FORMALIZED MATHEMATICS
Vol. 23, No. 3, Pages 253–277, 2015
DOI: 10.1515/forma-2015-0021

DE GRUYTER OPEN
degruyter.com/view/j/forma

# Extended Real-Valued Double Sequence and Its Convergence[1]

Noboru Endou
Gifu National College of Technology
Gifu, Japan

**Summary.** In this article we introduce the convergence of extended real-valued double sequences [16], [17]. It is similar to our previous articles [15], [10]. In addition, we also prove Fatou's lemma and the monotone convergence theorem for double sequences.

MSC: 40A05 40B05 03B35

Keywords: double sequence; Fatou's lemma for double sequence; monotone convergence theorem for double sequence

MML identifier: DBLSEQ_3, version: 8.1.04 5.32.1246

The notation and terminology used in this paper have been introduced in the following articles: [5], [21], [15], [10], [12], [6], [7], [22], [13], [11], [14], [1], [2], [8], [18], [24], [25], [26], [20], [23], [3], [4], and [9].

## 1. Preliminaries

Let $X$ be a non empty set. One can verify that there exists a function from $X$ into $\mathbb{R}$ which is non-negative and non-positive and there exists a function from $X$ into $\overline{\mathbb{R}}$ which is without $-\infty$, without $+\infty$, non-negative, and non-positive and every function from $X$ into $\overline{\mathbb{R}}$ which is non-negative is also without $-\infty$ and every function from $X$ into $\overline{\mathbb{R}}$ which is non-positive is also without $+\infty$ and there exists a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$ which is without $-\infty$.

Let $f$ be a function from $X$ into $\overline{\mathbb{R}}$. Let us observe that the functor $-f$ yields a function from $X$ into $\overline{\mathbb{R}}$. Let $f$ be a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$. Note that $-f$ is without $+\infty$.

[1]This work was supported by JSPS KAKENHI 23500029.


ISSN 1426–2630(Print), 1898-9934(Online)

Let $f$ be a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let us observe that $-f$ is without $-\infty$.

Let $f$ be a non-negative function from $X$ into $\overline{\mathbb{R}}$. Note that $-f$ is non-positive.

Let $f$ be a non-positive function from $X$ into $\overline{\mathbb{R}}$. Let us observe that $-f$ is non-negative.

Let $A$, $B$ be non empty sets and $f$ be a without $-\infty$ function from $A \times B$ into $\overline{\mathbb{R}}$. Let us observe that $f^{\mathrm{T}}$ is without $-\infty$.

Let $f$ be a without $+\infty$ function from $A \times B$ into $\overline{\mathbb{R}}$. One can verify that $f^{\mathrm{T}}$ is without $+\infty$.

Let $f$ be a non-negative function from $A \times B$ into $\overline{\mathbb{R}}$. One can check that $f^{\mathrm{T}}$ is non-negative.

Let $f$ be a non-positive function from $A \times B$ into $\overline{\mathbb{R}}$. Note that $f^{\mathrm{T}}$ is non-positive.

Now we state the propositions:

(1) Let us consider a sequence $s$ of extended reals. Then $(\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}} = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$.

PROOF: Define $\mathcal{Q}[\text{natural number}] \equiv (-(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})(\$_1) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number $n$, $\mathcal{Q}[n]$ from [1, Sch. 2]. Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}}(\$_1) = (-(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})(\$_1)$. For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

(2) Let us consider a non empty set $X$, and a partial function $f$ from $X$ to $\overline{\mathbb{R}}$. Then $--f = f$.

(3) Let us consider non empty sets $X$, $Y$, and a function $f$ from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(-f)^{\mathrm{T}} = -f^{\mathrm{T}}$.

Let $s$ be a non-negative sequence of extended reals. One can verify that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is non-negative.

Let $s$ be a non-positive sequence of extended reals. Let us observe that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is non-positive.

Now we state the propositions:

(4) Let us consider a non-negative sequence $s$ of extended reals, and a natural number $m$. Then $s(m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv s(\$_1) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

(5) Let us consider a non-positive sequence $s$ of extended reals, and a natural number $m$. Then $s(m) \geqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m)$. The theorem is a consequence of (4), (1), and (2).

(6) Let us consider a non empty set $X$. Then every without $-\infty$, without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$ is a function from $X$ into $\mathbb{R}$.

Let $X$ be a non empty set and $f_1$, $f_2$ be without $-\infty$ functions from $X$ into $\overline{\mathbb{R}}$. One can verify that the functor $f_1 + f_2$ yields a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let $f_1$, $f_2$ be without $+\infty$ functions from $X$ into $\overline{\mathbb{R}}$. One can verify that the functor $f_1 + f_2$ yields a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let $f_1$ be a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$ and $f_2$ be a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let us observe that the functor $f_1 - f_2$ yields a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$. Let $f_1$ be a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$ and $f_2$ be a without $-\infty$ function from $X$ into $\overline{\mathbb{R}}$. Observe that the functor $f_1 - f_2$ yields a without $+\infty$ function from $X$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(7) Let us consider a non empty set $X$, an element $x$ of $X$, and functions $f_1$, $f_2$ from $X$ into $\overline{\mathbb{R}}$. Then

- (i) if $f_1$ is without $-\infty$ and $f_2$ is without $-\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and
- (ii) if $f_1$ is without $+\infty$ and $f_2$ is without $+\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and
- (iii) if $f_1$ is without $-\infty$ and $f_2$ is without $+\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$, and
- (iv) if $f_1$ is without $+\infty$ and $f_2$ is without $-\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$.

(8) Let us consider a non empty set $X$, and without $-\infty$ functions $f_1$, $f_2$ from $X$ into $\overline{\mathbb{R}}$. Then

- (i) $f_1 + f_2 = f_1 - -f_2$, and
- (ii) $-(f_1 + f_2) = -f_1 - f_2$.

The theorem is a consequence of (7).

(9) Let us consider a non empty set $X$, and without $+\infty$ functions $f_1$, $f_2$ from $X$ into $\overline{\mathbb{R}}$. Then

- (i) $f_1 + f_2 = f_1 - -f_2$, and
- (ii) $-(f_1 + f_2) = -f_1 - f_2$.

The theorem is a consequence of (7).

(10) Let us consider a non empty set $X$, a without $-\infty$ function $f_1$ from $X$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function $f_2$ from $X$ into $\overline{\mathbb{R}}$. Then

- (i) $f_1 - f_2 = f_1 + -f_2$, and
- (ii) $f_2 - f_1 = f_2 + -f_1$, and
- (iii) $-(f_1 - f_2) = -f_1 + f_2$, and

(iv) $-(f_2 - f_1) = -f_2 + f_1$.

The theorem is a consequence of (8), (2), and (9).

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and $n$, $m$ be natural numbers. One can check that the functor $f(n, m)$ yields an element of $\overline{\mathbb{R}}$. Now we state the propositions:

(11) Let us consider without $-\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$. The theorem is a consequence of (7).

(12) Let us consider without $+\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$. The theorem is a consequence of (7).

(13) Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a without $+\infty$ function $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) $(f_1 - f_2)(n, m) = f_1(n, m) - f_2(n, m)$, and

(ii) $(f_2 - f_1)(n, m) = f_2(n, m) - f_1(n, m)$.

The theorem is a consequence of (7).

(14) Let us consider non empty sets $X$, $Y$, and without $-\infty$ functions $f_1$, $f_2$ from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(f_1 + f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} + {f_2}^{\mathrm{T}}$. The theorem is a consequence of (7).

(15) Let us consider non empty sets $X$, $Y$, and without $+\infty$ functions $f_1$, $f_2$ from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(f_1 + f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} + {f_2}^{\mathrm{T}}$. The theorem is a consequence of (7).

(16) Let us consider non empty sets $X$, $Y$, a without $-\infty$ function $f_1$ from $X \times Y$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function $f_2$ from $X \times Y$ into $\overline{\mathbb{R}}$. Then

(i) $(f_1 - f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} - {f_2}^{\mathrm{T}}$, and

(ii) $(f_2 - f_1)^{\mathrm{T}} = {f_2}^{\mathrm{T}} - {f_1}^{\mathrm{T}}$.

The theorem is a consequence of (7).

One can verify that every sequence of extended reals which is convergent to $+\infty$ is also convergent and every sequence of extended reals which is convergent to $-\infty$ is also convergent and every sequence of extended reals which is convergent to a finite limit is also convergent and there exists a sequence of extended reals which is convergent and there exists a without $-\infty$ sequence of extended reals which is convergent and there exists a without $+\infty$ sequence of extended reals which is convergent.

Now we state the proposition:

(17) Let us consider a convergent sequence $s$ of extended reals. Then

(i) $s$ is convergent to a finite limit iff $-s$ is convergent to a finite limit, and

(ii) $s$ is convergent to $+\infty$ iff $-s$ is convergent to $-\infty$, and

(iii) $s$ is convergent to $-\infty$ iff $-s$ is convergent to $+\infty$, and

(iv) $-s$ is convergent, and

(v) $\lim(-s) = -\lim s$.

The theorem is a consequence of (2).

Let us consider without $-\infty$ sequences $s_1$, $s_2$ of extended reals. Now we state the propositions:

(18) Suppose $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to $+\infty$. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

(19) Suppose $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

Now we state the proposition:

(20) Let us consider without $+\infty$ sequences $s_1$, $s_2$ of extended reals. Suppose $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

Let us consider without $-\infty$ sequences $s_1$, $s_2$ of extended reals. Now we state the propositions:

(21) Suppose $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to $-\infty$. Then

(i) $s_1 + s_2$ is convergent to $-\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = -\infty$.

The theorem is a consequence of (7).

(22) Suppose $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $-\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = -\infty$.

The theorem is a consequence of (7).

(23) Suppose $s_1$ is convergent to a finite limit and $s_2$ is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to a finite limit and convergent, and

(ii) $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$.

The theorem is a consequence of (7).

Now we state the propositions:

(24) Let us consider without $+\infty$ sequences $s_1$, $s_2$ of extended reals. Then

(i) if $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to $+\infty$, then $s_1 + s_2$ is convergent to $+\infty$ and convergent and $\lim(s_1 + s_2) = +\infty$, and

(ii) if $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to a finite limit, then $s_1 + s_2$ is convergent to $+\infty$ and convergent and $\lim(s_1 + s_2) = +\infty$, and

(iii) if $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to $-\infty$, then $s_1 + s_2$ is convergent to $-\infty$ and convergent and $\lim(s_1 + s_2) = -\infty$, and

(iv) if $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to a finite limit, then $s_1 + s_2$ is convergent to $-\infty$ and convergent and $\lim(s_1 + s_2) = -\infty$, and

(v) if $s_1$ is convergent to a finite limit and $s_2$ is convergent to a finite limit, then $s_1 + s_2$ is convergent to a finite limit and convergent and $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$.

The theorem is a consequence of (17), (21), (10), (9), (2), (22), (18), (19), and (23).

(25) Let us consider a without $-\infty$ sequence $s_1$ of extended reals, and a without $+\infty$ sequence $s_2$ of extended reals. Then

(i) if $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to $-\infty$, then $s_1 - s_2$ is convergent to $+\infty$ and convergent and $s_2 - s_1$ is convergent to $-\infty$ and convergent and $\lim(s_1 - s_2) = +\infty$ and $\lim(s_2 - s_1) = -\infty$, and

(ii) if $s_1$ is convergent to $+\infty$ and $s_2$ is convergent to a finite limit, then $s_1 - s_2$ is convergent to $+\infty$ and convergent and $s_2 - s_1$ is convergent to $-\infty$ and convergent and $\lim(s_1 - s_2) = +\infty$ and $\lim(s_2 - s_1) = -\infty$, and

(iii) if $s_1$ is convergent to $-\infty$ and $s_2$ is convergent to a finite limit, then $s_1 - s_2$ is convergent to $-\infty$ and convergent and $s_2 - s_1$ is convergent to $+\infty$ and convergent and $\lim(s_1 - s_2) = -\infty$ and $\lim(s_2 - s_1) = +\infty$, and

(iv) if $s_1$ is convergent to a finite limit and $s_2$ is convergent to a finite limit, then $s_1 - s_2$ is convergent to a finite limit and convergent and $s_2 - s_1$ is convergent to a finite limit and convergent and $\lim(s_1 - s_2) = \lim s_1 - \lim s_2$ and $\lim(s_2 - s_1) = \lim s_2 - \lim s_1$.

The theorem is a consequence of (17), (24), (18), (10), (19), (22), (23), and (2).

## 2. Subsequences of Convergent Extended Real-Valued Sequences

Let us consider sequences $s_1$, $s_2$ of extended reals. Now we state the propositions:

(26) Suppose $s_2$ is a subsequence of $s_1$ and $s_1$ is convergent to a finite limit. Then

(i) $s_2$ is convergent to a finite limit, and

(ii) $\lim s_1 = \lim s_2$.

PROOF: Consider $g$ being a real number such that $\lim s_1 = g$ and for every real number $p$ such that $0 < p$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $|s_1(m) - \lim s_1| < p$ and $s_1$ is convergent to a finite limit. Reconsider $L = \lim s_1$ as an extended real number. There exists a real number $g$ such that for every real number $p$ such that $0 < p$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $|(s_2(m) - g$ **qua** extended real$)| < p$ by [19, (14)], [7, (15)]. For every real number $p$ such that $0 < p$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $|s_2(m) - L| < p$ by [19, (14)], [7, (15)]. □

(27) Suppose $s_2$ is a subsequence of $s_1$ and $s_1$ is convergent to $+\infty$. Then

(i) $s_2$ is convergent to $+\infty$, and

(ii) $\lim s_2 = +\infty$.

(28) Suppose $s_2$ is a subsequence of $s_1$ and $s_1$ is convergent to $-\infty$. Then

(i) $s_2$ is convergent to $-\infty$, and

(ii) $\lim s_2 = -\infty$.

## 3. Convergency for Extended Real-Valued Double Sequences

Let us consider a function $R$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Now we state the propositions:

(29) Suppose the lim in the first coordinate of $R$ is convergent. Then the first coordinate major iterated lim of $R =$ lim(the lim in the first coordinate of $R$).

(30) Suppose the lim in the second coordinate of $R$ is convergent. Then the second coordinate major iterated lim of $R =$ lim(the lim in the second coordinate of $R$).

Let $E$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that $E$ is P-convergent to a finite limit if and only if

(Def. 1) there exists a real number $p$ such that for every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|E(n,m) - (p$ **qua** extended real$)| < e$.

We say that $E$ is P-convergent to $+\infty$ if and only if

(Def. 2) for every real number $g$ such that $0 < g$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $g \leqslant E(n,m)$.

We say that $E$ is P-convergent to $-\infty$ if and only if

(Def. 3) for every real number $g$ such that $g < 0$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $E(n,m) \leqslant g$.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that $f$ is convergent in the first coordinate to $+\infty$ if and only if

(Def. 4) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}'(f,m)$ is convergent to $+\infty$.

We say that $f$ is convergent in the first coordinate to $-\infty$ if and only if

(Def. 5) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}'(f,m)$ is convergent to $-\infty$.

We say that $f$ is convergent in the first coordinate to a finite limit if and only if

(Def. 6) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}'(f,m)$ is convergent to a finite limit.

We say that $f$ is convergent in the first coordinate if and only if

(Def. 7) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}'(f,m)$ is convergent.

We say that $f$ is convergent in the second coordinate to $+\infty$ if and only if

(Def. 8) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}(f,m)$ is convergent to $+\infty$.

We say that $f$ is convergent in the second coordinate to $-\infty$ if and only if

(Def. 9) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}(f,m)$ is convergent to $-\infty$.

We say that $f$ is convergent in the second coordinate to a finite limit if and only if

(Def. 10) for every element $m$ of $\mathbb{N}$, $\operatorname{curry}(f, m)$ is convergent to a finite limit.

We say that $f$ is convergent in the second coordinate if and only if

(Def. 11) for every element $m$ of $\mathbb{N}$, $\operatorname{curry}(f, m)$ is convergent.

Now we state the propositions:

(31) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) if $f$ is convergent in the first coordinate to $+\infty$ or convergent in the first coordinate to $-\infty$ or convergent in the first coordinate to a finite limit, then $f$ is convergent in the first coordinate, and

(ii) if $f$ is convergent in the second coordinate to $+\infty$ or convergent in the second coordinate to $-\infty$ or convergent in the second coordinate to a finite limit, then $f$ is convergent in the second coordinate.

(32) Let us consider non empty sets $X$, $Y$, $Z$, a function $F$ from $X \times Y$ into $Z$, and an element $x$ of $X$. Then $\operatorname{curry}(F, x) = \operatorname{curry}'(F^{\mathrm{T}}, x)$.

(33) Let us consider non empty sets $X$, $Y$, $Z$, a function $F$ from $X \times Y$ into $Z$, and an element $y$ of $Y$. Then $\operatorname{curry}'(F, y) = \operatorname{curry}(F^{\mathrm{T}}, y)$.

(34) Let us consider non empty sets $X$, $Y$, a function $F$ from $X \times Y$ into $\overline{\mathbb{R}}$, and an element $x$ of $X$. Then $\operatorname{curry}(-F, x) = -\operatorname{curry}(F, x)$.

(35) Let us consider non empty sets $X$, $Y$, a function $F$ from $X \times Y$ into $\overline{\mathbb{R}}$, and an element $y$ of $Y$. Then $\operatorname{curry}'(-F, y) = -\operatorname{curry}'(F, y)$.

Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(36) (i) $f$ is convergent in the first coordinate to $+\infty$ iff $f^{\mathrm{T}}$ is convergent in the second coordinate to $+\infty$, and

(ii) $f$ is convergent in the second coordinate to $+\infty$ iff $f^{\mathrm{T}}$ is convergent in the first coordinate to $+\infty$, and

(iii) $f$ is convergent in the first coordinate to $-\infty$ iff $f^{\mathrm{T}}$ is convergent in the second coordinate to $-\infty$, and

(iv) $f$ is convergent in the second coordinate to $-\infty$ iff $f^{\mathrm{T}}$ is convergent in the first coordinate to $-\infty$, and

(v) $f$ is convergent in the first coordinate to a finite limit iff $f^{\mathrm{T}}$ is convergent in the second coordinate to a finite limit, and

(vi) $f$ is convergent in the second coordinate to a finite limit iff $f^{\mathrm{T}}$ is convergent in the first coordinate to a finite limit.

The theorem is a consequence of (33) and (32).

(37) (i) $f$ is convergent in the first coordinate to $+\infty$ iff $-f$ is convergent in the first coordinate to $-\infty$, and

(ii) $f$ is convergent in the first coordinate to $-\infty$ iff $-f$ is convergent in the first coordinate to $+\infty$, and

(iii) $f$ is convergent in the first coordinate to a finite limit iff $-f$ is convergent in the first coordinate to a finite limit, and

(iv) $f$ is convergent in the second coordinate to $+\infty$ iff $-f$ is convergent in the second coordinate to $-\infty$, and

(v) $f$ is convergent in the second coordinate to $-\infty$ iff $-f$ is convergent in the second coordinate to $+\infty$, and

(vi) $f$ is convergent in the second coordinate to a finite limit iff $-f$ is convergent in the second coordinate to a finite limit.

The theorem is a consequence of (35), (17), (2), and (34).

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The functors: the lim in the first coordinate of $f$ and the lim in the second coordinate of $f$ yielding sequences of extended reals are defined by conditions

(Def. 12) for every element $m$ of $\mathbb{N}$, the lim in the first coordinate of $f(m) = \lim \mathrm{curry}'(f, m)$,

(Def. 13) for every element $n$ of $\mathbb{N}$, the lim in the second coordinate of $f(n) = \lim \mathrm{curry}(f, n)$,

respectively. Now we state the proposition:

(38) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the lim in the first coordinate of $f =$ the lim in the second coordinate of $f^{\mathrm{T}}$, and

(ii) the lim in the second coordinate of $f =$ the lim in the first coordinate of $f^{\mathrm{T}}$.

The theorem is a consequence of (33) and (32).

Let $X$, $Y$ be non empty sets, $F$ be a without $+\infty$ function from $X \times Y$ into $\overline{\mathbb{R}}$, and $x$ be an element of $X$. Let us observe that $\mathrm{curry}(F, x)$ is without $+\infty$.

Let $y$ be an element of $Y$. One can verify that $\mathrm{curry}'(F, y)$ is without $+\infty$.

Let $F$ be a without $-\infty$ function from $X \times Y$ into $\overline{\mathbb{R}}$ and $x$ be an element of $X$. Let us note that $\mathrm{curry}(F, x)$ is without $-\infty$.

Let $y$ be an element of $Y$. Observe that $\mathrm{curry}'(F, y)$ is without $-\infty$.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The partial sums in the second coordinate of $f$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by

(Def. 14) for every natural numbers $n$, $m$, $it(n, 0) = f(n, 0)$ and $it(n, m+1) = it(n, m) + f(n, m+1)$.

The partial sums in the first coordinate of $f$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by

(Def. 15) for every natural numbers $n$, $m$, $it(0, m) = f(0, m)$ and $it(n+1, m) = it(n, m) + f(n+1, m)$.

Let $f$ be a without $-\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us note that the partial sums in the second coordinate of $f$ is without $-\infty$.

Let $f$ be a without $+\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Observe that the partial sums in the second coordinate of $f$ is without $+\infty$.

Let $f$ be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us observe that the partial sums in the second coordinate of $f$ is non-negative as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let $f$ be a non-positive function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. One can check that the partial sums in the second coordinate of $f$ is non-positive as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let $f$ be a without $-\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us note that the partial sums in the first coordinate of $f$ is without $-\infty$.

Let $f$ be a without $+\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Observe that the partial sums in the first coordinate of $f$ is without $+\infty$.

Let $f$ be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us observe that the partial sums in the first coordinate of $f$ is non-negative as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let $f$ be a non-positive function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. One can check that the partial sums in the first coordinate of $f$ is non-positive as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The functor $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by the term

(Def. 16) the partial sums in the second coordinate of the partial sums in the first coordinate of $f$.

Now we state the propositions:

(39) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) (the partial sums in the first coordinate of $f$)$(n, m) =$ (the partial sums in the second coordinate of $f^{\mathrm{T}}$)$(m, n)$, and

(ii) (the partial sums in the second coordinate of $f$)$(n, m) =$ (the partial sums in the first coordinate of $f^{\mathrm{T}}$)$(m, n)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the first coordinate of $f$)$(\$_1, m) =$ (the partial sums in the second coordinate of $f^{\mathrm{T}}$)$(m, \$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f$)$(n, \$_1) =$ (the partial sums in the first coordinate of $f^{\mathrm{T}})(\$_1, n)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

(40) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) (the partial sums in the first coordinate of $f)^{\mathrm{T}}$ = the partial sums in the second coordinate of $f^{\mathrm{T}}$, and

(ii) (the partial sums in the second coordinate of $f)^{\mathrm{T}}$ = the partial sums in the first coordinate of $f^{\mathrm{T}}$.

The theorem is a consequence of (39).

(41) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, an extended real-valued function $g$, and a natural number $n$. Suppose for every natural number $k$, (the partial sums in the first coordinate of $f)(n, k) = g(k)$. Then

(i) for every natural number $k$, $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, k) = (\sum_{\alpha=0}^{\kappa} g(\alpha))_{\kappa \in \mathbb{N}}(k)$, and

(ii) (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}})(n) = \sum g$.

(42) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $-f$ = $-$(the partial sums in the second coordinate of $f$), and

(ii) the partial sums in the first coordinate of $-f$ = $-$(the partial sums in the first coordinate of $f$).

PROOF: For every element $z$ of $\mathbb{N}\times \mathbb{N}$, ($-$(the partial sums in the second coordinate of $f))(z)$ = (the partial sums in the second coordinate of $-f)(z)$ by [9, (87)]. For every element $z$ of $\mathbb{N} \times \mathbb{N}$, ($-$(the partial sums in the first coordinate of $f))(z)$ = (the partial sums in the first coordinate of $-f)(z)$ by [9, (87)]. □

(43) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and elements $m$, $n$ of $\mathbb{N}$. Then

(i) (the partial sums in the first coordinate of $f)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(m)$, and

(ii) (the partial sums in the second coordinate of $f)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the first coordinate of $f)(\$_1, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(m, \$_1) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. $\square$

(44) Let us consider without $-\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 + f_2 =$ (the partial sums in the second coordinate of $f_1$)+(the partial sums in the second coordinate of $f_2$), and

(ii) the partial sums in the first coordinate of $f_1 + f_2 =$ (the partial sums in the first coordinate of $f_1$) + (the partial sums in the first coordinate of $f_2$).

The theorem is a consequence of (11).

(45) Let us consider without $+\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 + f_2 =$ (the partial sums in the second coordinate of $f_1$)+(the partial sums in the second coordinate of $f_2$), and

(ii) the partial sums in the first coordinate of $f_1 + f_2 =$ (the partial sums in the first coordinate of $f_1$) + (the partial sums in the first coordinate of $f_2$).

The theorem is a consequence of (10), (9), (2), (42), (44), and (8).

(46) Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 - f_2 =$ (the partial sums in the second coordinate of $f_1$)−(the partial sums in the second coordinate of $f_2$), and

(ii) the partial sums in the first coordinate of $f_1 - f_2 =$ (the partial sums in the first coordinate of $f_1$) − (the partial sums in the first coordinate of $f_2$), and

(iii) the partial sums in the second coordinate of $f_2 - f_1 =$ (the partial sums in the second coordinate of $f_2$)−(the partial sums in the second coordinate of $f_1$), and

(iv) the partial sums in the first coordinate of $f_2 - f_1 =$ (the partial sums in the first coordinate of $f_2$) − (the partial sums in the first coordinate of $f_1$).

The theorem is a consequence of (10), (44), (42), and (45).

(47) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) =$ (the partial sums in the second coordinate of $f)(n+1, m) + (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of $f$)$(n, m+1)$ = (the partial sums in the first coordinate of $f$)$(n, m+1)$ + (the partial sums in the first coordinate of the partial sums in the second coordinate of $f$)$(n, m)$.

PROOF: Set $R_1 = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$. Set $C_1$ = the partial sums in the first coordinate of the partial sums in the second coordinate of $f$. Set $R_2$ = the partial sums in the first coordinate of $f$. Set $C_2$ = the partial sums in the second coordinate of $f$. Define $\mathcal{P}$[natural number] $\equiv R_1(n+1, \$_1) = C_2(n+1, \$_1) + R_1(n, \$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv C_1(\$_1, m+1) = R_2(\$_1, m+1) + C_1(\$_1, m)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

(48) Let us consider a without $+\infty$ function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

(i) $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}(n+1, m)$ = (the partial sums in the second coordinate of $f$)$(n+1, m) + (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}(n, m)$, and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of $f$)$(n, m+1)$ = (the partial sums in the first coordinate of $f$)$(n, m+1)$ + (the partial sums in the first coordinate of the partial sums in the second coordinate of $f$)$(n, m)$.

The theorem is a consequence of (2), (42), and (47).

(49) Let us consider a function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $f$ is without $-\infty$ or without $+\infty$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$ = the partial sums in the first coordinate of the partial sums in the second coordinate of $f$.

(50) Let us consider without $-\infty$ functions $f_1$, $f_2$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa}(f_1+f_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (44).

(51) Let us consider without $+\infty$ functions $f_1$, $f_2$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa}(f_1+f_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (45).

(52) Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function $f_2$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) $(\sum_{\alpha=0}^{\kappa}(f_1-f_2)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}}$, and

(ii) $(\sum_{\alpha=0}^{\kappa}(f_2-f_1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa\in\mathbb{N}}$.

The theorem is a consequence of (46).

(53) Let us consider a function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$, and an element $k$ of $\mathbb{N}$. Then

(i) curry′(the partial sums in the first coordinate of $f, k$) = $(\sum_{\alpha=0}^{\kappa}(\text{curry}'(f,k))(\alpha))_{\kappa\in\mathbb{N}}$, and

(ii) curry(the partial sums in the second coordinate of $f, k$) = $(\sum_{\alpha=0}^{\kappa}(\text{curry}(f,k))(\alpha))_{\kappa\in\mathbb{N}}$.

The theorem is a consequence of (43).

(54) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $f$ is without $-\infty$ or without $+\infty$. Then

(i) $\text{curry}((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}, 0)$ = curry(the partial sums in the second coordinate of $f, 0$), and

(ii) $\text{curry}'((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}, 0)$ = curry′(the partial sums in the first coordinate of $f, 0$).

(55) Let us consider non empty sets $C$, $D$, without $-\infty$ functions $F_1$, $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $c$ of $C$. Then $\text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c)$. The theorem is a consequence of (7).

(56) Let us consider non empty sets $C$, $D$, without $-\infty$ functions $F_1$, $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $d$ of $D$. Then $\text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d)$. The theorem is a consequence of (7).

(57) Let us consider non empty sets $C$, $D$, without $+\infty$ functions $F_1$, $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $c$ of $C$. Then $\text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c)$. The theorem is a consequence of (7).

(58) Let us consider non empty sets $C$, $D$, without $+\infty$ functions $F_1$, $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $d$ of $D$. Then $\text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d)$. The theorem is a consequence of (7).

(59) Let us consider non empty sets $C$, $D$, a without $-\infty$ function $F_1$ from $C \times D$ into $\overline{\mathbb{R}}$, a without $+\infty$ function $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $c$ of $C$. Then

(i) $\text{curry}(F_1 - F_2, c) = \text{curry}(F_1, c) - \text{curry}(F_2, c)$, and

(ii) $\text{curry}(F_2 - F_1, c) = \text{curry}(F_2, c) - \text{curry}(F_1, c)$.

The theorem is a consequence of (7).

(60) Let us consider non empty sets $C$, $D$, a without $-\infty$ function $F_1$ from $C \times D$ into $\overline{\mathbb{R}}$, a without $+\infty$ function $F_2$ from $C \times D$ into $\overline{\mathbb{R}}$, and an element $d$ of $D$. Then

(i) $\text{curry}'(F_1 - F_2, d) = \text{curry}'(F_1, d) - \text{curry}'(F_2, d)$, and

(ii) $\text{curry}'(F_2 - F_1, d) = \text{curry}'(F_2, d) - \text{curry}'(F_1, d)$.

The theorem is a consequence of (7).

## 4. Non-Negative Extended Real-Valued Double Sequences

Now we state the propositions:

(61) Let us consider a non-negative sequence $s$ of extended reals. Suppose $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is not convergent to $+\infty$. Let us consider a natural number $n$. Then $s(n)$ is a real number.

(62) Let us consider a non-negative sequence $s$ of extended reals. Suppose $s$ is non-decreasing. Then $s$ is convergent to $+\infty$ or convergent to a finite limit.

Let $f$ be a non-negative function from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$ and $n$ be an element of $\mathbb{N}$. Let us observe that curry$(f,n)$ is non-negative and curry$'(f,n)$ is non-negative.

Now we state the propositions:

(63) Let us consider a non-negative function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$, and an element $m$ of $\mathbb{N}$. Then curry(the partial sums in the second coordinate of $f,m$) is non-decreasing.

Proof: Set $P=$ curry(the partial sums in the second coordinate of $f,m$). For every natural numbers $n$, $j$ such that $j\leqslant n$ holds $P(j)\leqslant P(n)$ by [4, (51)], [1, (13), (20)]. $\square$

(64) Let us consider a non-negative function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$, and an element $n$ of $\mathbb{N}$. Then curry$'$(the partial sums in the first coordinate of $f,n$) is non-decreasing. The theorem is a consequence of (63), (40), and (33).

Let $f$ be a non-negative function from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$ and $m$ be an element of $\mathbb{N}$. One can check that curry(the partial sums in the second coordinate of $f,m$) is non-decreasing and curry$'$(the partial sums in the first coordinate of $f,m$) is non-decreasing.

Let us consider a non-negative function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(65) (i) if $f$ is convergent in the first coordinate, then the lim in the first coordinate of $f$ is non-negative, and

(ii) if $f$ is convergent in the second coordinate, then the lim in the second coordinate of $f$ is non-negative.

(66) (i) the partial sums in the first coordinate of $f$ is convergent in the first coordinate, and

(ii) the partial sums in the second coordinate of $f$ is convergent in the second coordinate.

Let us consider a non-negative function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$, an element $m$ of $\mathbb{N}$, and a natural number $n$.

Let us assume that curry$'$(the partial sums in the first coordinate of $f,m$) is not convergent to $+\infty$. Now we state the propositions:

(67) $f(n, m)$ is a real number.

(68) $f(m, n)$ is a real number.

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and natural numbers $n$, $m$. Now we state the propositions:

(69) Suppose for every natural number $i$ such that $i \leqslant n$ holds $f(i, m)$ is a real number. Then (the partial sums in the first coordinate of $f)(n, m) < +\infty$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then (the partial sums in the first coordinate of $f)(\$_1, m) < +\infty$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)], [1, (13)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(70) Suppose for every natural number $i$ such that $i \leqslant m$ holds $f(n, i)$ is a real number. Then (the partial sums in the second coordinate of $f)(n, m) < +\infty$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant m$, then (the partial sums in the second coordinate of $f)(n, \$_1) < +\infty$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)], [1, (13)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

Now we state the proposition:

(71) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to $-\infty$. Then there exists an element $m$ of $\mathbb{N}$ such that curry$'$(the partial sums in the first coordinate of $f, m)$ is convergent to $-\infty$. The theorem is a consequence of (54).

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a natural number $m$. Now we state the propositions:

(72) for every element $k$ of $\mathbb{N}$ such that $k \leqslant m$ holds curry(the partial sums in the second coordinate of $f, k)$ is not convergent to $+\infty$ if and only if for every element $k$ of $\mathbb{N}$ such that $k \leqslant m$ holds lim curry(the partial sums in the second coordinate of $f, k) < +\infty$. The theorem is a consequence of (62).

(73) for every element $k$ of $\mathbb{N}$ such that $k \leqslant m$ holds curry$'$(the partial sums in the first coordinate of $f, k)$ is not convergent to $+\infty$ if and only if for every element $k$ of $\mathbb{N}$ such that $k \leqslant m$ holds lim curry$'$(the partial sums in the first coordinate of $f, k) < +\infty$. The theorem is a consequence of (62).

(74) $(\sum_{\alpha=0}^{\kappa}$(the lim in the second coordinate of the partial sums in the second coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$ if and only if there exists an element $k$ of $\mathbb{N}$ such that $k \leqslant m$ and curry(the partial sums in the second coordinate

of $f, k)$ is convergent to $+\infty$. The theorem is a consequence of (72), (65), and (4).

(75) $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa\in\mathbb{N}}(m) = +\infty$ if and only if there exists an element $k$ of $\mathbb{N}$ such that $k \leqslant m$ and curry′(the partial sums in the first coordinate of $f, k)$ is convergent to $+\infty$. The theorem is a consequence of (38), (40), (74), and (32).

Now we state the proposition:

(76) Let us consider a non-negative function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then

 (i) (the partial sums in the first coordinate of $f)(n, m) \geqslant f(n, m)$, and

 (ii) (the partial sums in the second coordinate of $f)(n, m) \geqslant f(n, m)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then (the partial sums in the first coordinate of $f)(\$_1, m) \geqslant f(\$_1, m)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)]. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ if $\$_1 \leqslant m$, then (the partial sums in the second coordinate of $f)(n, \$_1) \geqslant f(n, \$_1)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$ by [4, (51)]. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

Let us consider a non-negative function $f$ from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$ and an element $m$ of $\mathbb{N}$. Now we state the propositions:

(77) Suppose there exists an element $k$ of $\mathbb{N}$ such that $k \leqslant m$ and curry(the partial sums in the second coordinate of $f, k)$ is convergent to $+\infty$. Then

 (i) curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m)$ is convergent to $+\infty$, and

 (ii) lim curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m) = +\infty$.

Proof: For every real number $g$ such that $0 < g$ there exists a natural number $N$ such that for every natural number $n$ such that $N \leqslant n$ holds $g \leqslant$ (curry(the par- tial sums in the second coordinate of the partial sums in the first coordinate of $f, m))(n)$ by [26, (7)], (76). □

(78) Suppose there exists an element $k$ of $\mathbb{N}$ such that $k \leqslant m$ and curry′(the partial sums in the first coordinate of $f, k)$ is convergent to $+\infty$. Then

 (i) curry′(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m)$ is convergent to $+\infty$, and

 (ii) lim curry′(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m) = +\infty$.

The theorem is a consequence of (40), (32), and (77).

Now we state the propositions:

(79) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit if and only if the partial sums in the first coordinate of $f$ is convergent in the first coordinate to a finite limit. The theorem is a consequence of (54), (47), (7), and (23).

(80) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit. Let us consider an element $m$ of $\mathbb{N}$. Then $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) =$ lim curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m$).

PROOF: The partial sums in the first coordinate of $f$ is convergent in the first coordinate to a finite limit. Define $\mathcal{P}$[natural number] $\equiv$ for every element $k$ of $\mathbb{N}$ such that $k \leqslant \$_1$ holds $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(k) =$ lim curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, k$). For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$ by [1, (13)], [14, (7)], (47), [4, (51)]. For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(81) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit if and only if the partial sums in the second coordinate of $f$ is convergent in the second coordinate to a finite limit. The theorem is a consequence of (36), (40), and (79).

(82) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit. Let us consider an element $m$ of $\mathbb{N}$. Then $(\sum_{\alpha=0}^{\kappa}$(the lim in the second coordinate of the partial sums in the second coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) =$ lim curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m$). The theorem is a consequence of (36), (40), (38), (80), and (32).

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence $s$ of extended reals. Now we state the propositions:

(83) Suppose for every element $m$ of $\mathbb{N}$, $s(m) = \liminf \text{curry}'(f, m)$. Then $\sum s \leqslant$ lim inf(the lim in the second coordinate of the partial sums in the second coordinate of $f$).

PROOF: For every element $m$ of $\mathbb{N}$ and for every elements $N$, $n$ of $\mathbb{N}$

such that $n \geqslant N$ holds (the inferior realsequence $\mathrm{curry}'(f, m))(N) \leqslant f(n, m)$ by [26, (7), (8)]. Define $\mathcal{F}$(element of $\mathbb{N}$) = the inferior realsequence $\mathrm{curry}'(f, \$_1)$. Define $\mathcal{G}$(element of $\mathbb{N}$, element of $\mathbb{N}$) = (the inferior realsequence $\mathrm{curry}'(f, \$_2))(\$_1)$. Consider $g$ being a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element $n$ of $\mathbb{N}$ and for every element $m$ of $\mathbb{N}$, $g(n, m) = \mathcal{G}(n, m)$ from [5, Sch. 4]. For every element $m$ of $\mathbb{N}$ and for every elements $N$, $n$ of $\mathbb{N}$ such that $n \geqslant N$ holds (the partial sums in the second coordinate of $g)(N, m) \leqslant$ (the partial sums in the second coordinate of $f)(n, m)$. For every element $m$ of $\mathbb{N}$ and for every elements $N$, $n$ of $\mathbb{N}$ such that $n \geqslant N$ holds (the partial sums in the second coordinate of $g)(N, m) \leqslant$ (the inferior realsequence the lim in the second coordinate of the partial sums in the second coordinate of $f)(n)$ by [26, (37), (23)]. Define $\mathcal{Q}$[natural number] $\equiv$ for every element $m$ of $\mathbb{N}$ such that $m = \$_1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) = \lim \mathrm{curry}'$(the partial sums in the second coordinate of $g, m)$. For every element $m$ of $\mathbb{N}$, $\mathrm{curry}'$(the partial sums in the second coordinate of $g, m)$ is convergent by [26, (7), (37)]. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$ by [26, (37)], [4, (51), (52)], [14, (11)]. For every natural number $k$, $\mathcal{Q}[k]$ from [1, Sch. 2]. For every natural number $m$, $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) \leqslant \liminf$(the lim in the second coordinate of the partial sums in the second coordinate of $f$) by [26, (37), (38)]. For every object $m$ such that $m \in \mathrm{dom}\, s$ holds $0 \leqslant s(m)$ by [4, (51), (52)], [26, (23)]. □

(84) Suppose for every element $m$ of $\mathbb{N}$, $s(m) = \liminf \mathrm{curry}(f, m)$. Then $\sum s \leqslant \liminf$(the lim in the first coordinate of the partial sums in the first coordinate of $f$). The theorem is a consequence of (32), (83), (38), and (40).

Now we state the proposition:

(85) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a sequence $s$ of extended reals, and natural numbers $n$, $m$. Then

(i) if for every natural numbers $i$, $j$, $f(i, j) \leqslant s(i)$, then (the partial sums in the first coordinate of $f)(n, m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)$, and

(ii) if for every natural numbers $i$, $j$, $f(i, j) \leqslant s(j)$, then (the partial sums in the second coordinate of $f)(n, m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(n, \$_1) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 2]. □

Let us consider a sequence $s$ of extended reals and an extended real number $r$. Now we state the propositions:

(86) If for every natural number $n$, $s(n) \leqslant r$, then $\limsup s \leqslant r$.
PROOF: Define $\mathcal{F}$(element of $\mathbb{N}$) $= r$. Consider $f$ being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element $n$ of $\mathbb{N}$, $f(n) = \mathcal{F}(n)$ from [7, Sch. 4]. For every natural number $n$, $f(n) = r$. For every natural number $n$, $s(n) \leqslant f(n)$. $\square$

(87) If for every natural number $n$, $r \leqslant s(n)$, then $r \leqslant \liminf s$.
PROOF: Define $\mathcal{F}$(element of $\mathbb{N}$) $= r$. Consider $f$ being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element $n$ of $\mathbb{N}$, $f(n) = \mathcal{F}(n)$ from [7, Sch. 4]. For every natural number $n$, $f(n) = r$. For every natural number $n$, $f(n) \leqslant s(n)$. $\square$

Now we state the proposition:

(88) Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) for every natural numbers $i_1, i_2, j$ such that $i_1 \leqslant i_2$ holds (the partial sums in the first coordinate of $f)(i_1, j) \leqslant$ (the partial sums in the first coordinate of $f)(i_2, j)$, and

(ii) for every natural numbers $i, j_1, j_2$ such that $j_1 \leqslant j_2$ holds (the partial sums in the second coordinate of $f)(i, j_1) \leqslant$ (the partial sums in the second coordinate of $f)(i, j_2)$.

Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and natural numbers $i$, $j$, $k$. Now we state the propositions:

(89) Suppose for every element $m$ of $\mathbb{N}$, $\text{curry}'(f, m)$ is non-decreasing and $i \leqslant j$. Then (the partial sums in the second coordinate of $f)(i, k) \leqslant$ (the partial sums in the second coordinate of $f)(j, k)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(i, \$_1) \leqslant$ (the partial sums in the second coordinate of $f)(j, \$_1)$. For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [26, (7)]. For every natural number $n$, $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

(90) Suppose for every element $n$ of $\mathbb{N}$, $\text{curry}(f, n)$ is non-decreasing and $i \leqslant j$. Then (the partial sums in the first coordinate of $f)(k, i) \leqslant$ (the partial sums in the first coordinate of $f)(k, j)$. The theorem is a consequence of (32), (89), and (39).

Let us consider a non-negative function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence $s$ of extended reals. Now we state the propositions:

(91) Suppose for every element $m$ of $\mathbb{N}$, $\text{curry}'(f, m)$ is non-decreasing and $s(m) = \lim \text{curry}'(f, m)$. Then

(i) the lim in the second coordinate of the partial sums in the second coordinate of $f$ is non-decreasing, and

(ii) $\sum s = \lim(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$.

PROOF: $\sum s \leqslant \liminf(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$. For every natural numbers $n$, $m$, $f(n, m) \leqslant s(m)$ by [26, (37)], [6, (3)]. For every natural numbers $n$, $m$ such that $m \leqslant n$ holds (the lim in the second coordinate of the partial sums in the second coordinate of $f)(m) \leqslant$ (the lim in the second coordinate of the partial sums in the second coordinate of $f)(n)$ by [26, (37)], (89), [26, (38)]. For every natural number $n$, (the lim in the second coordinate of the partial sums in the second coordinate of $f)(n) \leqslant \sum s$ by [26, (37)], [4, (39)], (87), [26, (41)]. $\limsup(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f) \leqslant \sum s$. $\square$

(92) Suppose for every element $m$ of $\mathbb{N}$, $\operatorname{curry}(f, m)$ is non-decreasing and $s(m) = \lim \operatorname{curry}(f, m)$. Then

(i) the lim in the first coordinate of the partial sums in the first coordinate of $f$ is non-decreasing, and

(ii) $\sum s = \lim(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)$.

The theorem is a consequence of (32), (91), (33), and (40).

## 5. Pringsheim Sense Convergence for Extended Real-Valued Double Sequences

Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(93) If $f$ is P-convergent to $+\infty$, then $f$ is not P-convergent to $-\infty$ and $f$ is not P-convergent to a finite limit.

(94) If $f$ is P-convergent to $-\infty$, then $f$ is not P-convergent to $+\infty$ and $f$ is not P-convergent to a finite limit.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that $f$ is P-convergent if and only if

(Def. 17) $f$ is P-convergent to a finite limit or P-convergent to $+\infty$ or P-convergent to $-\infty$.

Assume $f$ is P-convergent. The functor P-lim $f$ yielding an extended real is defined by

(Def. 18) there exists a real number $p$ such that $it = p$ and for every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|f(n, m) - it| < e$

and $f$ is P-convergent to a finite limit or $it = +\infty$ and $f$ is P-convergent to $+\infty$ or $it = -\infty$ and $f$ is P-convergent to $-\infty$.

Now we state the propositions:

(95) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a real number $r$. Suppose for every natural numbers $n$, $m$, $f(n, m) = r$. Then

(i) $f$ is P-convergent to a finite limit, and

(ii) $\text{P-lim}\, f = r$.

(96) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f(n_1, m_1) \leqslant f(n_2, m_2)$. Then

(i) $f$ is P-convergent, and

(ii) $\text{P-lim}\, f = \sup \operatorname{rng} f$.

(97) Let us consider functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers $n$, $m$, $f_1(n, m) \leqslant f_2(n, m)$. Then $\sup \operatorname{rng} f_1 \leqslant \sup \operatorname{rng} f_2$.

(98) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers $n$, $m$. Then $f(n, m) \leqslant \sup \operatorname{rng} f$.

Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and an extended real number $K$. Now we state the propositions:

(99) If for every natural numbers $n$, $m$, $f(n, m) \leqslant K$, then $\sup \operatorname{rng} f \leqslant K$.

(100) If $K \neq +\infty$ and for every natural numbers $n$, $m$, $f(n, m) \leqslant K$, then $\sup \operatorname{rng} f < +\infty$.

Now we state the propositions:

(101) Let us consider a without $-\infty$ function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\sup \operatorname{rng} f \neq +\infty$ if and only if there exists a real number $K$ such that $0 < K$ and for every natural numbers $n$, $m$, $f(n, m) \leqslant K$.

(102) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an extended real $c$. Suppose for every natural numbers $n$, $m$, $f(n, m) = c$. Then

(i) $f$ is P-convergent, and

(ii) $\text{P-lim}\, f = c$, and

(iii) $\text{P-lim}\, f = \sup \operatorname{rng} f$.

(103) Let us consider a function $f$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and without $-\infty$ functions $f_1$, $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_1(n_1, m_1) \leqslant f_1(n_2, m_2)$ and for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_2(n_1, m_1) \leqslant f_2(n_2, m_2)$ and for every natural numbers $n$, $m$, $f_1(n, m) + f_2(n, m) = f(n, m)$. Then

(i) $f$ is P-convergent, and

(ii) P-lim $f = \sup \operatorname{rng} f$, and

(iii) P-lim $f =$ P-lim $f_1 +$ P-lim $f_2$, and

(iv) $\sup \operatorname{rng} f = \sup \operatorname{rng} f_1 + \sup \operatorname{rng} f_2$.

The theorem is a consequence of (96) and (101).

Let us consider a without $-\infty$ function $f_1$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a function $f_2$ from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a real number $c$. Now we state the propositions:

(104) Suppose $0 \leqslant c$ and for every natural numbers $n$, $m$, $f_2(n, m) = c \cdot f_1(n, m)$. Then

(i) $\sup \operatorname{rng} f_2 = c \cdot \sup \operatorname{rng} f_1$, and

(ii) $f_2$ is without $-\infty$.

The theorem is a consequence of (102) and (101).

(105) Suppose $0 \leqslant c$ and for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_1(n_1, m_1) \leqslant f_1(n_2, m_2)$ and for every natural numbers $n$, $m$, $f_2(n, m) = c \cdot f_1(n, m)$. Then

(i) for every natural numbers $n_1$, $m_1$, $n_2$, $m_2$ such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_2(n_1, m_1) \leqslant f_2(n_2, m_2)$, and

(ii) $f_2$ is without $-\infty$ and P-convergent, and

(iii) P-lim $f_2 = \sup \operatorname{rng} f_2$, and

(iv) P-lim $f_2 = c \cdot$ P-lim $f_1$.

The theorem is a consequence of (96) and (104).

## References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.

[4] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.

[5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[10] Noboru Endou. Double series and sums. *Formalized Mathematics*, 22(**1**):57–68, 2014. doi:10.2478/forma-2014-0006.

[11] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006. doi:10.2478/v10037-006-0008-x.

[12] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(**3**):491–494, 2001.

[13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(**3**):495–500, 2001.

[14] Noboru Endou, Keiko Narita, and Yasunari Shidama. The Lebesgue monotone convergence theorem. *Formalized Mathematics*, 16(**2**):167–175, 2008. doi:10.2478/v10037-008-0023-1.

[15] Noboru Endou, Hiroyuki Okazaki, and Yasunari Shidama. Double sequences and limits. *Formalized Mathematics*, 21(**3**):163–170, 2013. doi:10.2478/forma-2013-0018.

[16] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Wiley, 2 edition, 1999.

[17] D.J.H. Garling. *A Course in Mathematical Analysis: Volume 1, Foundations and Elementary Real Analysis*, volume 1. Cambridge University Press, 2013.

[18] Andrzej Kondracki. Basic properties of rational numbers. *Formalized Mathematics*, 1(**5**): 841–845, 1990.

[19] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1 (**3**):471–475, 1990.

[20] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[21] Adam Naumowicz. Conjugate sequences, bounded complex sequences and convergent complex sequences. *Formalized Mathematics*, 6(**2**):265–268, 1997.

[22] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

[25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[26] Hiroshi Yamazaki, Noboru Endou, Yasunari Shidama, and Hiroyuki Okazaki. Inferior limit, superior limit and convergence of sequences of extended real numbers. *Formalized Mathematics*, 15(**4**):231–236, 2007. doi:10.2478/v10037-007-0026-3.


*Received July 1, 2015*