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Some Facts about Trigonometry and Euclidean Geometry

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Summary. We calculate the values of the trigonometric functions for angles: $\frac{\pi}{3}$ and $\frac{\pi}{6}$, by [16]. After defining some trigonometric identities, we demonstrate conventional trigonometric formulas in the triangle, and the geometric property, by [14], of the triangle inscribed in a semicircle, by the proposition 3.31 in [15]. Then we define the diameter of the circumscribed circle of a triangle using the definition of the area of a triangle and prove some identities of a triangle [9]. We conclude by indicating that the diameter of a circle is twice the length of the radius.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [10], [11], [19], [25], [3], [12], [5], [21], [2], [28], [6], [7], [24], [29], [23], [18], [26], [27], [13], and [8].

1. Values of the Trigonometric Functions for Angles: $\frac{\pi}{3}$ and $\frac{\pi}{6}$

Let us consider a real number a. Now we state the propositions:

(1) $\sin(\pi - a) = \sin a$.

(2) $\cos(\pi - a) = -\cos a$.

(3) $\sin(2 \cdot \pi - a) = -\sin a$.

(4) $\cos(2 \cdot \pi - a) = \cos a$.

(5) $\sin(-2 \cdot \pi + a) = \sin a$.

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(6) $\cos(-2 \cdot \pi + a) = \cos a$.

(7) $\sin(\frac{3 \cdot \pi}{2} + a) = -\cos a$.

(8) $\cos(\frac{3 \cdot \pi}{2} + a) = \sin a$.

(9) $\sin(\frac{3 \cdot \pi}{2} + a) = -\sin(\frac{\pi}{2} - a)$. The theorem is a consequence of (7).

(10) $\cos(\frac{3 \cdot \pi}{2} + a) = \cos(\frac{\pi}{2} - a)$. The theorem is a consequence of (8).

(11) $\sin(\frac{2 \cdot \pi}{3} - a) = \sin(\frac{\pi}{3} + a)$.

(12) $\cos(\frac{2 \cdot \pi}{3} - a) = -\cos(\frac{\pi}{3} + a)$.

(13) $\sin(\frac{2 \cdot \pi}{3} + a) = \sin(\frac{\pi}{3} - a)$.

Now we state the propositions:

(14) $\cos \frac{\pi}{3} = \frac{1}{2}$.

(15) $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

PROOF: $\sin \frac{\pi}{3} \geqslant 0$ by [20, (5)], [29, (79), (81)]. □

(16) $\operatorname{tg} \frac{\pi}{3} = \sqrt{3}$. The theorem is a consequence of (14) and (15).

(17) $\sin \frac{\pi}{6} = \frac{1}{2}$. The theorem is a consequence of (14).

(18) $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. The theorem is a consequence of (15).

(19) $\operatorname{tg} \frac{\pi}{6} = \frac{\sqrt{3}}{3}$. The theorem is a consequence of (17) and (18).

(20) (i) $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$, and

(ii) $\cos(-\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, and

(iii) $\operatorname{tg}(-\frac{\pi}{6}) = -\frac{\sqrt{3}}{3}$, and

(iv) $\sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$, and

(v) $\cos(-\frac{\pi}{3}) = \frac{1}{2}$, and

(vi) $\operatorname{tg}(-\frac{\pi}{3}) = -\sqrt{3}$.

(21) (i) $\arcsin \frac{1}{2} = \frac{\pi}{6}$, and

(ii) $\arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

The theorem is a consequence of (15) and (17).

(22) $\sin \frac{2 \cdot \pi}{3} = \frac{\sqrt{3}}{2}$. The theorem is a consequence of (11) and (15).

(23) $\cos \frac{2 \cdot \pi}{3} = -\frac{1}{2}$. The theorem is a consequence of (12) and (14).

2. Some Trigonometric Identities

Now we state the proposition:

(24) Let us consider a real number x. Then $(\sin(-x))^{\mathbf{2}} = (\sin x)^{\mathbf{2}}$.

Let us consider real numbers x, y, z. Now we state the propositions:

(25) If $x + y + z = \pi$, then $(\sin x)^{\mathbf{2}} + (\sin y)^{\mathbf{2}} - 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^{\mathbf{2}}$.

(26) If $x - y + z = \pi$, then $(\sin x)^{\mathbf{2}} + (\sin y)^{\mathbf{2}} + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^{\mathbf{2}}$. The theorem is a consequence of (24) and (25).

(27) Suppose $x - (-2 \cdot \pi + y) + z = \pi$. Then $(\sin x)^{\mathbf{2}} + (\sin y)^{\mathbf{2}} + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^{\mathbf{2}}$. The theorem is a consequence of (24), (5), and (25).

(28) If $\pi - x - (\pi - y) + z = \pi$, then $(\sin x)^{\mathbf{2}} + (\sin y)^{\mathbf{2}} + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^{\mathbf{2}}$. The theorem is a consequence of (24), (1), and (25).

Now we state the proposition:

(29) Let us consider a real number a. Then $\sin(3 \cdot a) = 4 \cdot \sin a \cdot \sin(\frac{\pi}{3} + a) \cdot \sin(\frac{\pi}{3} - a)$. The theorem is a consequence of (15).

3. Trigonometric Functions and Right Triangle

Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$.

Let us assume that A, B, C form a triangle. Now we state the propositions:

(30) (i) $\measuredangle(A, B, C)$ is not zero, and

(ii) $\measuredangle(B, C, A)$ is not zero, and

(iii) $\measuredangle(C, A, B)$ is not zero, and

(iv) $\measuredangle(A, C, B)$ is not zero, and

(v) $\measuredangle(C, B, A)$ is not zero, and

(vi) $\measuredangle(B, A, C)$ is not zero.

(31) (i) $\measuredangle(A, B, C) = 2 \cdot \pi - \measuredangle(C, B, A)$, and

(ii) $\measuredangle(B, C, A) = 2 \cdot \pi - \measuredangle(A, C, B)$, and

(iii) $\measuredangle(C, A, B) = 2 \cdot \pi - \measuredangle(B, A, C)$, and

(iv) $\measuredangle(B, A, C) = 2 \cdot \pi - \measuredangle(C, A, B)$, and

(v) $\measuredangle(A, C, B) = 2 \cdot \pi - \measuredangle(B, C, A)$, and

(vi) $\measuredangle(C, B, A) = 2 \cdot \pi - \measuredangle(A, B, C)$.

Now we state the proposition:

(32) Suppose A, B, C form a triangle and $|(B - A, C - A)| = 0$. Then

(i) $|C - B| \cdot \sin \measuredangle(C, B, A) = |A - C|$, or

(ii) $|C - B| \cdot (-\sin \measuredangle(C, B, A)) = |A - C|$.

Let us assume that A, B, C form a triangle and $\measuredangle(B, A, C) = \frac{\pi}{2}$. Now we state the propositions:

(33) $\measuredangle(C, B, A) + \measuredangle(A, C, B) = \frac{\pi}{2}$.

(34) (i) $|C - B| \cdot \sin \measuredangle(C, B, A) = |A - C|$, and

(ii) $|C - B| \cdot \sin \measuredangle(A, C, B) = |A - B|$, and

(iii) $|C - B| \cdot \cos \measuredangle(C, B, A) = |A - B|$, and

(iv) $|C - B| \cdot \cos \measuredangle(A, C, B) = |A - C|$.

(35) (i) $\operatorname{tg} \measuredangle(A, C, B) = \frac{|A-B|}{|A-C|}$, and

(ii) $\operatorname{tg} \measuredangle(C, B, A) = \frac{|A-C|}{|A-B|}$.

The theorem is a consequence of (34).

4. Triangle Inscribed in a Semicircle is a Right Triangle

Let a, b be real numbers and r be a negative real number. Let us note that circle(a, b, r) is empty.

Now we state the proposition:

(36) Let us consider real numbers a, b. Then circle$(a, b, 0) = \{[a, b]\}$.

Let a, b be real numbers. One can verify that circle$(a, b, 0)$ is trivial.

Now we state the propositions:

(37) Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$, and real numbers a, b, r. Suppose A, B, C form a triangle and A, $B \in$ circle(a, b, r). Then r is positive. The theorem is a consequence of (36).

(38) Let us consider a point A of $\mathcal{E}^2_{\mathrm{T}}$, real numbers a, b, and a positive real number r. If $A \in$ circle(a, b, r), then $A \neq [a, b]$.

(39) Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$, and real numbers a, b, r. Suppose A, B, C form a triangle and $\measuredangle(C, B, A)$, $\measuredangle(B, A, C) \in]0, \pi[$ and A, B, $C \in$ circle(a, b, r) and $[a, b] \in \mathcal{L}(A, C)$. Then $\measuredangle(C, B, A) = \frac{\pi}{2}$.

Proof: Set $O = [a, b]$. Consider J_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $A = J_1$ and $|J_1 - [a, b]| = r$. Consider J_2 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $B = J_2$ and $|J_2 - [a, b]| = r$. Consider J_3 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $C = J_3$ and $|J_3 - [a, b]| = r$. r is positive. $O \neq A$ and $O \neq C$. $\measuredangle(C, B, O) < \pi$ by [25, (16), (9)], [19, (47)]. A, O, B form a triangle and C, O, B form a triangle by (37), (38), [6, (72), (75)]. $\measuredangle(C, B, O) + \measuredangle(O, C, B) + \measuredangle(O, B, A) + \measuredangle(B, A, O) = \pi$ or $\measuredangle(C, B, O) + \measuredangle(O, C, B) + \measuredangle(O, B, A) + \measuredangle(B, A, O) = -\pi$ by [25, (13)], [19, (47)]. $\measuredangle(O, C, B) = \measuredangle(C, B, O)$ and $\measuredangle(B, A, O) = \measuredangle(O, B, A)$. $\square$

(40) Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$, and a positive real number r. Suppose $\measuredangle(A, B, C)$ is not zero. Then $\sin(r \cdot \measuredangle(C, B, A)) = \sin(r \cdot 2 \cdot \pi) \cdot \cos(r \cdot \measuredangle(A, B, C)) - \cos(r \cdot 2 \cdot \pi) \cdot \sin(r \cdot \measuredangle(A, B, C))$.

(41) Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $\measuredangle(A, B, C)$ is not zero. Then $\sin \frac{\measuredangle(C,B,A)}{3} = \frac{\sqrt{3}}{2} \cdot \cos \frac{\measuredangle(A,B,C)}{3} + \frac{1}{2} \cdot \sin \frac{\measuredangle(A,B,C)}{3}$. The theorem is a consequence of (40), (22), and (23).

5. Diameter of the Circumcircle of a Triangle

Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$. Now we state the propositions:

(42) (i) area of $\triangle(A, B, C)$ = area of $\triangle(B, C, A)$, and

(ii) area of $\triangle(A, B, C)$ = area of $\triangle(C, A, B)$.

(43) area of $\triangle(A, B, C) = -$(area of $\triangle(B, A, C)$).

Let A, B, C be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\varnothing_{\mathbb{O}}(A, B, C)$ yielding a real number is defined by the term

(Def. 1) $\frac{\frac{|A-B|\cdot|B-C|\cdot|C-A|}{2}}{\text{area of } \triangle(A,B,C)}$.

Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$.

Let us assume that A, B, C form a triangle. Now we state the propositions:

(44) $\varnothing_{\mathbb{O}}(A, B, C) = \frac{|C-A|}{\sin \measuredangle(C,B,A)}$.

(45) $\varnothing_{\mathbb{O}}(A, B, C) = -\frac{|C-A|}{\sin \measuredangle(A,B,C)}$. The theorem is a consequence of (44).

Now we state the proposition:

(46) $\varnothing_{\mathbb{O}}(A, B, C) = \varnothing_{\mathbb{O}}(B, C, A)$.

Let us assume that A, B, C form a triangle. Now we state the propositions:

(47) $\varnothing_{\mathbb{O}}(A, B, C) = -\varnothing_{\mathbb{O}}(B, A, C)$. The theorem is a consequence of (43).

(48) $\varnothing_{\mathbb{O}}(A, B, C) = -\varnothing_{\mathbb{O}}(A, C, B)$. The theorem is a consequence of (42) and (47).

(49) $\varnothing_{\mathbb{O}}(A, B, C) = -\varnothing_{\mathbb{O}}(C, B, A)$. The theorem is a consequence of (48) and (42).

6. Some Identities of a Triangle

Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$.

Let us assume that A, B, C form a triangle. Now we state the propositions:

(50) (i) $|A - B| = \varnothing_{\mathbb{O}}(A, B, C) \cdot \sin \measuredangle(A, C, B)$, and

(ii) $|B - C| = \varnothing_{\mathbb{O}}(A, B, C) \cdot \sin \measuredangle(B, A, C)$, and

(iii) $|C - A| = \varnothing_{\mathbb{O}}(A, B, C) \cdot \sin \measuredangle(C, B, A)$.

The theorem is a consequence of (42).

(51) $|A-B| = \varnothing_{\mathbb{O}}(A, B, C)\cdot 4\cdot \sin \frac{\measuredangle(A,C,B)}{3} \cdot \sin(\frac{\pi}{3} + \frac{\measuredangle(A,C,B)}{3}) \cdot \sin(\frac{\pi}{3} - \frac{\measuredangle(A,C,B)}{3})$. The theorem is a consequence of (29).

Let us consider points A, B, C, P of $\mathcal{E}^2_{\mathrm{T}}$. Now we state the propositions:

(52) Suppose A, B, P are mutually different and $\measuredangle(P, B, A) = \frac{\measuredangle(C,B,A)}{3}$ and $\measuredangle(B, A, P) = \frac{\measuredangle(B,A,C)}{3}$ and $\measuredangle(A, P, B) < \pi$. Then $|A - P| \cdot \sin(\pi - (\frac{\measuredangle(C,B,A)}{3} + \frac{\measuredangle(B,A,C)}{3})) = |A - B| \cdot \sin \frac{\measuredangle(C,B,A)}{3}$.

(53) Suppose A, B, P are mutually different and $\measuredangle(P,B,A) = \frac{\measuredangle(C,B,A)}{3}$ and $\measuredangle(B,A,P) = \frac{\measuredangle(B,A,C)}{3}$ and $\measuredangle(A,P,B) < \pi$ and $\frac{\measuredangle(C,B,A)}{3} + \frac{\measuredangle(B,A,C)}{3} + \frac{\measuredangle(A,C,B)}{3} = \frac{\pi}{3}$. Then $|A-P| \cdot \sin(\frac{2\cdot\pi}{3} + \frac{\measuredangle(A,C,B)}{3}) = |A-B| \cdot \sin \frac{\measuredangle(C,B,A)}{3}$.

Now we state the proposition:

(54) Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A, B, C form a triangle and $\measuredangle(C,A,B) < \pi$. Then

(i) $\measuredangle(C,B,A) + \measuredangle(B,A,C) + \measuredangle(A,C,B) = 5 \cdot \pi$, and

(ii) $\measuredangle(C,A,B) + \measuredangle(A,B,C) + \measuredangle(B,C,A) = \pi$.

Let us consider points A, B, C, P of $\mathcal{E}^2_{\mathrm{T}}$. Now we state the propositions:

(55) Suppose A, B, C form a triangle and $\measuredangle(C,B,A) < \pi$ and A, B, P are mutually different and $\measuredangle(P,B,A) = \frac{\measuredangle(C,B,A)}{3}$ and $\measuredangle(B,A,P) = \frac{\measuredangle(B,A,C)}{3}$ and $\measuredangle(A,P,B) < \pi$. Then $|A-P| \cdot \sin(\frac{\pi}{3} - \frac{\measuredangle(A,C,B)}{3}) = |A-B| \cdot \sin \frac{\measuredangle(C,B,A)}{3}$. The theorem is a consequence of (1).

(56) Suppose A, B, C form a triangle and A, B, P form a triangle and $\measuredangle(C,B,A) < \pi$ and $\measuredangle(A,P,B) < \pi$ and $\measuredangle(P,B,A) = \frac{\measuredangle(C,B,A)}{3}$ and $\measuredangle(B,A,P) = \frac{\measuredangle(B,A,C)}{3}$ and $\sin(\frac{\pi}{3} - \frac{\measuredangle(A,C,B)}{3}) \neq 0$. Then $|A-P| = -\varnothing_{\triangleleft}(C,B,A) \cdot 4 \cdot \sin \frac{\measuredangle(A,C,B)}{3} \cdot \sin(\frac{\pi}{3} + \frac{\measuredangle(A,C,B)}{3}) \cdot \sin \frac{\measuredangle(C,B,A)}{3}$. The theorem is a consequence of (53), (29), (50), (13), and (49).

7. Diameter of a Circle

Now we state the propositions:

(57) Let us consider points A, B, C of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A, B, C are mutually different and $C \in \mathcal{L}(A,B)$. Then $|A-B| = |A-C| + |C-B|$.

(58) Let us consider points A, B of $\mathcal{E}^2_{\mathrm{T}}$, real numbers a, b, and a positive real number r. Suppose A, B, $[a,b]$ are mutually different and A, $B \in \mathrm{circle}(a,b,r)$ and $[a,b] \in \mathcal{L}(A,B)$. Then $|A-B| = 2 \cdot r$. The theorem is a consequence of (57).

(59) Let us consider real numbers a, b, a positive real number r, and a subset C of $\mathcal{E}^2$. If $C = \mathrm{circle}(a,b,r)$, then $\varnothing C = 2 \cdot r$.

PROOF: For every points x, y of $\mathcal{E}^2$ such that $x, y \in C$ holds $\rho(x,y) \leqslant 2 \cdot r$ by [11, (22), (67)], [17, (4)], [22, (5)]. For every real number s such that for every points x, y of $\mathcal{E}^2$ such that $x, y \in C$ holds $\rho(x,y) \leqslant s$ holds $2 \cdot r \leqslant s$ by [11, (62)], [4, (12)], [19, (24)], [26, (22)]. $\square$

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