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Some Remarkable Identities Involving Numbers

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Summary. The article focuses on simple identities found for binomials, their divisibility, and basic inequalities. A general formula allowing factorization of the sum of like powers is introduced and used to prove elementary theorems for natural numbers.

Formulas for short multiplication are sometimes referred in English or French as remarkable identities. The same formulas could be found in works concerning polynomial factorization, where there exists no single term for various identities. Their usability is not questionable, and they have been successfully utilized since for ages. For example, in his books published in 1731 (p. 385), Edward Hatton [3] wrote: "Note, that the differences of any two like powers of two quantities, will always be divided by the difference of the quantities without any remainer...".

Despite of its conceptual simplicity, the problem of factorization of sums/differences of two like powers could still be analyzed [7], giving new and possibly interesting results [6].

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The notation and terminology used in this paper have been introduced in the following articles: [8], [9], [5], [4], and [2].

From now on a, b, c, d, x, j, k, l, m, n denote natural numbers, p, q, t, z, u, v denote integers, and a_1, b_1, c_1, d_1 denote complex numbers.

Let u, v be even integers. One can check that $u - v$ is even.

Let u be an odd integer. Let us consider k. Observe that u^k is odd.

Let k be a positive natural number and u be an even integer. Let us observe that u^k is even.

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Now we state the propositions:

(1) ${a_1}^2 - {b_1}^2 = (a_1 - b_1) \cdot (a_1 + b_1)$.

(2) $(2 \cdot a_1 + 1)^2 + (2 \cdot {a_1}^2 + 2 \cdot a_1)^2 = (2 \cdot {a_1}^2 + 2 \cdot a_1 + 1)^2$.

(3) ${a_1}^2 + a_1 \cdot b_1 + {b_1}^2 = \frac{3 \cdot (a_1+b_1)^2 + (a_1-b_1)^2}{4}$.

(4) If a is odd, then there exists b such that $a^2 + b^2 = (b+1)^2$. The theorem is a consequence of (2).

(5) $\frac{({a_1}^m+{b_1}^m) \cdot ({a_1}^n-{b_1}^n) + ({a_1}^n+{b_1}^n) \cdot ({a_1}^m-{b_1}^m)}{2} = {a_1}^{m+n} - {b_1}^{m+n}$.

(6) If $a^m + b^m \leqslant c^m$, then $a \leqslant c$.

(7) Suppose $(a_1 + b_1)^{n+1} = {a_1}^{n+1} + {b_1}^{n+1} + a_1 \cdot b_1 \cdot c_1$. Then $(a_1 + b_1)^{n+2} = {a_1}^{n+2} + {b_1}^{n+2} + a_1 \cdot b_1 \cdot ({a_1}^n + {b_1}^n + c_1 \cdot (a_1 + b_1))$.

(8) $\frac{({a_1}^m+{b_1}^m) \cdot ({a_1}^n+{b_1}^n) + ({a_1}^n-{b_1}^n) \cdot ({a_1}^m-{b_1}^m)}{2} = {a_1}^{m+n} + {b_1}^{m+n}$.

(9) ${a_1}^{m+1} + {b_1}^{m+1} = \frac{({a_1}^m+{b_1}^m) \cdot (a_1+b_1) + (a_1-b_1) \cdot ({a_1}^m-{b_1}^m)}{2}$. The theorem is a consequence of (8).

(10) $(a - b) \cdot (a^m - b^m) \geqslant 0$.

(11) $a^{m+1} + b^{m+1} \geqslant \frac{(a^m+b^m) \cdot (a+b)}{2}$. The theorem is a consequence of (9) and (10).

(12) If $a^m + b^m \leqslant c^m$, then there exists x such that $a^m + b^m \leqslant (a + x)^m$. The theorem is a consequence of (6).

(13) ${a_1}^{m+1} - {b_1}^{m+1} = \frac{({a_1}^m+{b_1}^m) \cdot (a_1-b_1) + (a_1+b_1) \cdot ({a_1}^m-{b_1}^m)}{2}$. The theorem is a consequence of (5).

(14) $a^{m+1} - b^{m+1} = \frac{(a-b) \cdot (t \cdot (a+b) + a^m + b^m)}{2}$ if and only if $a^m - b^m = (a - b) \cdot t$.

(15) $(\frac{{c_1}^n}{2} + \frac{c_1}{2})^2 - (\frac{{c_1}^n}{2} - \frac{c_1}{2})^2 = {c_1}^{n+1}$.

(16) $a^3 - b^3 = \frac{(a-b) \cdot ((a+b) \cdot (a+b) + a^2 + b^2)}{2}$. The theorem is a consequence of (1) and (14).

(17) If $c^m \geqslant a^m + b^m$ and $a > 0$ and $b > 0$, then $c^{m+1} > a^{m+1} + b^{m+1}$. The theorem is a consequence of (6).

(18) If $c^m \geqslant a^m + b^m$ and $a > 0$ and $b > 0$ and $k > 0$, then $c^{k+m} > a^{k+m} + b^{k+m}$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv c^{\$_1+m+1} > a^{\$_1+m+1} + b^{\$_1+m+1}$ and $a > 0$ and $b > 0$. $\mathcal{P}[0]$. If $\mathcal{P}[x]$, then $\mathcal{P}[x + 1]$. For every j, $\mathcal{P}[j]$ from [1, Sch. 2]. □

(19) If $c^m \geqslant a^m + b^m$, then $c^{k+m} \geqslant a^{k+m} + b^{k+m}$. The theorem is a consequence of (18).

(20) If $c^n > a^n + b^n$, then $c^{k+n} > a^{k+n} + b^{k+n}$.

PROOF: Consider m such that $n = 1 + m$. Define $\mathcal{P}$[natural number] $\equiv c^{\$_1+m+1} > a^{\$_1+m+1} + b^{\$_1+m+1}$. For every j, $\mathcal{P}[j]$ from [1, Sch. 2]. □

(21) $a_1{}^{m+2} - b_1{}^{m+2} = (a_1{}^{m+1} + b_1{}^{m+1}) \cdot (a_1 - b_1) + a_1 \cdot b_1 \cdot (a_1{}^{m} - b_1{}^{m})$.

(22) $a_1{}^{m+2} + b_1{}^{m+2} = (a_1{}^{m+1} - b_1{}^{m+1}) \cdot (a_1 - b_1) + a_1 \cdot b_1 \cdot (a_1{}^{m} + b_1{}^{m})$.

(23) $a^{2 \cdot m+2} - b^{2 \cdot m+2} = \frac{(a^2 - b^2) \cdot (c \cdot (a^2 + b^2) + a^{2 \cdot m} + b^{2 \cdot m})}{2}$ if and only if $a^{2 \cdot m} - b^{2 \cdot m} = (a^2 - b^2) \cdot c$. The theorem is a consequence of (14).

(24) $a_1{}^{2 \cdot m+3} + b_1{}^{2 \cdot m+3} = (a_1{}^{2 \cdot m+2} + b_1{}^{2 \cdot m+2}) \cdot (a_1 + b_1) - a_1 \cdot b_1 \cdot (a_1{}^{2 \cdot m+1} + b_1{}^{2 \cdot m+1})$.

(25) If $a_1{}^{m} - b_1{}^{m} = (a_1 - b_1) \cdot k$, then $a_1{}^{m+2} - b_1{}^{m+2} = (a_1{}^{m+1} + b_1{}^{m+1} + a_1 \cdot b_1 \cdot k) \cdot (a_1 - b_1)$. The theorem is a consequence of (21).

(26) If $a_1{}^{m+2} - b_1{}^{m+2} = (a_1{}^{m+1} + b_1{}^{m+1} + a_1 \cdot b_1 \cdot k) \cdot (a_1 - b_1)$ and $a_1 \cdot b_1 \neq 0$, then $a_1{}^{m} - b_1{}^{m} = (a_1 - b_1) \cdot k$. The theorem is a consequence of (21).

(27) If $b > 0$ and $a > b$, then $(a^n - b^n) \cdot (a + b) = (a^n + b^n) \cdot (a - b)$ iff $n = 1$.

(28) If $n > 1$ and $b > 0$ and $a > b$, then $(a^n - b^n) \cdot (a + b) > a^{n+1} - b^{n+1}$.

(29) If $n > 0$ and $a > b$, then $(a^n + b^n) \cdot (a - b) \leqslant a^{n+1} - b^{n+1}$.

(30) If $p + q \mid p \cdot u + q \cdot v$, then $p + q \mid p \cdot (u + z) + q \cdot (v + z)$.

(31) $p + q \mid p \cdot (t \cdot (p + q) + z) + q \cdot z$. The theorem is a consequence of (30).

(32) If $p + q \mid u - v$, then $p + q \mid p \cdot (u + t) + q \cdot (v + t)$. The theorem is a consequence of (30).

(33) $a - b \mid a^n - b^n$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv a - b \mid a^{\$_1} - b^{\$_1}$. If $\mathcal{P}[x]$, then $\mathcal{P}[x+1]$. For every m, $\mathcal{P}[m]$ from [1, Sch. 2]. $\square$

(34) $a^2 - b^2 \mid a^{2 \cdot m} - b^{2 \cdot m}$. The theorem is a consequence of (33).

(35) $a + b \mid a^{2 \cdot m+1} + b^{2 \cdot m+1}$. The theorem is a consequence of (21), (34), and (1).

(36) $a + b \mid a^{2 \cdot m} - b^{2 \cdot m}$. The theorem is a consequence of (34).

(37) If $a + b \mid a^n - b^n$, then $a + b \mid a^{n+1} + b^{n+1}$. The theorem is a consequence of (32).

(38) (i) $a + b \mid a^n + b^n$, or

(ii) $a + b \mid a^n - b^n$.

The theorem is a consequence of (35) and (34).

(39) If $a \geqslant b$ and $c^n - b^n = a^n$, then $\gcd(c - b, a^n) = c - b$ and $\gcd(c - a, b^n) = c - a$. The theorem is a consequence of (6) and (33).

(40) If a and b are relatively prime and $a + b \mid a \cdot c + b \cdot d$, then $a + b \mid c - d$. The theorem is a consequence of (32).

(41) If $a \cdot b$ and $c \cdot d$ are relatively prime, then a and c are relatively prime.

(42) Suppose $a > 0$ and $b > 0$ and $a^n + b^n = c^n$. Then there exists j and there exists k and there exists l such that $j^n + k^n = l^n$ and j and k are

relatively prime and j and l are relatively prime and k and l are relatively prime and $a = (\gcd(a,b)) \cdot j$ and $b = (\gcd(a,b)) \cdot k$ and $c = (\gcd(a,b)) \cdot l$.

(43) If $a > 0$, then $a^{n+2} + a^{n+2} \neq b^{n+2}$. The theorem is a consequence of (42).

(44) If $x > 0$ and $b < c$ and $a + b^2 = c^2$, then $a + (b+x)^2 < (c+x)^2$.

(45) If $q < 0$ and $b < c$ and $a^2 + b^2 = c^2$, then $a^2 + (b+q)^2 > (c+q)^2$.

(46) If $x > 0$ and $a^2 + b^2 = (b+1)^2$, then $a^2 + (b-x)^2 > (b+1-x)^2$. The theorem is a consequence of (45).

(47) If $a \geqslant 1$ and $(a+1)^2 + (a+1+x)^2 \leqslant (a+1+x+1)^2$, then $a^2 + (a+x)^2 < (a+x+1)^2$.

(48) If $a \geqslant 1$ and $a^2 + (a+x)^2 \geqslant (a+x+1)^2$,
then $(a+l+1)^2 + (a+l+1+x)^2 > (a+l+1+x+1)^2$.
Proof: Define $\mathcal{P}$[natural number] $\equiv (a + \$_1 + 1)^2 + (a + \$_1 + 1 + x)^2 > (a + \$_1 + 1 + x + 1)^2$. $\mathcal{P}[0]$. If $\mathcal{P}[k]$, then $\mathcal{P}[k+1]$. For every j, $\mathcal{P}[j]$ from [1, Sch. 2]. $\square$

(49) $a \geqslant 3$ if and only if $a^2 + a^2 > (a+1)^2$.
Proof: If $a \geqslant 3$, then $a^2 + a^2 > (a+1)^2$ by [1, (10)], [4, (81)], (48). $\square$

(50) $2^{3+m} + 2^{3+m} < 3^{3+m}$.

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