

# Finite Product of Semiring of Sets

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**Summary.** We formalize that the image of a semiring of sets [17] by an injective function is a semiring of sets. We offer a non-trivial example of a semiring of sets in a topological space [21]. Finally, we show that the finite product of a semiring of sets is also a semiring of sets [21] and that the finite product of a classical semiring of sets [8] is a classical semiring of sets. In this case, we use here the notation from the book of Aliprantis and Border [1].

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The notation and terminology used in this paper have been introduced in the following articles: [9], [2], [3], [4], [22], [7], [15], [23], [10], [11], [6], [12], [20], [26], [27], [19], [14], [16], [25], [18], and [13].

## 1. PRELIMINARIES

From now on  $X_1, X_2, X_3, X_4$  denote sets.

Now we state the propositions:

- (1) (i)  $X_1 \cap X_4 \setminus (X_2 \cup X_3)$  misses  $X_1 \setminus ((X_2 \cup X_3) \cup X_4)$ , and  
(ii)  $X_1 \cap X_4 \setminus (X_2 \cup X_3)$  misses  $(X_1 \cap X_3) \cap X_4 \setminus X_2$ , and  
(iii)  $X_1 \setminus ((X_2 \cup X_3) \cup X_4)$  misses  $(X_1 \cap X_3) \cap X_4 \setminus X_2$ .
- (2)  $(X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = (X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2)$ .
- (3)  $(X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)$ .

(4)  $(X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)$ . The theorem is a consequence of (2) and (3).

(5)  $\cup\{X_1, X_2, X_3\} = (X_1 \cup X_2) \cup X_3$ .

## 2. THE DIRECT IMAGE OF A SEMIRING OF SETS BY AN INJECTIVE FUNCTION

Now we state the proposition:

(6) Let us consider sets  $T, S$ , a function  $f$  from  $T$  into  $S$ , and a family  $G$  of subsets of  $T$ . Then  $f^\circ G = \{f^\circ A, \text{ where } A \text{ is a subset of } T : A \in G\}$ .

Let  $T, S$  be sets,  $f$  be a function from  $T$  into  $S$ , and  $G$  be a finite family of subsets of  $T$ . Let us note that  $f^\circ G$  is finite.

Let  $f$  be a function and  $A$  be a countable set. Let us note that  $f^\circ A$  is countable.

The scheme *FraenkelCountable* deals with a set  $\mathcal{A}$  and a set  $\mathcal{X}$  and a unary functor  $\mathcal{F}$  yielding a set and states that

(Sch. 1)  $\{\mathcal{F}(w), \text{ where } w \text{ is an element of } \mathcal{A} : w \in \mathcal{X}\}$  is countable provided

- $\mathcal{X}$  is countable.

Let  $T, S$  be sets,  $f$  be a function from  $T$  into  $S$ , and  $G$  be a countable family of subsets of  $T$ . Let us note that  $f^\circ G$  is countable.

Let  $X, Y$  be sets,  $S$  be a family of subsets of  $X$  with the empty element, and  $f$  be a function from  $X$  into  $Y$ . One can verify that  $f^\circ S$  has the empty element.

Now we state the propositions:

(7) Let us consider sets  $X, Y$ , a function  $f$  from  $X$  into  $Y$ , and families  $S_1, S_2$  of subsets of  $X$ . If  $S_1 \subseteq S_2$ , then  $f^\circ S_1 \subseteq f^\circ S_2$ . The theorem is a consequence of (6).

(8) Let us consider sets  $X, Y$ , a  $\cap$ -closed family  $S$  of subsets of  $X$ , and a function  $f$  from  $X$  into  $Y$ . Suppose  $f$  is one-to-one. Then  $f^\circ S$  is a  $\cap$ -closed family of subsets of  $Y$ .

(9) Let us consider non empty sets  $X, Y$ , a  $\cap_{fp}$ -closed family  $S$  of subsets of  $X$ , and a function  $f$  from  $X$  into  $Y$ . Suppose  $f$  is one-to-one. Then  $f^\circ S$  is a  $\cap_{fp}$ -closed family of subsets of  $Y$ .

(10) Let us consider non empty sets  $X, Y$ , a  $\setminus_{fp}^\subseteq$ -closed family  $S$  of subsets of  $X$ , and a function  $f$  from  $X$  into  $Y$ . Suppose  $f$  is one-to-one and  $f^\circ S$  is not empty. Then  $f^\circ S$  is a  $\setminus_{fp}^\subseteq$ -closed family of subsets of  $Y$ .

PROOF: Reconsider  $f_1 = f^\circ S$  as a family of subsets of  $Y$ .  $f_1$  is  $\setminus_{fp}$ -closed by [10, (64), (87)], [11, (103)], [26, (123)].  $\square$

- (11) Let us consider non empty sets  $X, Y$ , a  $\setminus_{fp}$ -closed family  $S$  of subsets of  $X$ , and a function  $f$  from  $X$  into  $Y$ . Suppose  $f$  is one-to-one. Then  $f^\circ S$  is a  $\setminus_{fp}$ -closed family of subsets of  $Y$ .
- (12) Let us consider non empty sets  $X, Y$ , a semiring  $S$  of sets of  $X$ , and a function  $f$  from  $X$  into  $Y$ . If  $f$  is one-to-one, then  $f^\circ S$  is a semiring of sets of  $Y$ .

### 3. THE SET OF SET DIFFERENCES OF ALL ELEMENTS OF A SEMIRING OF SETS

Now we state the proposition:

- (13) Let us consider a 1-element finite sequence  $X$ . Suppose  $X(1)$  is not empty. Then there exists a function  $I$  from  $X(1)$  into  $\prod X$  such that
  - (i)  $I$  is one-to-one and onto, and
  - (ii) for every object  $x$  such that  $x \in X(1)$  holds  $I(x) = \langle x \rangle$ .

Let  $X$  be a set. Observe that  $2_*^X$  is  $\cap$ -closed and there exists a  $\cap$ -closed family of subsets of  $X$  which has the empty element and there exists a  $\cap$ -closed family of subsets of  $X$  with the empty element which is  $\cup$ -closed.

Let  $X, Y$  be non empty sets. Let us observe that  $X \parallel Y$  is non empty.

Now we state the proposition:

- (14) Let us consider a set  $X$ , and a family  $S$  of subsets of  $X$  with the empty element. Then  $S \parallel S =$  the set of all  $A \setminus B$  where  $A, B$  are elements of  $S$ .

Let  $X$  be a set and  $S$  be a family of subsets of  $X$  with the empty element. The functor semidiff  $S$  yielding a family of subsets of  $X$  is defined by the term (Def. 1)  $S \parallel S$ .

Now we state the proposition:

- (15) Let us consider a set  $X$ , a family  $S$  of subsets of  $X$  with the empty element, and an object  $x$ . Suppose  $x \in \text{semidiff } S$ . Then there exist elements  $A, B$  of  $S$  such that  $x = A \setminus B$ . The theorem is a consequence of (14).

Let  $X$  be a set and  $S$  be a family of subsets of  $X$  with the empty element. Observe that semidiff  $S$  has the empty element.

Let  $S$  be a  $\cap$ -closed,  $\cup$ -closed family of subsets of  $X$  with the empty element. Note that semidiff  $S$  is  $\cap$ -closed and  $\setminus_{fp}$ -closed.

Now we state the proposition:

- (16) Let us consider a set  $X$ , and a  $\cap$ -closed,  $\cup$ -closed family  $S$  of subsets of  $X$  with the empty element. Then semidiff  $S$  is a semiring of sets of  $X$ .

#### 4. THE COLLECTION OF ALL LOCALLY CLOSED SETS $LC(X, \tau)$ OF A TOPOLOGICAL SPACE $(X, \tau)$

Let  $T$  be a non empty topological space. The functor  $LC(T)$  yielding a family of subsets of  $\Omega_T$  is defined by the term

(Def. 2)  $\{A \cap B, \text{ where } A, B \text{ are subsets of } T : A \text{ is open and } B \text{ is closed}\}.$

Let us note that  $LC(T)$  is  $\cap$ -closed and  $\setminus_{fp}$ -closed and has the empty element.

(17) Let us consider a non empty topological space  $T$ . Then  $LC(T)$  is a semiring of sets of  $\Omega_T$ .

#### 5. THE FINITE PRODUCT OF SEMIRINGS OF SETS

Let  $n$  be a natural number. Note that there exists an  $n$ -element finite sequence which is non-empty.

Let  $n$  be a non zero natural number and  $X$  be a non-empty,  $n$ -element finite sequence.

A semiring family of  $X$  is an  $n$ -element finite sequence and is defined by

(Def. 3) for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $it(i)$  is a semiring of sets of  $X(i)$ .

In the sequel  $n$  denotes a non zero natural number and  $X$  denotes a non-empty,  $n$ -element finite sequence. Now we state the propositions:

(18) Let us consider a semiring family  $S$  of  $X$ . Then  $\text{dom } S = \text{dom } X$ .

(19) Let us consider a semiring family  $S$  of  $X$ , and a natural number  $i$ . If  $i \in \text{Seg } n$ , then  $\bigcup(S(i)) \subseteq X(i)$ .

(20) Let us consider a function  $f$ , and an  $n$ -element finite sequence  $X$ . If  $f \in \prod X$ , then  $f$  is an  $n$ -element finite sequence.

Let  $n$  be a non zero natural number and  $X$  be an  $n$ -element finite sequence.

The functor  $\text{SemiringProduct } X$  yielding a set is defined by

(Def. 4) for every object  $f$ ,  $f \in it$  iff there exists a function  $g$  such that  $f = \prod g$  and  $g \in \prod X$ .

Now we state the propositions:

(21) Let us consider an  $n$ -element finite sequence  $X$ .

Then  $\text{SemiringProduct } X \subseteq 2(\bigcup \bigcup X)^{\text{dom } X}$ .

(22) Let us consider a semiring family  $S$  of  $X$ . Then  $\text{SemiringProduct } S$  is a family of subsets of  $\prod X$ .

PROOF: Reconsider  $S_1 = \text{SemiringProduct } S$  as a subset of  $2(\bigcup \bigcup S)^{\text{dom } S}$ .  $S_1 \subseteq 2\prod X$  by [3, (9)], (18), [7, (89)], (19).  $\square$

- (23) Let us consider a non-empty, 1-element finite sequence  $X$ . Then  $\prod X =$  the set of all  $\langle x \rangle$  where  $x$  is an element of  $X(1)$ . The theorem is a consequence of (13).

One can check that  $\prod \langle \emptyset \rangle$  is empty. Now we state the propositions:

- (24) Let us consider a non empty set  $x$ . Then  $\prod \langle x \rangle =$  the set of all  $\langle y \rangle$  where  $y$  is an element of  $x$ . The theorem is a consequence of (23).
- (25) Let us consider a non-empty, 1-element finite sequence  $X$ , and a semiring family  $S$  of  $X$ . Then  $\text{SemiringProduct } S =$  the set of all  $\prod \langle s \rangle$  where  $s$  is an element of  $S(1)$ . PROOF:  $S$  is non-empty by (18), [7, (3)].  $\prod S =$  the set of all  $\langle s \rangle$  where  $s$  is an element of  $S(1)$ .  $\square$

Let us consider sets  $x, y$ . Now we state the propositions:

- (26)  $\prod \langle x \rangle \cap \prod \langle y \rangle = \prod \langle x \cap y \rangle$ . The theorem is a consequence of (24).
- (27)  $\prod \langle x \rangle \setminus \prod \langle y \rangle = \prod \langle x \setminus y \rangle$ . The theorem is a consequence of (24).

Let us consider a non-empty, 1-element finite sequence  $X$  and a semiring family  $S$  of  $X$ . Now we state the propositions:

- (28) the set of all  $\prod \langle s \rangle$  where  $s$  is an element of  $S(1)$  is a semiring of sets of the set of all  $\langle x \rangle$  where  $x$  is an element of  $X(1)$ . The theorem is a consequence of (24), (26), and (27).
- (29)  $\text{SemiringProduct } S$  is a semiring of sets of  $\prod X$ . The theorem is a consequence of (23), (25), and (28).
- (30) Let us consider sets  $X_1, X_2$ , a semiring  $S_1$  of sets of  $X_1$ , and a semiring  $S_2$  of sets of  $X_2$ . Then the set of all  $s_1 \times s_2$  where  $s_1$  is an element of  $S_1$ ,  $s_2$  is an element of  $S_2$  is a semiring of sets of  $X_1 \times X_2$ .
- (31) Let us consider a non-empty,  $n$ -element finite sequence  $X_3$ , a non-empty, 1-element finite sequence  $X_1$ , a semiring family  $S_3$  of  $X_3$ , and a semiring family  $S_1$  of  $X_1$ . Suppose  $\text{SemiringProduct } S_3$  is a semiring of sets of  $\prod X_3$  and  $\text{SemiringProduct } S_1$  is a semiring of sets of  $\prod X_1$ . Let us consider a family  $S_4$  of subsets of  $\prod X_3 \times \prod X_1$ . Suppose  $S_4 =$  the set of all  $s_1 \times s_2$  where  $s_1$  is an element of  $\text{SemiringProduct } S_3$ ,  $s_2$  is an element of  $\text{SemiringProduct } S_1$ . Then there exists a function  $I$  from  $\prod X_3 \times \prod X_1$  into  $\prod (X_3 \cap X_1)$  such that

- (i)  $I$  is one-to-one and onto, and
- (ii) for every finite sequences  $x, y$  such that  $x \in \prod X_3$  and  $y \in \prod X_1$  holds  $I(x, y) = x \cap y$ , and
- (iii)  $I^\circ S_4 = \text{SemiringProduct}(S_3 \cap S_1)$ .

PROOF:  $\bigcup (S_1(1)) \subseteq X_1(1)$ . Consider  $I$  being a function from  $\prod X_3 \times \prod X_1$  into  $\prod (X_3 \cap X_1)$  such that  $I$  is one-to-one and  $I$  is onto and for every finite

sequences  $x, y$  such that  $x \in \prod X_3$  and  $y \in \prod X_1$  holds  $I(x, y) = x \cap y$ .  $I^\circ S_4 = \text{SemiringProduct}(S_3 \cap S_1)$  by (25), (20), [7, (89)], [24, (153)].  $\square$

(32) Let us consider a non-empty,  $n$ -element finite sequence  $X_3$ , a non-empty, 1-element finite sequence  $X_1$ , a semiring family  $S_3$  of  $X_3$ , and a semiring family  $S_1$  of  $X_1$ . Suppose  $\text{SemiringProduct } S_3$  is a semiring of sets of  $\prod X_3$  and  $\text{SemiringProduct } S_1$  is a semiring of sets of  $\prod X_1$ . Then  $\text{SemiringProduct}(S_3 \cap S_1)$  is a semiring of sets of  $\prod(X_3 \cap X_1)$ . The theorem is a consequence of (30), (31), (9), and (10).

(33) Let us consider a semiring family  $S$  of  $X$ . Then  $\text{SemiringProduct } S$  is a semiring of sets of  $\prod X$ . PROOF: Define  $\mathcal{P}[\text{non zero natural number}] \equiv$  for every non-empty, 1-element finite sequence  $X$  for every semiring family  $S$  of  $X$ ,  $\text{SemiringProduct } S$  is a semiring of sets of  $\prod X$ .  $\mathcal{P}[1]$ . For every non zero natural number  $n$ ,  $\mathcal{P}[n]$  from [5, Sch. 10].  $\square$

Let  $n$  be a non zero natural number,  $X$  be a non-empty,  $n$ -element finite sequence, and  $S$  be a semiring family of  $X$ . We say that  $S$  is  $\cap$ -closed yielding if and only if

(Def. 5) for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $S(i)$  is  $\cap$ -closed.

Note that there exists a semiring family of  $X$  which is  $\cap$ -closed yielding.

## 6. THE FINITE PRODUCT OF CLASSICAL SEMIRINGS OF SETS

Let  $X$  be a set. Note that there exists a semiring of sets of  $X$  which is  $\cap$ -closed.

Let us consider a non-empty, 1-element finite sequence  $X$  and a  $\cap$ -closed yielding semiring family  $S$  of  $X$ . Now we state the propositions:

(34) the set of all  $\prod \langle s \rangle$  where  $s$  is an element of  $S(1)$  is a  $\cap$ -closed semiring of sets of the set of all  $\langle x \rangle$  where  $x$  is an element of  $X(1)$ . The theorem is a consequence of (26) and (28).

(35)  $\text{SemiringProduct } S$  is a  $\cap$ -closed semiring of sets of  $\prod X$ . The theorem is a consequence of (23), (25), and (34).

Now we state the propositions:

(36) Let us consider sets  $X_1, X_2$ , a  $\cap$ -closed semiring  $S_1$  of sets of  $X_1$ , and a  $\cap$ -closed semiring  $S_2$  of sets of  $X_2$ . Then the set of all  $s_1 \times s_2$  where  $s_1$  is an element of  $S_1$ ,  $s_2$  is an element of  $S_2$  is a  $\cap$ -closed semiring of sets of  $X_1 \times X_2$ .

(37) Let us consider a non-empty,  $n$ -element finite sequence  $X_3$ , a non-empty, 1-element finite sequence  $X_1$ , a  $\cap$ -closed yielding semiring family  $S_3$  of  $X_3$ , and a  $\cap$ -closed yielding semiring family  $S_1$  of  $X_1$ . Suppose  $\text{SemiringProduct}$

$S_3$  is a  $\cap$ -closed semiring of sets of  $\prod X_3$  and  $\text{SemiringProduct } S_1$  is a  $\cap$ -closed semiring of sets of  $\prod X_1$ . Then  $\text{SemiringProduct}(S_3 \cap S_1)$  is a  $\cap$ -closed semiring of sets of  $\prod(X_3 \cap X_1)$ . The theorem is a consequence of (30), (31), (36), (8), and (10).

Let us consider  $n$  and  $X$ . Let  $S$  be a  $\cap$ -closed yielding semiring family of  $X$ . One can check that  $\text{SemiringProduct } S$  is  $\cap$ -closed.

(38) Let us consider a  $\cap$ -closed yielding semiring family  $S$  of  $X$ .

Then  $\text{SemiringProduct } S$  is a  $\cap$ -closed semiring of sets of  $\prod X$ .

## 7. MEASURABLE RECTANGLE

Let  $n$  be a non zero natural number and  $X$  be a non-empty,  $n$ -element finite sequence.

A classical semiring family of  $X$  is an  $n$ -element finite sequence and is defined by

(Def. 6) for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $it(i)$  is a semi-diff-closed,  $\cap$ -closed family of subsets of  $X(i)$  with the empty element.

Let  $X$  be an  $n$ -element finite sequence. We introduce  $\text{MeasurableRectangle } X$  as a synonym of  $\text{SemiringProduct } X$ . Now we state the propositions:

(39) Every classical semiring family of  $X$  is a  $\cap$ -closed yielding semiring family of  $X$ .

(40) Let us consider a classical semiring family  $S$  of  $X$ .

Then  $\text{MeasurableRectangle } S$  is a semi-diff-closed,  $\cap$ -closed family of subsets of  $\prod X$  with the empty element. The theorem is a consequence of (39) and (33).

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