname{len} x = \operatorname{len} y$ and for every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{dom} x$ there exists a real number $v$ such that $v = x(i)$ and $y(i) = |v|$ holds $|\sum x| \leqslant \sum y$. $\mathcal{P}[0]$ by [7, (72)], [3, (44)]. For every natural number $i$, $\mathcal{P}[i]$ from [1, Sch. 2]. $\square$

(3) Let us consider finite sequences $p$, $q$ of elements of $\mathbb{R}$. Suppose $\operatorname{len} p = \operatorname{len} q$ and for every natural number $j$ such that $j \in \operatorname{dom} p$ holds $|p(j)| \leqslant q(j)$. Then $|\sum p| \leqslant \sum q$.

Proof: Define $\mathcal{P}[\text{natural number}, \text{set}] \equiv$ there exists a real number $v$ such that $v = p(\$_1)$ and $\$_2 = |v|$. For every natural number $i$ such that $i \in \operatorname{Seg} \operatorname{len} p$ there exists an element $x$ of $\mathbb{R}$ such that $\mathcal{P}[i, x]$. Consider $u$ being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} u = \operatorname{Seg} \operatorname{len} p$ and for every natural number $i$ such that $i \in \operatorname{Seg} \operatorname{len} p$ holds $\mathcal{P}[i, u(i)]$ from [2, Sch. 5]. For every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{dom} p$ there exists a real number $v$ such that $v = p(i)$ and $u(i) = |v|$. $|\sum p| \leqslant \sum u$. $\square$

(4) Let us consider a natural number $n$, and an object $a$. Then $\operatorname{len}(n \mapsto a) = n$.

(5) Let us consider a finite sequence $p$, and an object $a$. Then $p = \operatorname{len} p \mapsto a$ if and only if for every object $k$ such that $k \in \operatorname{dom} p$ holds $p(k) = a$.

Proof: If $p = \operatorname{len} p \mapsto a$, then for every object $k$ such that $k \in \operatorname{dom} p$ holds $p(k) = a$ by [4, (57)]. $\square$

(6) Let us consider a finite sequence $p$ of elements of $\mathbb{R}$, a natural number $i$, and a real number $r$. Suppose $i \in \operatorname{dom} p$ and $p(i) = r$ and for every natural number $k$ such that $k \in \operatorname{dom} p$ and $k \neq i$ holds $p(k) = 0$. Then $\sum p = r$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence $p$ of elements of $\mathbb{R}$ for every natural number $i$ for every real number $r$ such that $\operatorname{len} p = \$_1$ and $i \in \operatorname{dom} p$ and $p(i) = r$ and for every natural number $k$ such that $k \in \operatorname{dom} p$ and $k \neq i$ holds $p(k) = 0$ holds $\sum p = r$. $\mathcal{P}[0]$. For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [4, (19), (16)], [18, (25)], [17, (7)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

(7) Let us consider finite sequences $p$, $q$ of elements of $\mathbb{R}$. Suppose $\operatorname{len} p \leqslant$

$\operatorname{len} q$ and for every natural number $i$ such that $i \in \operatorname{dom} q$ holds if $i \leqslant \operatorname{len} p$, then $q(i) = p(i)$ and if $\operatorname{len} p < i$, then $q(i) = 0$. Then $\sum q = \sum p$.

Proof: Consider $i_1$ being a natural number such that $i_1 = \operatorname{len} q - \operatorname{len} p$. Set $x = i_1 \mapsto (0$ **qua** real number). $q = p \frown x$ by (4), [18, (25)], [16, (13)], [4, (57)]. $\square$

(8) Let us consider real numbers $a$, $b$, $c$, $d$. If $b \leqslant c$, then $[a, b] \cap [c, d] \subseteq [b, b]$.

(9) Let us consider a real number $a$, a subset $A$ of $\mathbb{R}$, and a real-valued function $\varrho$. If $A \subseteq [a, a]$, then $\operatorname{vol}(A, \varrho) = 0$.

(10) Let us consider a non empty, increasing finite sequence $s$ of elements of $\mathbb{R}$, and a natural number $m$. Suppose $m \in \operatorname{dom} s$. Then $s \restriction m$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$.

Proof: Set $H = s \restriction m$. For every extended reals $e_1$, $e_2$ such that $e_1, e_2 \in \operatorname{dom} H$ and $e_1 < e_2$ holds $H(e_1) < H(e_2)$ by [19, (57)], [5, (47)]. $\square$

(11) Let us consider non empty, increasing finite sequences $s$, $t$ of elements of $\mathbb{R}$. Suppose $s(\operatorname{len} s) < t(1)$. Then $s \frown t$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$.

Proof: Set $H = s \frown t$. For every extended reals $e_1$, $e_2$ such that $e_1$, $e_2 \in \operatorname{dom} H$ and $e_1 < e_2$ holds $H(e_1) < H(e_2)$ by [18, (25)], [2, (25), (3)]. $\square$

(12) Let us consider a non empty, increasing finite sequence $s$ of elements of $\mathbb{R}$, and a real number $a$. Suppose $s(\operatorname{len} s) < a$. Then $s \frown \langle a \rangle$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$. The theorem is a consequence of (11).

(13) Let us consider a finite sequence $T$ of elements of $\mathbb{R}$, and natural numbers $n$, $m$. Suppose $n + 1 < m \leqslant \operatorname{len} T$. Then there exists a finite sequence $T_1$ of elements of $\mathbb{R}$ such that

(i) $\operatorname{len} T_1 = m - (n + 1)$, and

(ii) $\operatorname{rng} T_1 \subseteq \operatorname{rng} T$, and

(iii) for every natural number $i$ such that $i \in \operatorname{dom} T_1$ holds $T_1(i) = T(i + n)$.

Proof: Define $\mathcal{F}(\text{natural number}) = T(\$_1 + n)$. Reconsider $m_1 = m - (n + 1)$ as a natural number. Consider $p$ being a finite sequence such that $\operatorname{len} p = m_1$ and for every natural number $k$ such that $k \in \operatorname{dom} p$ holds $p(k) = \mathcal{F}(k)$ from [2, Sch. 2]. $\operatorname{rng} p \subseteq \operatorname{rng} T$ by [18, (25)], [5, (3)]. $\square$

(14) Let us consider a non empty, increasing finite sequence $T$ of elements of $\mathbb{R}$, and natural numbers $n$, $m$. Suppose $n + 1 < m \leqslant \operatorname{len} T$. Then there exists a non empty, increasing finite sequence $T_1$ of elements of $\mathbb{R}$ such that

(i) $\operatorname{len} T_1 = m - (n+1)$, and

(ii) $\operatorname{rng} T_1 \subseteq \operatorname{rng} T$, and

(iii) for every natural number $i$ such that $i \in \operatorname{dom} T_1$ holds $T_1(i) = T(i + n)$.

PROOF: Consider $p$ being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{len} p = m-(n+1)$ and $\operatorname{rng} p \subseteq \operatorname{rng} T$ and for every natural number $i$ such that $i \in \operatorname{dom} p$ holds $p(i) = T(i+n)$. For every extended reals $e_1$, $e_2$ such that $e_1$, $e_2 \in \operatorname{dom} p$ and $e_1 < e_2$ holds $p(e_1) < p(e_2)$ by [18, (25)]. □

(15) Let us consider a finite sequence $p$ of elements of $\mathbb{R}$, and natural numbers $n$, $m$. Suppose $n + 1 < m \leqslant \operatorname{len} p$. Then there exists a finite sequence $p_1$ of elements of $\mathbb{R}$ such that

(i) $\operatorname{len} p_1 = m - (n+1) - 1$, and

(ii) $\operatorname{rng} p_1 \subseteq \operatorname{rng} p$, and

(iii) for every natural number $i$ such that $i \in \operatorname{dom} p_1$ holds $p_1(i) = p(i + n + 1)$.

The theorem is a consequence of (13).

## 2. Existence of Riemann-Stieltjes Integral for Continuous Functions

Now we state the propositions:

(16) Let us consider a non empty, closed interval subset $A$ of $\mathbb{R}$, a partition $T$ of $A$, a real-valued function $\varrho$, a non empty, closed interval subset $B$ of $\mathbb{R}$, a non empty, increasing finite sequence $S_0$ of elements of $\mathbb{R}$, and a finite sequence $S_1$ of elements of $\mathbb{R}$.
Suppose $B \subseteq A$ and $\inf B = \inf A$ and there exists a partition $S$ of $B$ such that $S = S_0$ and $\operatorname{len} S_1 = \operatorname{len} S$ and for every natural number $j$ such that $j \in \operatorname{dom} S$ there exists a finite sequence $p$ of elements of $\mathbb{R}$ such that $S_1(j) = \sum p$ and $\operatorname{len} p = \operatorname{len} T$ and for every natural number $i$ such that $i \in \operatorname{dom} T$ holds $p(i) = |\operatorname{vol}(\operatorname{divset}(T, i) \cap \operatorname{divset}(S, j), \varrho)|$.
Then there exists a partition $H$ of $B$ and there exists a var-volume $F$ of $\varrho$ and $H$ such that $\sum S_1 = \sum F$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non empty, closed interval subset $B$ of $\mathbb{R}$ for every non empty, increasing finite sequence $S_0$ of elements of $\mathbb{R}$ for every finite sequence $S_1$ of elements of $\mathbb{R}$ such that $B \subseteq A$ and $\inf B = \inf A$ and $\operatorname{len} S_0 = \$_1$ and there exists a partition $S$ of $B$ such that $S = S_0$ and $\operatorname{len} S_1 = \operatorname{len} S$ and for every natural number $j$ such

that $j \in \operatorname{dom} S$ there exists a finite sequence $p$ of elements of $\mathbb{R}$ such that $S_1(j) = \sum p$ and $\operatorname{len} p = \operatorname{len} T$ and for every natural number $i$ such that $i \in \operatorname{dom} T$ holds $p(i) = |\operatorname{vol}(\operatorname{divset}(T, i) \cap \operatorname{divset}(S, j), \varrho)|$ there exists a partition $H$ of $B$ and there exists a var-volume $F$ of $\varrho$ and $H$ such that $\sum S_1 = \sum F$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [18, (29)], [1, (14)], [18, (25)], [2, (40)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(17) Let us consider a non empty, closed interval subset $A$ of $\mathbb{R}$, a function $\varrho$ from $A$ into $\mathbb{R}$, and partitions $T$, $S$ of $A$. Suppose $\varrho$ is bounded-variation. Then there exists a finite sequence $S_1$ of elements of $\mathbb{R}$ such that

(i) $\operatorname{len} S_1 = \operatorname{len} S$, and

(ii) $\sum S_1 \leqslant \operatorname{TotalVD}(\varrho)$, and

(iii) for every natural number $j$ such that $j \in \operatorname{dom} S$ there exists a finite sequence $p$ of elements of $\mathbb{R}$ such that $S_1(j) = \sum p$ and $\operatorname{len} p = \operatorname{len} T$ and for every natural number $i$ such that $i \in \operatorname{dom} T$ holds $p(i) = |\operatorname{vol}(\operatorname{divset}(T, i) \cap \operatorname{divset}(S, j), \varrho)|$.

PROOF: Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv$ there exists a finite sequence $p$ of elements of $\mathbb{R}$ such that $\$_2 = \sum p$ and $\operatorname{len} p = \operatorname{len} T$ and for every natural number $i$ such that $i \in \operatorname{dom} T$ holds $p(i) = |\operatorname{vol}(\operatorname{divset}(T, i) \cap \operatorname{divset}(S, \$_1), \varrho)|$. For every natural number $j$ such that $j \in \operatorname{Seg} \operatorname{len} S$ there exists an element $x$ of $\mathbb{R}$ such that $\mathcal{P}[j, x]$. Consider $S_1$ being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} S_1 = \operatorname{Seg} \operatorname{len} S$ and for every natural number $j$ such that $j \in \operatorname{Seg} \operatorname{len} S$ holds $\mathcal{P}[j, S_1(j)]$ from [2, Sch. 5]. Consider $H$ being a partition of $A$, $F$ being a var-volume of $\varrho$ and $H$ such that $\sum S_1 = \sum F$. □

(18) Let us consider a non empty, closed interval subset $A$ of $\mathbb{R}$, a function $\varrho$ from $A$ into $\mathbb{R}$, and a partial function $u$ from $\mathbb{R}$ to $\mathbb{R}$.
Suppose $\varrho$ is bounded-variation and $\operatorname{dom} u = A$ and $u \restriction A$ is uniformly continuous. Let us consider a division sequence $T$ of $A$, and a middle volume sequence $S$ of $\varrho$, $u$ and $T$. Suppose $\delta_T$ is convergent and $\lim \delta_T = 0$. Then middle-sum$(S)$ is convergent.

PROOF: For every division sequence $T$ of $A$ and for every middle volume sequence $S$ of $\varrho$, $u$ and $T$ such that $\delta_T$ is convergent and $\lim \delta_T = 0$ holds middle-sum$(S)$ is convergent by [14, (6)], [9, (9)], [8, (87)], [6, (5)]. □

(19) Let us consider a non empty, closed interval subset $A$ of $\mathbb{R}$, a function $\varrho$ from $A$ into $\mathbb{R}$, a partial function $u$ from $\mathbb{R}$ to $\mathbb{R}$, division sequences $T_0$, $T$, $T_1$ of $A$, a middle volume sequence $S_0$ of $\varrho$, $u$ and $T_0$, and a middle volume sequence $S$ of $\varrho$, $u$ and $T$.

Suppose for every natural number $i$, $T_1(2\cdot i) = T_0(i)$ and $T_1(2\cdot i+1) = T(i)$. Then there exists a middle volume sequence $S_1$ of $\varrho$, $u$ and $T_1$ such that for every natural number $i$, $S_1(2 \cdot i) = S_0(i)$ and $S_1(2 \cdot i + 1) = S(i)$.

PROOF: Reconsider $S_3 = S_0$, $S_2 = S$ as a sequence of $\mathbb{R}^*$. Define $\mathcal{F}$(natural number) $= S_3(\$_1)(\in \mathbb{R}^*)$. Define $\mathcal{G}$(natural number) $= S_2(\$_1)(\in \mathbb{R}^*)$. Consider $S_1$ being a sequence of $\mathbb{R}^*$ such that for every natural number $n$, $S_1(2 \cdot n) = \mathcal{F}(n)$ and $S_1(2 \cdot n + 1) = \mathcal{G}(n)$ from [13, Sch. 1]. For every element $i$ of $\mathbb{N}$, $S_1(i)$ is a middle volume of $\varrho$, $u$ and $T_1(i)$ by [13, (14)], [6, (5)]. $\square$

(20) Let us consider sequences $S_1$, $S_2$, $S_3$ of real numbers. Suppose $S_3$ is convergent and for every natural number $i$, $S_3(2 \cdot i) = S_1(i)$ and $S_3(2 \cdot i + 1) = S_2(i)$. Then

(i) $S_1$ is convergent, and

(ii) $\lim S_1 = \lim S_3$, and

(iii) $S_2$ is convergent, and

(iv) $\lim S_2 = \lim S_3$.

PROOF: For every real number $r$ such that $0 < r$ there exists a natural number $m_1$ such that for every natural number $i$ such that $m_1 \leqslant i$ holds $|S_1(i) - \lim S_3| < r$ by [13, (14)], [1, (11)]. For every real number $r$ such that $0 < r$ there exists a natural number $m_1$ such that for every natural number $i$ such that $m_1 \leqslant i$ holds $|S_2(i) - \lim S_3| < r$ by [13, (14)], [1, (11)]. $\square$

(21) Let us consider a non empty, closed interval subset $A$ of $\mathbb{R}$, a function $\varrho$ from $A$ into $\mathbb{R}$, and a continuous partial function $u$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\varrho$ is bounded-variation and $\operatorname{dom} u = A$. Then $u$ is Riemann-Stieltjes integrable with $\varrho$.

PROOF: Consider $T_0$ being a division sequence of $A$ such that $\delta_{T_0}$ is convergent and $\lim \delta_{T_0} = 0$. Set $S_0 =$ the middle volume sequence of $\varrho$, $u$ and $T_0$. Set $I = \lim \text{middle-sum}(S_0)$. For every division sequence $T$ of $A$ and for every middle volume sequence $S$ of $\varrho$, $u$ and $T$ such that $\delta_T$ is convergent and $\lim \delta_T = 0$ holds middle-sum$(S)$ is convergent and $\lim \text{middle-sum}(S) = I$ by (18), [13, (15)], (19), [13, (16)]. $\square$

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