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Term Context

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Summary. Two construction functors: simple term with a variable and compound term with an operation and argument terms and schemes of term induction are introduced. The degree of construction as a number of used operation symbols is defined. Next, the term context is investigated. An x-context is a term which includes a variable x once only. The compound term is x-context iff the argument terms include an x-context once only. The context induction is shown and used many times. As a key concept, the context substitution is introduced. Finally, the translations and endomorphisms are expressed by context substitution.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [3], [4], [6], [43], [24], [22], [26], [53], [33], [45], [27], [28], [29], [8], [25], [9], [51], [39], [46], [47], [41], [48], [23], [10], [11], [49], [36], [37], [12], [13], [14], [15], [31], [50], [34], [55], [56], [16], [38], [54], [17], [18], [19], [20], [21], [35], and [32].

1. Preliminaries

Let Σ be a non empty non void many sorted signature, $\mathfrak{A}$ be a non-empty algebra over Σ, and σ be a sort symbol of Σ.

An element of $\mathfrak{A}$ from σ is an element of (the sorts of $\mathfrak{A}$)(σ). From now on a, b denote objects, I, J denote sets, f denotes a function, R denotes a binary relation, i, j, n denote natural numbers, m denotes an element of $\mathbb{N}$, Σ denotes a non empty non void many sorted signature, σ, σ_1, σ_2 denote sort symbols of Σ, o denotes an operation symbol of Σ, X denotes a non-empty many sorted set

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indexed by the carrier of Σ, x, x_1, x_2 denote elements of $X(\sigma)$, x_{11} denotes an element of $X(\sigma_1)$, T denotes a free in itself including Σ-terms over X algebra over Σ with all variables and inheriting operations, g denotes a translation in $\mathfrak{F}_\Sigma(X)$ from σ_1 into σ_2, and h denotes an endomorphism of $\mathfrak{F}_\Sigma(X)$.

Let us consider Σ and X. Let T be an including Σ-terms over X algebra over Σ with all variables and ρ be an element of T. The functor ${}^@\rho$ yielding an element of $\mathfrak{F}_\Sigma(X)$ is defined by the term

(Def. 1) ρ.

Let us consider T. Observe that every element of T is finite and every set which is natural-membered is also $\subseteq$-linear.

In the sequel ρ, ρ_1, ρ_2 denote elements of T and τ, τ_1, τ_2 denote elements of $\mathfrak{F}_\Sigma(X)$.

Let us consider Σ. Let $\mathfrak{A}$ be an algebra over Σ. Let us consider a. We say that $a \in \mathfrak{A}$ if and only if

(Def. 2) $a \in \bigcup$(the sorts of $\mathfrak{A}$).

Let us consider b. We say that b is a-different if and only if

(Def. 3) $b \neq a$.

Let I be a non trivial set. Note that there exists an element of I which is a-different.

Now we state the proposition:

(1) Let us consider trees τ, τ_1 and finite sequences p, q of elements of $\mathbb{N}$. Suppose

(i) $p \in \tau$, and

(ii) $q \in \tau$ with-replacement(p, τ_1).

Then

(iii) if $p \npreceq q$, then $q \in \tau$, and

(iv) for every finite sequence ρ of elements of $\mathbb{N}$ such that $q = p \frown \rho$ holds $\rho \in \tau_1$.

Proof: If $p \npreceq q$, then $q \in \tau$ by [17, (1)]. □

Let R be a finite binary relation. Let us consider a. Let us note that $\mathrm{Coim}(R, a)$ is finite.

Let us consider finite sequences p, q, ρ. Now we state the propositions:

(2) If $p \frown q \preceq \rho$, then $p \preceq \rho$.

(3) If $p \frown q \preceq p \frown \rho$, then $q \preceq \rho$.

Now we state the propositions:

(4) Let us consider finite sequences p, q. Suppose $i \leqslant \mathrm{len}\, p$. Then $(p \frown q) \restriction \mathrm{Seg}\, i = p \restriction \mathrm{Seg}\, i$.

(5) Let us consider finite sequences p, q, ρ. If $q \preceq p \frown \rho$, then $q \preceq p$ or $p \preceq q$. The theorem is a consequence of (4).

Let us consider Σ. We say that Σ is sufficiently rich if and only if

(Def. 4) There exists o such that $\sigma \in \operatorname{rng} \operatorname{Arity}(o)$.

We say that Σ is growable if and only if

(Def. 5) There exists τ such that $\operatorname{height} \operatorname{dom} \tau = n$.

Let us consider n. We say that Σ is n-ary operation including if and only if

(Def. 6) There exists o such that $\operatorname{len} \operatorname{Arity}(o) = n$.

Let us note that there exists a non empty non void many sorted signature which is n-ary operation including and there exists a non empty non void many sorted signature which is sufficiently rich.

Let us consider R. We say that R is nontrivial if and only if

(Def. 7) If $I \in \operatorname{rng} R$, then I is not trivial.

We say that R is infinite-yielding if and only if

(Def. 8) If $I \in \operatorname{rng} R$, then I is infinite.

Let us observe that every binary relation which is nontrivial is also non-empty and every binary relation which is infinite-yielding is also nontrivial.

Let I be a set. Observe that there exists a many sorted set indexed by I which is infinite-yielding and there exists a finite sequence which is infinite-yielding.

Let I be a non empty set, f be a nontrivial many sorted set indexed by I, and a be an element of I. Let us note that $f(a)$ is non trivial.

Let f be an infinite-yielding many sorted set indexed by I. Note that $f(a)$ is infinite.

Let us consider Σ, X, and o. Let us note that every element of $\operatorname{Args}(o, \mathfrak{F}_\Sigma(X))$ is decorated tree yielding.

In the sequel Y denotes an infinite-yielding many sorted set indexed by the carrier of Σ, y, y_1 denote elements of $Y(\sigma)$, y_{11} denotes an element of $Y(\sigma_1)$, Q denotes a free in itself including Σ-terms over Y algebra over Σ with all variables and inheriting operations, q, q_1 denote elements of $\operatorname{Args}(o, \mathfrak{F}_\Sigma(Y))$, u, $u1$, $u2$ denote elements of Q, v, v_1, v_2 denote elements of $\mathfrak{F}_\Sigma(Y)$, Z denotes a nontrivial many sorted set indexed by the carrier of Σ, z, z_1 denote elements of $Z(\sigma)$, l, $l1$ denote elements of $\mathfrak{F}_\Sigma(Z)$, R denotes a free in itself including Σ-terms over Z algebra over Σ with all variables and inheriting operations, and k, $k1$ denote elements of $\operatorname{Args}(o, \mathfrak{F}_\Sigma(Z))$.

Let p be a finite sequence. Note that $p \frown \emptyset$ reduces to p and $\emptyset \frown p$ reduces to p.

Let I be a finite sequence-membered set. The functor $p \frown I$ yielding a set is defined by the term

(Def. 9) $\{p \frown q$, where q is an element of $I : q \in I\}$.

Let us observe that $p \frown I$ is finite sequence-membered.

Let f be a finite sequence and E be an empty set. One can verify that $f \frown E$ reduces to E.

Let p be a decorated tree yielding finite sequence. Let us consider a. Let us note that $p(a)$ is relation-like and every set which is tree-like is also finite sequence-membered.

Let p be a decorated tree yielding finite sequence. Let us consider a. One can check that $\mathrm{dom}(p(a))$ is finite sequence-membered.

Let τ, τ_1 be trees. One can check that τ_1 with-replacement$(\varepsilon_{\mathbb{N}}, \tau)$ reduces to τ.

Let d, d_1 be decorated trees. One can check that d_1 with-replacement$(\varepsilon_{\mathbb{N}}, d)$ reduces to d.

Now we state the proposition:

(6) Let us consider finite sequences ξ, w of elements of $\mathbb{N}$, tree yielding finite sequences p, q, and trees d, τ. Suppose

(i) $i < \mathrm{len}\, p$, and

(ii) $\xi = \langle i \rangle \frown w$, and

(iii) $d = p(i+1)$, and

(iv) $q = p +\cdot (i+1, d\,\text{with-replacement}(w, \tau))$, and

(v) $\xi \in \overbrace{p}$.

Then $\overbrace{p}$ with-replacement$(\xi, \tau) = \overbrace{q}$. The theorem is a consequence of (2).

Let F be a function yielding function and f be a function. Let us consider a. Note that $F +\cdot (a, f)$ is function yielding.

Now we state the propositions:

(7) Let us consider a function yielding function F and a function f. Then $\mathrm{dom}_\kappa(F +\cdot (a, f))(\kappa) = \mathrm{dom}_\kappa F(\kappa) +\cdot (a, \mathrm{dom}\, f)$.

(8) Let us consider finite sequences ξ, w of elements of $\mathbb{N}$, decorated tree yielding finite sequences p, q, and decorated trees d, τ. Suppose

(i) $i < \mathrm{len}\, p$, and

(ii) $\xi = \langle i \rangle \frown w$, and

(iii) $d = p(i+1)$, and

(iv) $q = p +\cdot (i+1, d\,\text{with-replacement}(w, \tau))$, and

(v) $\xi \in \overbrace{\mathop{\mathrm{dom}}_{\kappa} p(\kappa)}$.

Then $(a\text{-tree}(p))$ with-replacement$(\xi, \tau) = a\text{-tree}(q)$. The theorem is a consequence of (7), (6), (2), and (3).

(9) Let us consider a set a and a decorated tree yielding finite sequence w. Then $\mathrm{dom}(a\text{-tree}(w)) = \{\emptyset\} \cup \bigcup \{\langle i \rangle \frown \mathrm{dom}(w(i+1)) : i < \mathrm{len}\, w\}$. Proof: Set $\mathfrak{A} = \{\langle i \rangle \frown \mathrm{dom}(w(i+1)) : i < \mathrm{len}\, w\}$. $\mathrm{dom}(a\text{-tree}(w)) \subseteq \{\emptyset\} \cup \bigcup \mathfrak{A}$ by [20, (11)]. $\square$

Let p be a decorated tree yielding finite sequence. Let us consider a and I. Note that $p(a)^{-1}(I)$ is finite sequence-membered.

Now we state the proposition:

(10) Let us consider a finite sequence-membered set I and a finite sequence p. Then $\overline{\overline{p \frown I}} = \overline{\overline{I}}$. Proof: Define $\mathcal{F}(\text{element of } I) = p \frown \$_1$. Consider f such that $\operatorname{dom} f = I$ and for every element q of I such that $q \in I$ holds $f(q) = \mathcal{F}(q)$ from [7, Sch. 2]. $\operatorname{rng} f = p \frown I$. f is one-to-one by [22, (33)]. $\square$

Let I be a finite finite sequence-membered set and p be a finite sequence. Note that $p \frown I$ is finite.

Now we state the proposition:

(11) Let us consider finite sequence-membered sets I, J and finite sequences p, q. Suppose

(i) $\operatorname{len} p = \operatorname{len} q$, and

(ii) $p \neq q$.

Then $p \frown I$ misses $q \frown J$.

Let us consider i. Let us note that $\overline{\overline{i}}$ reduces to i. Let us consider j. We identify $i + j$ with $i + j$.

The scheme *CardUnion* deals with a unary functor $\mathcal{I}$ yielding a set and a finite sequence f of elements of $\mathbb{N}$ and states that

(Sch. 1) $\overline{\overline{\bigcup\{\mathcal{I}(i) : i < \operatorname{len} f\}}} = \sum f$

provided

- for every i and j such that $i < \operatorname{len} f$ and $j < \operatorname{len} f$ and $i \neq j$ holds $\mathcal{I}(i)$ misses $\mathcal{I}(j)$ and
- for every i such that $i < \operatorname{len} f$ holds $\overline{\overline{\mathcal{I}(i)}} = f(i+1)$.

Let f be a finite sequence. Note that $\{f\}$ is finite sequence-membered.

Now we state the propositions:

(12) Let us consider finite sequences f, g. Then $f \frown \{g\} = \{f \frown g\}$.

(13) Let us consider finite sequence-membered sets I, J and a finite sequence f. Then $I \subseteq J$ if and only if $f \frown I \subseteq f \frown J$.

In the sequel c, c_1, c_2 denote sets and d, d_1 denote decorated trees.

Now we state the proposition:

(14) Leaves(the elementary tree of 0) $= \{\emptyset\}$.

Let us note that sethood property holds for trees.

Now we state the propositions:

(15) Let us consider a non empty tree yielding finite sequence p. Then $\text{Leaves}(\widehat{p}) = \{\langle i \rangle \frown q$, where q is a finite sequence of elements of $\mathbb{N}, d$ is a tree $: q \in \text{Leaves}(d)$ and $i + 1 \in \operatorname{dom} p$ and $d = p(i+1)\}$.

Proof: Set i_0 = the element of $\operatorname{dom} p$. Leaves($\widehat{p}$) $\subseteq \{\langle i \rangle \frown q$, where q is a finite sequence of elements of $\mathbb{N}, d$ is a tree : $q \in$ Leaves(d) and $i+1 \in \operatorname{dom} p$ and $d = p(i+1)\}$ by [13, (11), (13)], [52, (25)], [17, (1)].□

(16) Leaves(the root tree of c) $= \{c\}$.

(17) $\operatorname{dom} d \subseteq \operatorname{dom} d_{c \leftarrow d_1}$.

Let us consider c and d. Observe that (the root tree of $c)_{c \leftarrow d}$ reduces to d.

Now we state the proposition:

(18) Suppose $c_1 \neq c_2$. Then (the root tree of $c_1)_{c_2 \leftarrow d}$ = the root tree of c_1. Proof: dom(the root tree of $c_1)_{c_2 \leftarrow d}$ = dom(the root tree of c_1) by [20, (3)], [17, (29)], [40, (15)]. □

Let f be a non empty function yielding function. Note that $\operatorname{dom}_\kappa f(\kappa)$ is non empty and $\operatorname{rng}_\kappa f(\kappa)$ is non empty.

Now we state the proposition:

(19) Let us consider non empty decorated tree yielding finite sequences p, q. Suppose

(i) $\operatorname{dom} q = \operatorname{dom} p$, and

(ii) for every i and d_1 such that $i \in \operatorname{dom} p$ and $d_1 = p(i)$ holds $q(i) = d_{1c \leftarrow d}$.

Then (b-tree(p))$_{c \leftarrow d}$ = b-tree(q). Proof: Leaves($\widehat{\operatorname{dom}_\kappa p(\kappa)}$) $= \{\langle i \rangle ^\frown q$, where q is a finite sequence of elements of $\mathbb{N}, d$ is a tree : $q \in$ Leaves(d) and $i+1 \in \operatorname{dom}(\operatorname{dom}_\kappa p(\kappa))$ and $d = (\operatorname{dom}_\kappa p(\kappa))(i+1)\}$. dom($b$-tree($p$))$_{c \leftarrow d}$ = dom(b-tree(q)) by [17, (22)], [13, (11), (13)], [52, (25)]. □

Let us consider Σ and σ. Let $\mathfrak{A}$ be a non empty algebra over Σ and a be an element of $\mathfrak{A}$. We say that a is σ-sort if and only if

(Def. 10) $a \in$ (the sorts of $\mathfrak{A})(\sigma)$.

Let $\mathfrak{A}$ be a non-empty algebra over Σ. One can verify that there exists an element of $\mathfrak{A}$ which is σ-sort and every element of (the sorts of $\mathfrak{A})(\sigma)$ is σ-sort.

Let $\mathfrak{A}$ be a non empty algebra over Σ. Assume $\mathfrak{A}$ is disjoint valued. Let a be an element of $\mathfrak{A}$. The functor the sort of a yielding a sort symbol of Σ is defined by

(Def. 11) $a \in$ (the sorts of $\mathfrak{A})(it)$.

Now we state the propositions:

(20) Let us consider a disjoint valued non-empty algebra $\mathfrak{A}$ over Σ and a σ-sort element a of $\mathfrak{A}$. Then the sort of $a = \sigma$.

(21) Let us consider a disjoint valued non empty algebra $\mathfrak{A}$ over Σ. Then every element of $\mathfrak{A}$ is (the sort of a)-sort.

(22) The sort of ${}^{@}\rho$ = the sort of ρ.

(23) Let us consider an element ρ of (the sorts of T)(σ). Then the sort of $\rho = \sigma$.

(24) Let us consider a term u of Σ over X. Suppose $\tau = u$. Then the sort of $\tau =$ the sort of u.

Let us consider Σ, X, o, and T. One can verify that every element of $\mathrm{Args}(o, T)$ is ($\bigcup$(the sorts of T))-valued.

Now we state the proposition:

(25) Let us consider an element q of $\mathrm{Args}(o, T)$. Suppose $i \in \mathrm{dom}\, q$. Then the sort of $q_i = \mathrm{Arity}(o)_i$.

Let us consider Σ. Let $\mathfrak{A}$, $\mathcal{B}$ be non-empty algebras over Σ and f be a many sorted function from $\mathfrak{A}$ into $\mathcal{B}$. Assume $\mathfrak{A}$ is disjoint valued. Let a be an element of $\mathfrak{A}$. The functor $f(a)$ yielding an element of $\mathcal{B}$ is defined by the term

(Def. 12) f(the sort of a)(a).

Let us consider a disjoint valued non-empty algebra $\mathfrak{A}$ over Σ, a non-empty algebra $\mathcal{B}$ over Σ, a many sorted function f from $\mathfrak{A}$ into $\mathcal{B}$, and an element a of (the sorts of $\mathfrak{A}$)(σ). Now we state the propositions:

(26) $f(a) = f(\sigma)(a)$.

(27) $f(a)$ is an element of (the sorts of $\mathcal{B}$)(σ). The theorem is a consequence of (26).

Now we state the propositions:

(28) Let us consider disjoint valued non-empty algebras $\mathfrak{A}$, $\mathcal{B}$ over Σ, a many sorted function f from $\mathfrak{A}$ into $\mathcal{B}$, and an element a of $\mathfrak{A}$. Then the sort of $f(a) =$ the sort of a.

(29) Let us consider disjoint valued non-empty algebras $\mathfrak{A}$, $\mathcal{B}$ over Σ, a non-empty algebra $\mathcal{C}$ over Σ, a many sorted function f from $\mathfrak{A}$ into $\mathcal{B}$, a many sorted function g from $\mathcal{B}$ into $\mathcal{C}$, and an element a of $\mathfrak{A}$. Then $(g \circ f)(a) = g(f(a))$. The theorem is a consequence of (28).

(30) Let us consider a disjoint valued non-empty algebra $\mathfrak{A}$ over Σ, a non-empty algebra $\mathcal{B}$ over Σ, and many sorted functions f_1, f_2 from $\mathfrak{A}$ into $\mathcal{B}$. If for every element a of $\mathfrak{A}$, $f_1(a) = f_2(a)$, then $f_1 = f_2$. The theorem is a consequence of (26).

Let us consider Σ. Let $\mathfrak{A}$, $\mathcal{B}$ be algebras over Σ. Assume there exists a many sorted function h from $\mathfrak{A}$ into $\mathcal{B}$ such that h is a homomorphism of $\mathfrak{A}$ into $\mathcal{B}$.

A homomorphism from $\mathfrak{A}$ to $\mathcal{B}$ is a many sorted function from $\mathfrak{A}$ into $\mathcal{B}$ and is defined by

(Def. 13) *it* is a homomorphism of $\mathfrak{A}$ into $\mathcal{B}$.

Now we state the proposition:

(31) Let us consider a many sorted function h from $\mathfrak{F}_\Sigma(X)$ into T. Then h is a homomorphism from $\mathfrak{F}_\Sigma(X)$ to T if and only if h is a homomorphism of

$\mathfrak{F}_\Sigma(X)$ into T.

Let us consider Σ, X, and T. Observe that the functor the canonical homomorphism of T yields a homomorphism from $\mathfrak{F}_\Sigma(X)$ to T. Let us consider ρ. One can check that (the canonical homomorphism of T)$({}^{@}\rho)$ reduces to ρ.

Now we state the proposition:

(32) Suppose $\tau_2 =$ (the canonical homomorphism of T)(τ_1). Then (the canonical homomorphism of T)$(\tau_1) =$ (the canonical homomorphism of T)(τ_2). The theorem is a consequence of (22) and (28).

2. Constructing Terms

In the sequel w denotes an element of $\mathrm{Args}(o, T)$ and p, p_1 denote elements of $\mathrm{Args}(o, \mathfrak{F}_\Sigma(X))$.

Let us consider Σ, X, σ, and x. The functor x-term yielding an element of (the sorts of $\mathfrak{F}_\Sigma(X)$)(σ) is defined by the term

(Def. 14) The root tree of $\langle x, \sigma \rangle$.

Let us consider o and p. The functor o-term p yielding an element of $\mathfrak{F}_\Sigma(X)$ from the result sort of o is defined by the term

(Def. 15) $\langle o$, the carrier of $\Sigma\rangle$-tree(p).

Now we state the propositions:

(33) The sort of x-term $= \sigma$.

(34) The sort of o-term $p =$ the result sort of o. The theorem is a consequence of (24).

(35) Let us consider an object i. Then $i \in (\mathrm{FreeGenerator}(T))(\sigma)$ if and only if there exists x such that $i = x$-term.

Let us consider Σ, X, σ, and x. Let us note that x-term is non compound.

Let us consider o and p. One can check that o-term p is compound and (the result sort of o)-sort.

Now we state the propositions:

(36) (i) there exists σ and there exists x such that $\tau = x$-term, or

(ii) there exists o and there exists p such that $\tau = o$-term p.

(37) If τ is not compound, then there exists σ and there exists x such that $\tau = x$-term.

(38) If τ is compound, then there exists o and there exists p such that $\tau = o$-term p.

(39) x-term $\neq o$-term p.

Let us consider Σ. Let X be a non-empty many sorted set indexed by the carrier of Σ. Note that there exists an element of $\mathfrak{F}_\Sigma(X)$ which is compound.

Let us consider X. Let e be a compound element of $\mathfrak{F}_\Sigma(X)$. Let us note that the functor main-constr e yields an operation symbol of Σ. One can check that the functor args e yields an element of Args(main-constr $e, \mathfrak{F}_\Sigma(X)$). Now we state the propositions:

(40) $\text{args}(x\text{-term}) = \emptyset$.

(41) Let us consider a compound element τ of $\mathfrak{F}_\Sigma(X)$.
Then $\tau = \text{main-constr}\,\tau\text{-term}\,\text{args}\,\tau$. The theorem is a consequence of (38).

(42) $x\text{-term} \in T$.

Let us consider Σ, X, T, σ, and x. Note that (the canonical homomorphism of T)(x-term) reduces to x-term.

The scheme *TermInd* deals with a unary predicate $\mathcal{P}$ and a non empty non void many sorted signature Σ and a non-empty many sorted set $\mathcal{X}$ indexed by the carrier of Σ and an element τ of $\mathfrak{F}_\Sigma(\mathcal{X})$ and states that

(Sch. 2) $\mathcal{P}[\tau]$

provided

- for every sort symbol σ of Σ and for every element x of $\mathcal{X}(\sigma)$, $\mathcal{P}[x\text{-term}]$ and
- for every operation symbol o of Σ and for every element p of $\text{Args}(o, \mathfrak{F}_\Sigma(\mathcal{X}))$ such that for every element τ of $\mathfrak{F}_\Sigma(\mathcal{X})$ such that $\tau \in \text{rng}\,p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\,p]$.

The scheme *TermAlgebraInd* deals with a unary predicate $\mathcal{P}$ and a non empty non void many sorted signature Σ and a non-empty many sorted set $\mathcal{X}$ indexed by the carrier of Σ and a free in itself including Σ-terms over $\mathcal{X}$ algebra $\mathfrak{A}$ over Σ with all variables and inheriting operations and an element τ of $\mathfrak{A}$ and states that

(Sch. 3) $\mathcal{P}[\tau]$

provided

- for every sort symbol σ of Σ and for every element x of $\mathcal{X}(\sigma)$ and for every element ρ of $\mathfrak{A}$ such that $\rho = x\text{-term}$ holds $\mathcal{P}[\rho]$ and
- for every operation symbol o of Σ and for every element p of $\text{Args}(o, \mathfrak{F}_\Sigma(\mathcal{X}))$ and for every element ρ of $\mathfrak{A}$ such that $\rho = o\text{-term}\,p$ and for every element τ of $\mathfrak{A}$ such that $\tau \in \text{rng}\,p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[\rho]$.

3. Construction Degree

Let us consider Σ, X, T, and ρ. The functors: the construction degree of ρ and height ρ yielding natural numbers are defined by terms,

(Def. 16) $\overline{\overline{\rho^{-1}(\alpha \times \{\beta\})}}$, where α is the carrier' of Σ and β is the carrier of Σ,

(Def. 17) height dom ρ,

respectively. We introduce $\deg \rho$ as a synonym of the construction degree of ρ.

Now we state the propositions:

(43) $\deg {}^{@}\rho = \deg \rho$.

(44) height ${}^{@}\rho =$ height ρ.

(45) height(x-term) $= 0$.

One can verify that every set which is natural-membered is also ordinal-membered and finite-membered.

Let I be a finite natural-membered set. One can verify that $\bigcup I$ is natural.

Let I be a non empty finite natural-membered set. We identify $\bigcup I$ with $\max I$. Now we state the propositions:

(46) (i) $\{\text{height}\,\tau_1 : \tau_1 \in \text{rng}\,p\}$ is natural-membered and finite, and

(ii) $\bigcup\{\text{height}\,\tau : \tau \in \text{rng}\,p\}$ is a natural number.

Proof: Set $I = \{\text{height}\,\tau : \tau \in \text{rng}\,p\}$. I is natural-membered. Define $\mathcal{F}(\text{element of } \mathfrak{F}_\Sigma(X)) = \text{height}\,\$_1$. $\{\mathcal{F}(\tau_1) : \tau_1 \in \text{rng}\,p\}$ is finite from [44, Sch. 21]. □

(47) Suppose $\text{Arity}(o) \neq \emptyset$ and $n = \bigcup\{\text{height}\,\tau_1 : \tau_1 \in \text{rng}\,p\}$. Then height($o$-term p) $= n + 1$. Proof: Set $I = \{\text{height}\,\tau_1 : \tau_1 \in \text{rng}\,p\}$. I is natural-membered. Define $\mathcal{F}(\text{element of } \mathfrak{F}_\Sigma(X)) = \text{height}\,\$_1$. $\{\mathcal{F}(\tau_1) : \tau_1 \in \text{rng}\,p\}$ is finite from [44, Sch. 21]. □

(48) If $\text{Arity}(o) = \emptyset$, then height($o$-term p) $= 0$.

(49) $\deg(x\text{-term}) = 0$.

(50) $\deg \tau \neq 0$ if and only if there exists o and there exists p such that $\tau = o$-term p. Proof: Define $\mathcal{P}[\text{element of } \mathfrak{F}_\Sigma(X)] \equiv \deg \$_1 \neq 0$ iff there exists o and there exists p such that $\$_1 = o$-term p. $\mathcal{P}[x\text{-term}]$. $\mathcal{P}[\tau]$ from *TermInd*. □

Let τ be a decorated tree. Let us consider I. Observe that $\tau^{-1}(I)$ is finite sequence-membered.

Let us consider a. Let J, K be sets. Let us observe that the functor $\text{IFIN}(a, I, J, K)$ yields a set. Now we state the propositions:

(51) Suppose $J = \langle o, \text{the carrier of } \Sigma \rangle$. Then $(o\text{-term}\,p)^{-1}(I) = \text{IFIN}(J, I, \{\emptyset\}, \emptyset) \cup \bigcup\{\langle i \rangle \frown p(i+1)^{-1}(I) : i < \text{len}\,p\}$. Proof: Set $X = \{\langle i \rangle \frown p(i+1)^{-1}(I) : i < \text{len}\,p\}$. $(o\text{-term}\,p)^{-1}(I) \subseteq \text{IFIN}(J, I, \{\emptyset\}, \emptyset) \cup \bigcup X$ by [20, (10)], [13, (11), (13)], [52, (25)]. □

(52) Suppose there exists a finite sequence f of elements of $\mathbb{N}$ such that $i = \sum f$ and $\operatorname{dom} f = \operatorname{dom} \operatorname{Arity}(o)$ and for every i and τ such that $i \in \operatorname{dom} \operatorname{Arity}(o)$ and $\tau = p(i)$ holds $f(i) = \deg \tau$. Then $\deg(o\text{-term}\, p) = i+1$. PROOF: Set $\tau = o$-term p. Set $I =$ (the carrier' of Σ) $\times$ {the carrier of Σ}. Set $\mathfrak{A} = \{\langle i \rangle \frown p(i+1)^{-1}(I) : i < \operatorname{len} p\}$. $\emptyset \notin \bigcup \mathfrak{A}$. $\tau^{-1}(I) = \{\emptyset\} \cup \bigcup \mathfrak{A}$. Define $\mathcal{J}$(natural number) $= \langle \$_1 \rangle \frown p(\$_1 + 1)^{-1}(I)$. For every i and j such that $i < \operatorname{len} f$ and $j < \operatorname{len} f$ and $i \neq j$ holds $\mathcal{J}(i)$ misses $\mathcal{J}(j)$ by [22, (40)], (11). For every i such that $i < \operatorname{len} f$ holds $\overline{\overline{\mathcal{J}(i)}} = f(i+1)$ by [13, (12), (13)], [52, (25)], [12, (2)]. $\overline{\overline{\bigcup\{\mathcal{J}(i) : i < \operatorname{len} f\}}} = \sum f$ from *CardUnion*. $\square$

Let us consider Σ, X, T, and i. The functor $T \deg_{\leqslant} i$ yielding a subset of T is defined by the term

(Def. 18) $\{\rho : \deg \rho \leqslant i\}$.

The functor $T \operatorname{height}_{\leqslant} i$ yielding a subset of T is defined by the term

(Def. 19) $\{\tau : \tau \in T \text{ and } \operatorname{height} \tau \leqslant i\}$.

Now we state the propositions:

(53) $\rho \in T \deg_{\leqslant} i$ if and only if $\deg \rho \leqslant i$.

(54) $T \deg_{\leqslant} 0 =$ the set of all x-term. PROOF: $T \deg_{\leqslant} 0 \subseteq$ the set of all x-term by [10, (39)], (36), (50). Consider σ, x such that $a = x$-term. $\deg(x\text{-term}) = 0 \leqslant 0$ and x-term $\in T$. Reconsider $\rho = x$-term as an element of T. $\deg \rho = \deg^{@}\rho = 0$. $\square$

(55) $T \operatorname{height}_{\leqslant} 0 =$ the set of all x-term $\cup$ $\{o\text{-term}\, p : o\text{-term}\, p \in T$ and $\operatorname{Arity}(o) = \emptyset\}$. The theorem is a consequence of (36), (46), (47), (42), and (48).

(56) $T \deg_{\leqslant} 0 = \bigcup \operatorname{FreeGenerator}(T)$. PROOF: $T \deg_{\leqslant} 0 =$ the set of all x-term. $T \deg_{\leqslant} 0 \subseteq \bigcup \operatorname{FreeGenerator}(T)$ by [5, (2)]. Consider b such that $b \in \operatorname{dom} \operatorname{FreeGenerator}(T)$ and $a \in (\operatorname{FreeGenerator}(T))(b)$. Consider y being a set such that $y \in X(b)$ and $a =$ the root tree of $\langle y, b \rangle$. $\square$

(57) $\rho \in T \operatorname{height}_{\leqslant} i$ if and only if $\operatorname{height} \rho \leqslant i$.

Let us consider Σ, X, T, and i. One can check that $T \deg_{\leqslant} i$ is non empty and $T \operatorname{height}_{\leqslant} i$ is non empty.

Let us assume that $i \leqslant j$. Now we state the propositions:

(58) $T \deg_{\leqslant} i \subseteq T \deg_{\leqslant} j$.

(59) $T \operatorname{height}_{\leqslant} i \subseteq T \operatorname{height}_{\leqslant} j$.

Now we state the propositions:

(60) $T \deg_{\leqslant}(i+1) = (T \deg_{\leqslant} 0) \cup \{o\text{-term}\, p :$ there exists a finite sequence f of elements of $\mathbb{N}$ such that $i \geqslant \sum f$ and $\operatorname{dom} f = \operatorname{dom} \operatorname{Arity}(o)$ and for every i and τ such that $i \in \operatorname{dom} \operatorname{Arity}(o)$ and $\tau = p(i)$ holds $f(i) = \deg \tau\} \cap \bigcup$(the sorts of T). PROOF: Set $I = \{o\text{-term}\, p :$ there exists a finite sequence f of elements of $\mathbb{N}$ such that $i \geqslant \sum f$ and $\operatorname{dom} f =$

$\operatorname{dom}\operatorname{Arity}(o)$ and for every i and τ such that $i \in \operatorname{dom}\operatorname{Arity}(o)$ and $\tau = p(i)$ holds $f(i) = \deg\tau\}$. $T\deg_{\leqslant}(i+1) \subseteq (T\deg_{\leqslant} 0) \cup I \cap \bigcup(\text{the sorts of } T)$ by [10, (39)], (36), (54), [36, (6)]. $T\deg_{\leqslant} 0 \subseteq T\deg_{\leqslant}(i+1)$. $I \cap \bigcup(\text{the sorts of } T) \subseteq T\deg_{\leqslant}(i+1)$. $\square$

(61) $T\operatorname{height}_{\leqslant}(i+1) = (T\operatorname{height}_{\leqslant} 0) \cup \{o\text{-term}\, p : \bigcup\{\operatorname{height}\tau : \tau \in \operatorname{rng} p\} \subseteq i\} \cap \bigcup(\text{the sorts of } T)$. Proof: Set $I = \{o\text{-term}\, p : \bigcup\{\operatorname{height}\tau : \tau \in \operatorname{rng} p\} \subseteq i\}$. $T\operatorname{height}_{\leqslant}(i+1) \subseteq (T\operatorname{height}_{\leqslant} 0) \cup I \cap \bigcup(\text{the sorts of } T)$ by (36), (55), (46), (47). $T\operatorname{height}_{\leqslant} 0 \subseteq T\operatorname{height}_{\leqslant}(i+1)$. $I \cap \bigcup(\text{the sorts of } T) \subseteq T\operatorname{height}_{\leqslant}(i+1)$ by (46), (47), [13, (39)], (48). $\square$

(62) $\deg\tau \geqslant \operatorname{height}\tau$. Proof: Define $\mathcal{P}[\text{element of } \mathfrak{F}_{\Sigma}(X)] \equiv \deg \$_1 \geqslant \operatorname{height} \$_1$. For every operation symbol o of Σ and for every element p of $\operatorname{Args}(o, \mathfrak{F}_{\Sigma}(X))$ such that for every element τ of $\mathfrak{F}_{\Sigma}(X)$ such that $\tau \in \operatorname{rng} p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by (48), [36, (6)], (46), [42, (9)]. $\mathcal{P}[\tau]$ from *TermInd*. $\square$

(63) $\bigcup(\text{the sorts of } T) = \bigcup\{T\deg_{\leqslant} i : \text{not contradiction}\}$.

(64) $\bigcup(\text{the sorts of } T) = \bigcup\{T\operatorname{height}_{\leqslant} i : \text{not contradiction}\}$. The theorem is a consequence of (57).

(65) $T\deg_{\leqslant} i \subseteq \mathfrak{F}_{\Sigma}(X)\deg_{\leqslant} i$. Proof: Define $\mathcal{P}[\text{natural number}] \equiv T\deg_{\leqslant} \$_1 \subseteq \mathfrak{F}_{\Sigma}(X)\deg_{\leqslant} \$_1$. $T\deg_{\leqslant} 0 = \bigcup\operatorname{FreeGenerator}(T)$ and $\mathfrak{F}_{\Sigma}(X)\deg_{\leqslant} 0 = \bigcup\operatorname{FreeGenerator}(\mathfrak{F}_{\Sigma}(X))$. For every i, $\mathcal{P}[i]$ from [13, Sch. 2]. $\square$

4. Context

Let us consider Σ, X, T, σ, x, and ρ. We say that ρ is x-context if and only if

(Def. 20) $\overline{\overline{\operatorname{Coim}(\rho, \langle x, \sigma\rangle)}} = 1$.

We say that ρ is x-omitting if and only if

(Def. 21) $\operatorname{Coim}(\rho, \langle x, \sigma\rangle) = \emptyset$.

The functor $\operatorname{vf}\rho$ yielding a set is defined by the term

(Def. 22) $\pi_1(\operatorname{rng}\rho \cap (\bigcup X \times (\text{the carrier of } \Sigma)))$.

Now we state the propositions:

(66) $\operatorname{vf}\rho = \bigcup\operatorname{Var}_X \rho$. Proof: $\operatorname{vf}\rho \subseteq \bigcup\operatorname{Var}_X\rho$ by [32, (87)], [5, (2)], [10, (44)], [23, (9)]. $\square$

(67) $\operatorname{vf}(x\text{-term}) = \{x\}$.

(68) $\operatorname{vf}(o\text{-term}\, p) = \bigcup\{\operatorname{vf}\tau : \tau \in \operatorname{rng} p\}$. Proof: $\operatorname{vf}(o\text{-term}\, p) \subseteq \bigcup\{\operatorname{vf}\tau : \tau \in \operatorname{rng} p\}$ by (66), [5, (2)], [23, (13)], [55, (167)]. $\square$

Let us consider Σ, X, T, and ρ. Note that $\operatorname{vf}\rho$ is finite.

Now we state the proposition:

(69) If $x \notin \operatorname{vf}\rho$, then ρ is x-omitting.

Let us consider Σ, X, σ, and τ. We say that τ is σ-context if and only if

(Def. 23) There exists x such that τ is x-context.

Let us consider x. Let us observe that every element of $\mathfrak{F}_\Sigma(X)$ which is x-context is also σ-context.

One can verify that x-term is x-context.

One can check that there exists an element of $\mathfrak{F}_\Sigma(X)$ which is x-context and non compound and every element of $\mathfrak{F}_\Sigma(X)$ which is x-omitting is also non x-context.

Now we state the proposition:

(70) Let us consider sort symbols σ_1, σ_2 of Σ, an element x_1 of $X(\sigma_1)$, and an element x_2 of $X(\sigma_2)$. Then $\sigma_1 \neq \sigma_2$ or $x_1 \neq x_2$ if and only if x_1-term is x_2-omitting.

Let us consider Σ, σ, σ_1, Z, and z. Let z' be a z-different element of $Z(\sigma_1)$. One can check that z'-term is z-omitting.

One can check that there exists an element of $\mathfrak{F}_\Sigma(Z)$ which is z-omitting.

Let us consider σ_1. Let z_1 be a z-different element of $Z(\sigma_1)$. Observe that there exists an element of $\mathfrak{F}_\Sigma(Z)$ which is z-omitting and z_1-context.

Let us consider X. Let us consider x.

A context of x is an x-context element of $\mathfrak{F}_\Sigma(X)$. Now we state the proposition:

(71) Let us consider a sort symbol ρ of Σ and an element y of $X(\rho)$. Then x-term is a context of y if and only if $\rho = \sigma$ and $x = y$.

Let us consider Σ, X, and σ.

A context of σ and X is a σ-context element of $\mathfrak{F}_\Sigma(X)$. In the sequel $\mathcal{C}$ denotes a context of x, $\mathcal{C}_1$ denotes a context of y, $\mathcal{C}'$ denotes a context of z, $\mathcal{C}_{11}$ denotes a context of x_{11}, $\mathcal{C}_{12}$ denotes a context of y_{11}, and D denotes a context of σ and X.

Now we state the propositions:

(72) $\mathcal{C}$ is a context of σ and X.

(73) $x \in \operatorname{vf} \mathcal{C}$.

Let us consider Σ, o, σ, X, x, and p. We say that p is x-context including once only if and only if

(Def. 24) There exists i such that

(i) $i \in \operatorname{dom} p$, and

(ii) $p(i)$ is a context of x, and

(iii) for every j and τ such that $j \in \operatorname{dom} p$ and $j \neq i$ and $\tau = p(j)$ holds τ is x-omitting.

Let us note that every element of $\operatorname{Args}(o, \mathfrak{F}_\Sigma(X))$ which is x-context including once only is also non empty.

Now we state the propositions:

(74) p is x-context including once only if and only if o-term p is a context of x. PROOF: Set $I = \{\langle x, \sigma \rangle\}$. Set $k = p$. $(o\text{-term}\, k)^{-1}(I) = \emptyset \cup \bigcup\{\langle i \rangle ^\frown k(i+1)^{-1}(I) : i < \operatorname{len} k\}$. If k is x-context including once only, then o-term k is a context of x by [3, (42)], [52, (25)], [13, (10), (13), (11)]. □

(75) for every i such that $i \in \operatorname{dom} p$ holds p_i is x-omitting if and only if o-term p is x-omitting. The theorem is a consequence of (51) and (13).

(76) for every τ such that $\tau \in \operatorname{rng} p$ holds τ is x-omitting if and only if o-term p is x-omitting. The theorem is a consequence of (75).

Let us consider Σ, σ, and o. We say that o is σ-dependent if and only if

(Def. 25) $\sigma \in \operatorname{rng} \operatorname{Arity}(o)$.

Let Σ be a sufficiently rich non void non empty many sorted signature and σ be a sort symbol of Σ. Let us note that there exists an operation symbol of Σ which is σ-dependent.

In the sequel Σ' denotes a sufficiently rich non empty non void many sorted signature, σ' denotes a sort symbol of Σ', o' denotes a σ'-dependent operation symbol of Σ', X' denotes a nontrivial many sorted set indexed by the carrier of Σ', and x' denotes an element of $X'(\sigma')$.

Let us consider Σ', σ', o', X', and x'. Let us observe that there exists an element of $\operatorname{Args}(o', \mathfrak{F}_{\Sigma'}(X'))$ which is x'-context including once only.

Let p' be an x'-context including once only element of $\operatorname{Args}(o', \mathfrak{F}_{\Sigma'}(X'))$. One can check that o'-term p' is x'-context.

Let us consider Σ, o, σ, X, x, and p. Assume p is x-context including once only. The functor the x-context position in p yielding a natural number is defined by

(Def. 26) $p(it)$ is a context of x.

The functor the x-context in p yielding a context of x is defined by

(Def. 27) $it \in \operatorname{rng} p$.

Now we state the propositions:

(77) Suppose p is x-context including once only. Then

(i) the x-context position in $p \in \operatorname{dom} p$, and

(ii) the x-context in $p = p$(the x-context position in p).

(78) Suppose p is x-context including once only and the x-context position in $p \neq i \in \operatorname{dom} p$. Then p_i is x-omitting.

Let us assume that p is x-context including once only. Now we state the propositions:

(79) p yields the x-context in p just once. The theorem is a consequence of (77).

(80) $p \leftarrow$ (the x-context in p) = the x-context position in p. The theorem is a consequence of (79).

Now we state the proposition:

(81) (i) $\mathcal{C} = x$-term, or

(ii) there exists o and there exists p such that p is x-context including once only and $\mathcal{C} = o$-term p.

The theorem is a consequence of (36), (71), and (74).

Let us consider Σ', X', σ', and x'. One can verify that there exists an element of $\mathfrak{F}_{\Sigma'}(X')$ which is x'-context and compound.

The scheme *ContextInd* deals with a unary predicate $\mathcal{P}$ and a non empty non void many sorted signature Σ and a sort symbol σ of Σ and a non-empty many sorted set $\mathcal{X}$ indexed by the carrier of Σ and an element x of $\mathcal{X}(\sigma)$ and a context $\mathcal{C}$ of x and states that

(Sch. 4) $\mathcal{P}[\mathcal{C}]$

provided

- $\mathcal{P}[x\text{-term}]$ and
- for every operation symbol o of Σ and for every element w of $\mathrm{Args}(o, \mathfrak{F}_{\Sigma}(\mathcal{X}))$ such that w is x-context including once only holds if $\mathcal{P}$[the x-context in w], then for every context $\mathcal{C}$ of x such that $\mathcal{C} = o$-term w holds $\mathcal{P}[\mathcal{C}]$.

Now we state the propositions:

(82) If τ is x-omitting, then $\tau_{\langle x, \sigma \rangle \leftarrow \tau_1} = \tau$.

(83) Suppose the sort of $\tau_1 = \sigma$. Then $\tau_{\langle x, \sigma \rangle \leftarrow \tau_1} \in$ (the sorts of $\mathfrak{F}_{\Sigma}(X)$)(the sort of τ). Proof: Define $\mathcal{P}$[element of $\mathfrak{F}_{\Sigma}(X)$] $\equiv \$_1{}_{\langle x, \sigma \rangle \leftarrow \tau_1} \in$ (the sorts of $\mathfrak{F}_{\Sigma}(X)$)(the sort of $\$_1$). For every σ_1 and for every element y of $X(\sigma_1)$, $\mathcal{P}[y\text{-term}]$. For every o and p such that for every τ_2 such that $\tau_2 \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau_2]$ holds $\mathcal{P}[o\text{-term}\, p]$ by [20, (20)], (18), [52, (29)], [12, (2)]. $\mathcal{P}[\tau]$ from *TermInd*. $\square$

Let us consider Σ, X, σ, x, $\mathcal{C}$, and τ. Assume the sort of $\tau = \sigma$. The functor $\mathcal{C}[\tau]$ yielding an element of (the sorts of $\mathfrak{F}_{\Sigma}(X)$)(the sort of $\mathcal{C}$) is defined by the term

(Def. 28) $\mathcal{C}_{\langle x, \sigma \rangle \leftarrow \tau}$.

Now we state the proposition:

(84) If the sort of $\tau = \sigma$, then x-term$[\tau] = \tau$.

Let us consider Σ, X, σ, x, and $\mathcal{C}$. Observe that $\mathcal{C}[x\text{-term}]$ reduces to $\mathcal{C}$.

Now we state the propositions:

(85) Let us consider an element w of $\mathrm{Args}(o, \mathfrak{F}_{\Sigma}(Z))$ and an element τ of $\mathfrak{F}_{\Sigma}(Z)$. Suppose

(i) w is z-context including once only, and

(ii) the sort of $\tau = \mathrm{Arity}(o)$(the z-context position in w).

Then $w +\cdot$ (the z-context position in $w, \tau) \in \operatorname{Args}(o, \mathfrak{F}_\Sigma(Z))$.

(86) Suppose the sort of $\mathcal{C}' = \sigma_1$. Let us consider a z-different element z_1 of $Z(\sigma_1)$ and a z-omitting context $\mathcal{C}_1$ of z_1. Then $\mathcal{C}_1[\mathcal{C}']$ is a context of z. Proof: Define $\mathcal{P}$[element of $\mathfrak{F}_\Sigma(Z)] \equiv$ if $\$_1$ is z-omitting, then $\$_{1\langle z_1, \sigma_1\rangle \leftarrow \mathcal{C}'}$ is a context of z. For every o and k such that k is z_1-context including once only holds if $\mathcal{P}$[the z_1-context in k], then for every context $\mathcal{C}$ of z_1 such that $\mathcal{C} = o$-term k holds $\mathcal{P}[\mathcal{C}]$. $\mathcal{P}[\mathcal{C}_1]$ from *ContextInd*. □

(87) Let us consider elements w, p of $\operatorname{Args}(o, \mathfrak{F}_\Sigma(Z))$ and an element τ of $\mathfrak{F}_\Sigma(Z)$. Suppose

(i) w is z-context including once only, and

(ii) $\mathcal{C}' = o$-term w, and

(iii) $p = w +\cdot$ (the z-context position in w, (the z-context in $w)[\tau]$), and

(iv) the sort of $\tau = \sigma$.

Then $\mathcal{C}'[\tau] = o$-term p. The theorem is a consequence of (77), (78), (82), and (19).

(88) The sort of $\mathcal{C}[\tau] =$ the sort of $\mathcal{C}$.

(89) If $\tau(a) = \langle x, \sigma\rangle$, then $a \in \operatorname{Leaves}(\operatorname{dom} \tau)$. The theorem is a consequence of (36).

(90) Let us consider a sort symbol σ_0 of Σ and an element x_0 of $X(\sigma_0)$. Suppose

(i) the sort of $\tau = \sigma$, and

(ii) $\mathcal{C}$ is x_0-omitting, and

(iii) τ is x_0-omitting.

Then $\mathcal{C}[\tau]$ is x_0-omitting. The theorem is a consequence of (89).

(91) Suppose p is x-context including once only. Then the sort of the x-context in $p = \operatorname{Arity}(o)$(the x-context position in p). The theorem is a consequence of (77).

(92) Let us consider a disjoint valued non-empty algebra $\mathfrak{A}$ over Σ, a non-empty algebra $\mathcal{B}$ over Σ, an operation symbol o of Σ, elements p, q of $\operatorname{Args}(o, \mathfrak{A})$, a many sorted function h from $\mathfrak{A}$ into $\mathcal{B}$, an element a of $\mathfrak{A}$, and i. Suppose

(i) $i \in \operatorname{dom} p$, and

(ii) $q = p +\cdot (i, a)$.

Then $h\#q = h\#p +\cdot (i, h(a))$.

(93) Let us consider an element τ of $\mathfrak{F}_\Sigma(Z)$. Suppose the sort of $\tau = \sigma$. Then (the canonical homomorphism of $R)(\mathcal{C}'[\tau])$ = (the canonical homomorphism of $R)(\mathcal{C}'[^{@}$((the canonical homomorphism of $R)(\tau))]). Proof: Set $H =$

the canonical homomorphism of R. Define $\mathcal{P}[\text{context of } z] \equiv H(\$_1[\tau]) = H(\$_1[{}^{@}(H(\tau))])$. The sort of ${}^{@}(H(\tau))$ = the sort of $H(\tau)$. $\mathcal{P}[z\text{-term}]$ by (84), [10, (48)], [28, (15)]. $\mathcal{P}[\mathcal{C}']$ from *ContextInd*. $\square$

Let us consider Σ, X, T, σ, and x. Let h be a many sorted function from $\mathfrak{F}_\Sigma(X)$ into T. We say that h is x-constant if and only if

(Def. 29) (i) $h(x\text{-term}) = x\text{-term}$, and

(ii) for every σ_1 and for every element x_1 of $X(\sigma_1)$ such that $x_1 \neq x$ or $\sigma \neq \sigma_1$ holds $h(x_1\text{-term})$ is x-omitting.

Now we state the proposition:

(94) The canonical homomorphism of T is x-constant. The theorem is a consequence of (70).

Let us consider Σ, X, T, σ, and x. Note that there exists a homomorphism from $\mathfrak{F}_\Sigma(X)$ to T which is x-constant.

From now on h_1 denotes an x-constant homomorphism from $\mathfrak{F}_\Sigma(X)$ to T and h_2 denotes a y-constant homomorphism from $\mathfrak{F}_\Sigma(Y)$ to Q.

Let x, y be objects. The functor $x \leftrightarrow y$ yielding a function is defined by the term

(Def. 30) $\{\langle x, y\rangle, \langle y, x\rangle\}$.

Let us observe that the functor is commutative.

Now we state the proposition:

(95) (i) $\operatorname{dom}(a \leftrightarrow b) = \{a, b\}$, and

(ii) $(a \leftrightarrow b)(a) = b$, and

(iii) $(a \leftrightarrow b)(b) = a$, and

(iv) $\operatorname{rng}(a \leftrightarrow b) = \{a, b\}$.

Let $\mathfrak{A}$ be a non empty set and a, b be elements of $\mathfrak{A}$. One can verify that $a \leftrightarrow b$ is $\mathfrak{A}$-valued and $\mathfrak{A}$-defined.

Let $\mathfrak{A}$ be a set, $\mathcal{B}$ be a non empty set, f be a function from $\mathfrak{A}$ into $\mathcal{B}$, and g be a $\mathfrak{A}$-defined $\mathcal{B}$-valued function. Let us note that the functor $f{+}{\cdot}g$ yields a function from $\mathfrak{A}$ into $\mathcal{B}$. Let I be a non empty set, $\mathfrak{A}$, $\mathcal{B}$ be many sorted sets indexed by I, f be a many sorted function from $\mathfrak{A}$ into $\mathcal{B}$, x be an element of I, and g be a function from $\mathfrak{A}(x)$ into $\mathcal{B}(x)$. One can verify that the functor $f +\cdot (x, g)$ yields a many sorted function from $\mathfrak{A}$ into $\mathcal{B}$. Let us consider Σ, X, T, σ, x_1, and x_2. The functor $\operatorname{Hom}(T, x_1, x_2)$ yielding an endomorphism of T is defined by

(Def. 31) (i) $it(\sigma)(x_1\text{-term}) = x_2\text{-term}$, and

(ii) $it(\sigma)(x_2\text{-term}) = x_1\text{-term}$, and

(iii) for every σ_1 and for every element y of $X(\sigma_1)$ such that $\sigma_1 \neq \sigma$ or $y \neq x_1$ and $y \neq x_2$ holds $it(\sigma_1)(y\text{-term}) = y\text{-term}$.

Now we state the propositions:

(96) Let us consider an endomorphism h of T. Suppose $h(\sigma)(x\text{-term}) = x\text{-term}$. Then $h = \mathrm{id}_\alpha$, where α is the sorts of T. PROOF: $h \restriction$ FreeGenerator $(T) = \mathrm{id}_\alpha \restriction \mathrm{FreeGenerator}(T)$, where α is the sorts of T by [27, (49), (18)]. $\square$

(97) $\mathrm{Hom}(T, x, x) = \mathrm{id}_\alpha$, where α is the sorts of T. The theorem is a consequence of (96).

(98) $\mathrm{Hom}(T, x_1, x_2) = \mathrm{Hom}(T, x_2, x_1)$.

(99) $\mathrm{Hom}(T, x_1, x_2) \circ \mathrm{Hom}(T, x_1, x_2) = \mathrm{id}_\alpha$, where α is the sorts of T. PROOF: Set $h = \mathrm{Hom}(T, x_1, x_2)$. For every σ and x, $(h \circ h)(\sigma)(x\text{-term}) = x\text{-term}$ by [28, (15)], [36, (2)]. $\square$

(100) If ρ is x_1-omitting and x_2-omitting, then $(\mathrm{Hom}(T, x_1, x_2))(\rho) = \rho$. PROOF: Define $\mathcal{P}[\text{element of } T] \equiv$ if $\$_1$ is x_1-omitting and x_2-omitting, then $(\mathrm{Hom}(T, x_1, x_2))(\text{the sort of } \$_1)(\$_1) = \$_1$. For every σ, x, and ρ such that $\rho = x\text{-term}$ holds $\mathcal{P}[\rho]$. For every o, p, and ρ such that $\rho = o\text{-term}\, p$ and for every element τ of T such that $\tau \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[\rho]$ by (22), (34), [10, (13)], [36, (6)]. $\mathcal{P}[\rho]$ from *TermAlgebraInd*. $\square$

Let us consider Σ, X, T, σ, and x. Let us observe that (the canonical homomorphism of T)$(\sigma)(x\text{-term})$ reduces to x-term.

Now we state the propositions:

(101) (The canonical homomorphism of T) $\circ\, \mathrm{Hom}(\mathfrak{F}_\Sigma(X), x, x_1) = \mathrm{Hom}(T, x, x_1) \circ$ (the canonical homomorphism of T). PROOF: Set $H =$ the canonical homomorphism of T. Set $h = \mathrm{Hom}(T, x, x_1)$. Set $g = \mathrm{Hom}(\mathfrak{F}_\Sigma(X), x, x_1)$. Define $\mathcal{P}[\text{element of } \mathfrak{F}_\Sigma(X)] \equiv (H \circ g)(\$_1) = (h \circ H)(\$_1)$. For every σ and x, $\mathcal{P}[x\text{-term}]$ by [36, (2)], [28, (15)]. For every operation symbol o of Σ and for every element p of $\mathrm{Args}(o, \mathfrak{F}_\Sigma(X))$ such that for every element τ of $\mathfrak{F}_\Sigma(X)$ such that $\tau \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by [10, (13)], (34), [36, (6)], [52, (29), (25)]. $(H \circ g)(\sigma) = (h \circ H)(\sigma)$. $\square$

(102) Let us consider an element ρ of T from σ. Then $(\mathrm{Hom}(T, x_1, x_2))(\sigma)(\rho) = ((\text{the canonical homomorphism of } T) \circ \mathrm{Hom}(\mathfrak{F}_\Sigma(X), x_1, x_2))(\sigma)(\rho)$. The theorem is a consequence of (101).

(103) If $x_1 \neq x_2$ and τ is x_2-omitting, then $(\mathrm{Hom}(\mathfrak{F}_\Sigma(X), x_1, x_2))(\tau)$ is x_1-omitting. PROOF: Set $T = \mathfrak{F}_\Sigma(X)$. Set $h = \mathrm{Hom}(T, x_1, x_2)$. Define $\mathcal{P}[\text{element of } T] \equiv$ if $\$_1$ is x_2-omitting, then $h(\$_1)$ is x_1-omitting. For every σ and x, $\mathcal{P}[x\text{-term}]$. For every o and p such that for every element τ of T such that $\tau \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by (34), [10, (13)], [36, (6)], [12, (2)]. $\mathcal{P}[\tau]$ from *TermInd*. $\square$

(104) Let us consider a finite subset $\mathfrak{A}$ of $\bigcup$(the sorts of $\mathfrak{F}_\Sigma(Y)$). Then there exists y such that for every v such that $v \in \mathfrak{A}$ holds v is y-omitting. PROOF: Define $\mathcal{F}(\text{element of } \mathfrak{F}_\Sigma(Y)) = \mathrm{vf}\, \$_1$. $\{\mathcal{F}(v) : v \in \mathfrak{A}\}$ is finite from [44, Sch. 21]. $\square$

Let us consider Σ, X, and T. We say that T is structure-invariant if and only if

(Def. 32) Let us consider an element p of $\mathrm{Args}(o,T)$. Suppose $(\mathrm{Den}(o,T))(p) = (\mathrm{Den}(o,\mathfrak{F}_\Sigma(X)))(p)$. $(\mathrm{Den}(o,T))(\mathrm{Hom}(T,x_1,x_2)\#p) = (\mathrm{Den}(o,\mathfrak{F}_\Sigma(X)))(\mathrm{Hom}(T,x_1,x_2)\#p)$.

Now we state the propositions:

(105) Suppose T is structure-invariant. Let us consider an element ρ of T from σ. Then $(\mathrm{Hom}(T,x_1,x_2))(\sigma)(\rho) = (\mathrm{Hom}(\mathfrak{F}_\Sigma(X),x_1,x_2))(\sigma)(\rho)$. Proof: Set $h = \mathrm{Hom}(T,x_1,x_2)$. Set $g = \mathrm{Hom}(\mathfrak{F}_\Sigma(X),x_1,x_2)$. Define $\mathcal{P}$[element of T] $\equiv$ h(the sort of $\$_1$)($\$_1$) $=$ g(the sort of $\$_1$)($\$_1$). For every σ, x, and ρ such that $\rho = x$-term holds $\mathcal{P}[\rho]$. For every o, p, and ρ such that $\rho = o$-term p and for every element τ of T such that $\tau \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[\rho]$ by [10, (13)], (22), [36, (6)], [52, (29), (25)]. $\mathcal{P}[\rho]$ from *TermAlgebraInd*. □

(106) If T is structure-invariant and $x_1 \neq x_2$ and ρ is x_2-omitting, then $(\mathrm{Hom}(T,x_1,x_2))(\rho)$ is x_1-omitting. Proof: Set $h = \mathrm{Hom}(T,x_1,x_2)$. Define $\mathcal{P}$[element of T] $\equiv$ if $\$_1$ is x_2-omitting, then $h(\$_1)$ is x_1-omitting. For every σ, x, and ρ such that $\rho = x$-term holds $\mathcal{P}[\rho]$. For every o, p, and ρ such that $\rho = o$-term p and for every element τ of T such that $\tau \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[\rho]$ by (22), (34), [10, (13), (41)]. $\mathcal{P}[\rho]$ from *TermAlgebraInd*. □

(107) Suppose Q is structure-invariant and v is y-omitting. Then (the canonical homomorphism of Q)(v) is y-omitting. The theorem is a consequence of (104), (29), (101), (100), (98), and (106).

(108) Suppose Q is structure-invariant. Let us consider an element p of $\mathrm{Args}(o,Q)$. Suppose an element τ of Q. If $\tau \in \mathrm{rng}\, p$, then τ is y-omitting. Let us consider an element τ of Q. If $\tau = (\mathrm{Den}(o,Q))(p)$, then τ is y-omitting. The theorem is a consequence of (76), (34), and (107).

(109) If Q is structure-invariant and v is y-omitting, then $h_2(v)$ is y-omitting. Proof: Define $\mathcal{P}$[element of $\mathfrak{F}_\Sigma(Y)$] $\equiv$ if $\$_1$ is y-omitting, then $h_2(\$_1)$ is y-omitting. For every σ and y, $\mathcal{P}[y\text{-term}]$. For every o and q such that for every v such that $v \in \mathrm{rng}\, q$ holds $\mathcal{P}[v]$ holds $\mathcal{P}[o\text{-term}\, q]$ by (34), [10, (13)], [36, (6)], [12, (2)]. $\mathcal{P}[v]$ from *TermInd*. □

Let us consider a terminating invariant stable many sorted relation R indexed by $\mathfrak{F}_\Sigma(X)$ with NF-variables and unique normal form property. Now we state the propositions:

(110) (i) for every element τ of the algebra of normal forms of R, $(\mathrm{Hom}(\mathfrak{F}_\Sigma(X), x_1,x_2))$(the sort of τ)(τ) $=$ (Hom(the algebra of normal forms of R, x_1, x_2))(τ), and

(ii) $\mathrm{Hom}(\mathfrak{F}_\Sigma(X),x_1,x_2) \restriction \mathrm{NForms}(R)$ $=$ Hom(the algebra of normal forms of R, x_1, x_2).

Proof: Set $F = \mathfrak{F}_\Sigma(X)$. Set $T =$ the algebra of normal forms of R. Set $H_3 = \mathrm{Hom}(F, x_1, x_2)$. Set $H_2 = \mathrm{Hom}(T, x_1, x_2)$. Define $\mathcal{P}[\text{element of } T] \equiv H_3(\text{the sort of } \$_1)(\$_1) = H_2(\$_1)$. For every sort symbol σ of Σ and for every element x of $X(\sigma)$ and for every element ρ of T such that $\rho = x$-term holds $\mathcal{P}[\rho]$. For every operation symbol o of Σ and for every element p of $\mathrm{Args}(o, \mathfrak{F}_\Sigma(X))$ and for every element ρ of T such that $\rho = o$-term p and for every element τ of T such that $\tau \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[\rho]$ by (22), (34), [10, (13)], [16, (54)]. $(\mathrm{Hom}(\mathfrak{F}_\Sigma(X), x_1, x_2) \restriction \mathrm{NForms}(R))(\sigma) =$ (Hom(the algebra of normal forms of R, x_1, x_2))(σ) by [27, (49)]. □

(111) Suppose $i \in \mathrm{dom}\, p$ and $R(\mathrm{Arity}(o)_i)$ reduces τ_1 to τ_2. Then R(the result sort of o) reduces $(\mathrm{Den}(o, \mathfrak{F}_\Sigma(X)))(p +\cdot (i, \tau_1))$ to $(\mathrm{Den}(o, \mathfrak{F}_\Sigma(X)))(p +\cdot (i, \tau_2))$. Proof: Consider ρ being a reduction sequence w.r.t. $R(\mathrm{Arity}(o)_i)$ such that $\rho(1) = \tau_1$ and $\rho(\mathrm{len}\, \rho) = \tau_2$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \mathrm{len}\, \rho$, then R(the result sort of o) reduces $(\mathrm{Den}(o, \mathfrak{F}_\Sigma(X)))(p +\cdot (i, \tau_1))$ to $(\mathrm{Den}(o, \mathfrak{F}_\Sigma(X)))(p +\cdot (i, \rho(\$_1)))$. For every i such that $1 \leqslant i$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [13, (13)], [52, (25)], [32, (87)], [12, (7), (2)]. For every i such that $i \geqslant 1$ holds $\mathcal{P}[i]$ from [13, Sch. 8]. □

Now we state the propositions:

(112) Let us consider a terminating invariant stable many sorted relation R indexed by $\mathfrak{F}_\Sigma(X)$ with NF-variables and unique normal form property and τ. Then R(the sort of τ) reduces τ to (the canonical homomorphism of the algebra of normal forms of R)(τ). Proof: Set $T =$ the algebra of normal forms of R. Set $H =$ the canonical homomorphism of T. Define $\mathcal{P}[\text{element of } \mathfrak{F}_\Sigma(X)] \equiv R(\text{the sort of } \$_1)$ reduces $\$_1$ to $H(\$_1)$. For every o and p such that for every τ such that $\tau \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by [10, (13)], (34), [16, (54)], [12, (2)]. $\mathcal{P}[\tau]$ from *TermInd*. □

(113) Let us consider a terminating invariant stable many sorted relation R indexed by $\mathfrak{F}_\Sigma(X)$ with NF-variables and unique normal form property, o, and p. Then R(the result sort of o) reduces o-term p to (Den(o, the algebra of normal forms of R))((the canonical homomorphism of the algebra of normal forms of R)$\#p$). The theorem is a consequence of (34) and (112).

(114) Let us consider a terminating invariant stable many sorted relation R indexed by $\mathfrak{F}_\Sigma(X)$ with NF-variables and unique normal form property, o, p, and an element q of Args(o, the algebra of normal forms of R). Suppose $p = q$. Then R(the result sort of o) reduces o-term p to (Den(o, the algebra of normal forms of R))(q). The theorem is a consequence of (113).

Let us consider Σ and X. Let R be a terminating invariant stable many sorted relation indexed by $\mathfrak{F}_\Sigma(X)$ with NF-variables and unique normal form property. Observe that the algebra of normal forms of R is structure-invariant.

Let us note that there exists a free in itself including Σ-terms over X algebra over Σ with all variables and inheriting operations which is structure-invariant.

5. Context vs. Translations

Let us consider Σ, σ_1, and σ_2. We say that σ_2 is σ_1-reachable if and only if

(Def. 33) TranslRel(Σ) reduces σ_1 to σ_2.

One can verify that there exists a sort symbol of Σ which is σ_1-reachable.

From now on σ_2 denotes a σ_1-reachable sort symbol of Σ and g_1 denotes a translation in $\mathfrak{F}_\Sigma(Y)$ from σ_1 into σ_2.

Now we state the proposition:

(115) TranslRel(Σ) reduces σ to the sort of $\mathcal{C}'$. Proof: Define $\mathcal{P}$[element of $\mathfrak{F}_\Sigma(Z)$] $\equiv$ TranslRel(Σ) reduces σ to the sort of $\$_1$. $\mathcal{P}[\mathcal{C}']$ from *ContextInd*. $\square$

Let us consider Σ, X, σ, x, and $\mathcal{C}$. Observe that the sort of $\mathcal{C}$ is σ-reachable.

Let us consider σ_1, σ_2, and g. Let τ be an element of (the sorts of $\mathfrak{F}_\Sigma(X)$)(σ_1). One can check that the functor $g(\tau)$ yields an element of (the sorts of $\mathfrak{F}_\Sigma(X)$)(σ_2). Let us consider σ, x, and $\mathcal{C}$. We say that $\mathcal{C}$ is basic if and only if

(Def. 34) There exists o and there exists p such that $\mathcal{C} = o$-term p and the x-context in $p = x$-term.

The functor transl $\mathcal{C}$ yielding a function from (the sorts of $\mathfrak{F}_\Sigma(X)$)(σ) into (the sorts of $\mathfrak{F}_\Sigma(X)$)(the sort of $\mathcal{C}$) is defined by

(Def. 35) If the sort of $\tau = \sigma$, then $it(\tau) = \mathcal{C}[\tau]$.

Now we state the propositions:

(116) If $\mathcal{C} = x$-term, then transl $\mathcal{C} = \mathrm{id}_{\alpha(\sigma)}$, where α is the sorts of $\mathfrak{F}_\Sigma(X)$. The theorem is a consequence of (84).

(117) Suppose $\mathcal{C}' = o$-term k and the z-context in $k = z$-term and $k1 = k +\cdot$ (the z-context position in k, l). Then $\mathcal{C}'[l] = o$-term $k1$. The theorem is a consequence of (74), (77), (84), and (87).

(118) If $\mathcal{C}'$ is basic, then transl $\mathcal{C}'$ is an elementary translation in $\mathfrak{F}_\Sigma(Z)$ from σ into the sort of $\mathcal{C}'$. The theorem is a consequence of (34), (74), (77), and (117).

(119) Let us consider a finite set V. Suppose

(i) $m \in \operatorname{dom} q$, and

(ii) $\operatorname{Arity}(o)_m = \sigma$.

Then there exists y and there exists $\mathcal{C}_1$ and there exists q_1 such that $y \notin V$ and $\mathcal{C}_1 = o$-term q_1 and $q_1 = q +\cdot (m, y$-term) and q_1 is y-context including once only and $m =$ the y-context position in q_1 and the y-context in $q_1 =$ y-term. Proof: Set $y =$ the element of $Y(\sigma) \setminus (V \cup \pi_1(\operatorname{rng}(o\text{-term}\, q)))$.

Reconsider $q_1 = q +\cdot (m, y\text{-term})$ as an element of $\mathrm{Args}(o, \mathfrak{F}_\Sigma(Y))$. q_1 is y-context including once only by [25, (30), (31), (32)], [52, (25)]. □

(120) Let us consider sort symbols σ_1, σ_2 of Σ and a finite set V. Suppose

(i) $m \in \operatorname{dom} q$, and

(ii) $\sigma_1 = \mathrm{Arity}(o)_m$.

Then there exists an element y of $Y(\sigma_1)$ and there exists a context $\mathcal{C}$ of y and there exists q_1 such that $y \notin V$ and $q_1 = q +\cdot (m, y\text{-term})$ and q_1 is y-context including once only and the y-context in $q_1 = y$-term and $\mathcal{C} = o\text{-term}\, q_1$ and $m =$ the y-context position in q_1 and $\mathrm{transl}\, \mathcal{C} = o_m^{\mathfrak{F}_\Sigma(Y)}(q, -)$. The theorem is a consequence of (119) and (117).

Let us consider Σ, X, τ, and a. One can verify that $\mathrm{Coim}(\tau, a)$ is finite sequence-membered.

Now we state the propositions:

(121) Suppose X is nontrivial and the sort of $\tau = \sigma$. Then $\overline{\overline{\mathrm{Coim}(\tau, a)}} \subseteq \overline{\overline{\mathrm{Coim}(\mathcal{C}[\tau], a)}}$. Proof: Define $\mathcal{P}[\text{context of } x] \equiv$ for every $\mathcal{C}$ such that $\mathcal{C} = \$_1$ holds $\overline{\overline{\mathrm{Coim}(\tau, a)}} \subseteq \overline{\overline{\mathrm{Coim}(\mathcal{C}[\tau], a)}}$. $\mathcal{P}[x\text{-term}]$. For every o and p such that p is x-context including once only holds if $\mathcal{P}[\text{the } x\text{-context in } p]$, then for every context $\mathcal{C}$ of x such that $\mathcal{C} = o\text{-term}\, p$ holds $\mathcal{P}[\mathcal{C}]$ by (77), [36, (6)], [13, (10)], [52, (25)]. $\mathcal{P}[\mathcal{C}]$ from *ContextInd*. □

(122) If p is x-context including once only and $i \in \operatorname{dom} p$, then p_i is not x-omitting iff p_i is x-context.

Let us assume that X is nontrivial and the sort of $\mathcal{C} = \sigma_1$. Now we state the propositions:

(123) Let us consider an element x_1 of $X(\sigma_1)$, a context $\mathcal{C}_1$ of x_1, and a context $\mathcal{C}_2$ of x. Suppose $\mathcal{C}_2 = \mathcal{C}_1[\mathcal{C}]$. If the sort of $\tau = \sigma$, then $\mathcal{C}_2[\tau] = \mathcal{C}_1[\mathcal{C}[\tau]]$. Proof: Define $\mathcal{P}[\text{context of } x_1] \equiv$ for every context $\mathcal{C}_1$ of x_1 for every context $\mathcal{C}_2$ of x such that $\mathcal{C}_1 = \$_1$ and $\mathcal{C}_2 = \mathcal{C}_1[\mathcal{C}]$ holds $\mathcal{C}_2[\tau] = \mathcal{C}_1[\mathcal{C}[\tau]]$. $\mathcal{P}[x_1\text{-term}]$. For every o and for every element w of $\mathrm{Args}(o, \mathfrak{F}_\Sigma(X))$ such that w is x_1-context including once only holds if $\mathcal{P}[\text{the } x_1\text{-context in } w]$, then for every context $\mathcal{C}$ of x_1 such that $\mathcal{C} = o\text{-term}\, w$ holds $\mathcal{P}[\mathcal{C}]$ by (77), [36, (6)], [12, (2), (7)]. $\mathcal{P}[\mathcal{C}_1]$ from *ContextInd*. □

(124) Let us consider an element x_1 of $X(\sigma_1)$, a context $\mathcal{C}_1$ of x_1, and a context $\mathcal{C}_2$ of x. Suppose $\mathcal{C}_2 = \mathcal{C}_1[\mathcal{C}]$. Then $\mathrm{transl}\, \mathcal{C}_2 = \mathrm{transl}\, \mathcal{C}_1 \cdot \mathrm{transl}\, \mathcal{C}$. Proof: Reconsider $f = \mathrm{transl}\, \mathcal{C}$ as a function from (the sorts of $\mathfrak{F}_\Sigma(X))(\sigma)$ into (the sorts of $\mathfrak{F}_\Sigma(X))(\sigma_1)$. $\mathrm{transl}\, \mathcal{C}_2 = \mathrm{transl}\, \mathcal{C}_1 \cdot f$ by [28, (15)], (123). □

Now we state the proposition:

(125) There exists y_{11} and there exists $\mathcal{C}_{12}$ such that the sort of $\mathcal{C}_{12} = \sigma_2$ and $g_1 = \mathrm{transl}\, \mathcal{C}_{12}$. Proof: Define $\mathcal{P}[\text{function}, \text{sort symbol of } \Sigma, \text{sort symbol of } \Sigma] \equiv$ for every finite set V, there exists an element x of $Y(\$_2)$ and

there exists a context $\mathcal{C}$ of x such that $x \notin V$ and the sort of $\mathcal{C} = \$_3$ and $\$_1 = \mathrm{transl}\,\mathcal{C}$. For every σ, $\mathcal{P}[\mathrm{id}_{\alpha(\sigma)}, \sigma, \sigma]$, where α is the sorts of $\mathfrak{F}_\Sigma(Y)$. For every sort symbols σ_1, σ_2, σ_3 of Σ such that $\mathrm{TranslRel}(\Sigma)$ reduces σ_1 to σ_2 for every translation τ in $\mathfrak{F}_\Sigma(Y)$ from σ_1 into σ_2 such that $\mathcal{P}[\tau, \sigma_1, \sigma_2]$ for every function f such that f is an elementary translation in $\mathfrak{F}_\Sigma(Y)$ from σ_2 into σ_3 holds $\mathcal{P}[f \cdot \tau, \sigma_1, \sigma_3]$ by [12, (2)], (120), (73), (69). For every sort symbols σ_1, σ_2 of Σ such that $\mathrm{TranslRel}(\Sigma)$ reduces σ_1 to σ_2 for every translation τ in $\mathfrak{F}_\Sigma(Y)$ from σ_1 into σ_2, $\mathcal{P}[\tau, \sigma_1, \sigma_2]$ from [12, Sch. 1]. $\square$

The scheme *LambdaTerm* deals with a non empty non void many sorted signature Σ and a non-empty many sorted set $\mathcal{X}$ indexed by the carrier of Σ and including Σ-terms over $\mathcal{X}$ algebras T_1, T_2 over Σ with all variables and inheriting operations and a unary functor $\mathcal{F}$ yielding an element of T_2 and states that

(Sch. 5) There exists a many sorted function f from T_1 into T_2 such that for every element τ of T_1, $f(\tau) = \mathcal{F}(\tau)$

provided

- for every element τ of T_1, the sort of τ = the sort of $\mathcal{F}(\tau)$.

Now we state the propositions:

(126) There exists an endomorphism g of T such that

(i) (the canonical homomorphism of T) $\circ\, h =$ $g \circ$ (the canonical homomorphism of T), and

(ii) for every element τ of T, $g(\tau) =$ (the canonical homomorphism of T)($h({}^{@}\tau)$).

The theorem is a consequence of (29).

(127) (The canonical homomorphism of T)($h(\tau)$) = (the canonical homomorphism of T)($h({}^{@}$((the canonical homomorphism of T)(τ)))). The theorem is a consequence of (126) and (29).

6. CONTEXT VS. ENDOMORPHISM

Let us consider Σ. Let $\mathcal{B}$ be a non empty finite sequence of elements of the carrier of Σ and i be an element of $\mathrm{dom}\,\mathcal{B}$. Note that the functor $\mathcal{B}(i)$ yields a sort symbol of Σ. Let us consider X. Let $\mathcal{B}$ be a finite sequence of elements of the carrier of Σ and V be a finite sequence of elements of $\bigcup X$. We say that V is $\mathcal{B}$-sorting if and only if

(Def. 36) (i) $\mathrm{dom}\,V = \mathrm{dom}\,\mathcal{B}$, and

(ii) for every i such that $i \in \mathrm{dom}\,\mathcal{B}$ holds $V(i) \in X(\mathcal{B}(i))$.

Let us observe that there exists a finite sequence of elements of $\bigcup X$ which is $\mathcal{B}$-sorting.

Let $\mathcal{B}$ be a non empty finite sequence of elements of the carrier of Σ. One can check that every finite sequence of elements of $\bigcup X$ which is $\mathcal{B}$-sorting is also non empty.

Let V be a $\mathcal{B}$-sorting finite sequence of elements of $\bigcup X$ and i be an element of $\operatorname{dom}\mathcal{B}$. Note that the functor $V(i)$ yields an element of $X(\mathcal{B}(i))$. Let $\mathcal{B}$ be a finite sequence of elements of the carrier of Σ and D be a finite sequence of elements of $\mathfrak{F}_\Sigma(X)$. We say that D is $\mathcal{B}$-sorting if and only if

(Def. 37) (i) $\operatorname{dom} D = \operatorname{dom}\mathcal{B}$, and

(ii) for every i such that $i \in \operatorname{dom}\mathcal{B}$ holds $D(i) \in (\text{the sorts of } \mathfrak{F}_\Sigma(X))(\mathcal{B}(i))$.

Note that there exists a finite sequence of elements of $\mathfrak{F}_\Sigma(X)$ which is $\mathcal{B}$-sorting.

Let $\mathcal{B}$ be a non empty finite sequence of elements of the carrier of Σ. One can verify that every finite sequence of elements of $\mathfrak{F}_\Sigma(X)$ which is $\mathcal{B}$-sorting is also non empty.

Let D be a $\mathcal{B}$-sorting finite sequence of elements of $\mathfrak{F}_\Sigma(X)$ and i be an element of $\operatorname{dom}\mathcal{B}$. Let us note that the functor $D(i)$ yields an element of (the sorts of $\mathfrak{F}_\Sigma(X))(\mathcal{B}(i))$. Let V be a $\mathcal{B}$-sorting finite sequence of elements of $\bigcup X$ and F be a finite sequence of elements of $\mathfrak{F}_\Sigma(X)$. We say that F is V-context sequence if and only if

(Def. 38) (i) $\operatorname{dom} F = \operatorname{dom}\mathcal{B}$, and

(ii) for every element i of $\operatorname{dom}\mathcal{B}$, $F(i)$ is a context of $V(i)$.

Let us observe that every finite sequence of elements of $\mathfrak{F}_\Sigma(X)$ which is V-context sequence is also non empty.

The scheme *FinSeqLambda* deals with a non empty finite sequence $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding an object and states that

(Sch. 6) There exists a non empty finite sequence p such that $\operatorname{dom} p = \operatorname{dom}\mathcal{B}$ and for every element i of $\operatorname{dom}\mathcal{B}$, $p(i) = \mathcal{F}(i)$.

The scheme *FinSeqRecLambda* deals with a non empty finite sequence $\mathcal{B}$ and an object $\mathfrak{A}$ and a binary functor $\mathcal{F}$ yielding a set and states that

(Sch. 7) There exists a non empty finite sequence p such that $\operatorname{dom} p = \operatorname{dom}\mathcal{B}$ and $p(1) = \mathfrak{A}$ and for every elements i, j of $\operatorname{dom}\mathcal{B}$ such that $j = i + 1$ holds $p(j) = \mathcal{F}(i, p(i))$.

The scheme *FinSeqRec2Lambda* deals with a non empty finite sequence $\mathcal{B}$ and a decorated tree $\mathfrak{A}$ and a binary functor $\mathcal{F}$ yielding a decorated tree and states that

(Sch. 8) There exists a non empty decorated tree yielding finite sequence p such that $\operatorname{dom} p = \operatorname{dom}\mathcal{B}$ and $p(1) = \mathfrak{A}$ and for every elements i, j of $\operatorname{dom}\mathcal{B}$

such that $j = i + 1$ for every decorated tree d such that $d = p(i)$ holds $p(j) = \mathcal{F}(i, d)$.

Let us consider Σ and X. Let $\mathcal{B}$ be a non empty finite sequence of elements of the carrier of Σ and V be a $\mathcal{B}$-sorting finite sequence of elements of $\bigcup X$. One can check that there exists a finite sequence of elements of $\mathfrak{F}_\Sigma(X)$ which is V-context sequence.

Let F be a V-context sequence finite sequence of elements of $\mathfrak{F}_\Sigma(X)$ and i be an element of $\operatorname{dom}\mathcal{B}$. One can verify that the functor $F(i)$ yields a context of $V(i)$. Let V_1, V_2 be $\mathcal{B}$-sorting finite sequences of elements of $\bigcup X$. We say that V_2 is V_1-omitting if and only if

(Def. 39) $\operatorname{rng} V_1$ misses $\operatorname{rng} V_2$.

Let D be a $\mathcal{B}$-sorting finite sequence of elements of $\mathfrak{F}_\Sigma(X)$ and F be a V_2-context sequence finite sequence of elements of $\mathfrak{F}_\Sigma(X)$. We say that F is (V_1, V_2, D)-consequent context sequence if and only if

(Def. 40) Let us consider elements i, j of $\operatorname{dom}\mathcal{B}$. If $i+1 = j$, then $F(j)[V_1(j)\text{-term}] = F(i)[D(i)]$.

Let V be a $\mathcal{B}$-sorting finite sequence of elements of $\bigcup X$. We say that V is D-omitting if and only if

(Def. 41) If $\tau \in \operatorname{rng} D$, then $\operatorname{vf}\tau$ misses $\operatorname{rng} V$.

Now we state the proposition:

(128) Let us consider a non empty finite sequence $\mathcal{B}$ of elements of the carrier of Σa $\mathcal{B}$-sorting finite sequence D of elements of $\mathfrak{F}_\Sigma(X)$a $\mathcal{B}$-sorting finite sequence V of elements of $\bigcup X$. Suppose V is D-omitting. Let us consider elements b_1, b_2 of $\operatorname{dom}\mathcal{B}$. Then $D(b_1)$ is $(V(b_2))$-omitting. The theorem is a consequence of (69).

Let us consider Σ and Y. Let $\mathcal{B}$ be a non empty finite sequence of elements of the carrier of Σ, V be a $\mathcal{B}$-sorting finite sequence of elements of $\bigcup Y$, and D be a $\mathcal{B}$-sorting finite sequence of elements of $\mathfrak{F}_\Sigma(Y)$. Let us observe that there exists a $\mathcal{B}$-sorting finite sequence of elements of $\bigcup Y$ which is one-to-one, V-omitting, and D-omitting.

Let us consider X and τ.

A vf-sequence of τ is a finite sequence and is defined by

(Def. 42) There exists a one-to-one finite sequence f such that

(i) $\operatorname{rng} f = \{\xi$, where ξ is an element of $\operatorname{dom}\tau$: there exists σ and there exists x such that $\tau(\xi) = \langle x, \sigma\rangle\}$, and

(ii) $\operatorname{dom} it = \operatorname{dom} f$, and

(iii) for every i such that $i \in \operatorname{dom} it$ holds $it(i) = \tau(f(i))$.

Let f be a finite sequence. Let us observe that $\operatorname{pr1}(f)$ is finite sequence-like and $\operatorname{pr2}(f)$ is finite sequence-like.

Now we state the propositions:

(129) Let us consider a vf-sequence f of τ. Then $\mathrm{pr2}(f)$ is a finite sequence of elements of the carrier of Σ.

(130) Let us consider a vf-sequence f of τ and a finite sequence $\mathcal{B}$ of elements of the carrier of Σ. Suppose $\mathcal{B} = \mathrm{pr2}(f)$. Then $\mathrm{pr1}(f)$ is a $\mathcal{B}$-sorting finite sequence of elements of $\bigcup X$.

Let f be a non empty finite sequence. One can verify that $1(\in \mathrm{dom}\, f)$ reduces to 1 and $(\mathrm{len}\, f)(\in \mathrm{dom}\, f)$ reduces to $\mathrm{len}\, f$.

Now we state the propositions:

(131) Let us consider an element ξ of $\mathrm{dom}\,\tau$. Suppose $\tau(\xi) = \langle x, \sigma\rangle$. Suppose the sort of $\tau_1 = \sigma$. Then τ with-replacement(ξ, τ_1) is an element of $\mathfrak{F}_\Sigma(X)$ from the sort of τ. PROOF: Define $\mathcal{P}$[element of $\mathfrak{F}_\Sigma(X)$] $\equiv$ for every element ξ of dom $\$_1$ for every x_1 and τ such that $\$_1(\xi) = \langle x_1, \sigma\rangle$ and $\tau = \$_1$ holds $\$_1$ with-replacement(ξ, τ_1) is an element of $\mathfrak{F}_\Sigma(X)$ from the sort of τ. $\mathcal{P}[x_{11}\text{-term}]$ by [20, (3)], [17, (29)]. For every o and p such that for every τ such that $\tau \in \mathrm{rng}\, p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by [20, (10)], [13, (12), (13)], [52, (25)]. $\mathcal{P}[\tau]$ from *TermInd*. □

(132) Suppose X is nontrivial. Let us consider an element ξ of $\mathrm{dom}\,\mathcal{C}$. Suppose $\mathcal{C}(\xi) = \langle x, \sigma\rangle$. If the sort of $\tau = \sigma$, then $\mathcal{C}[\tau] = \mathcal{C}$ with-replacement(ξ, τ). PROOF: Define $\mathcal{P}$[element of $\mathfrak{F}_\Sigma(X)$] $\equiv$ for every context $\mathcal{C}$ of x such that $\mathcal{C} = \$_1$ for every element ξ of $\mathrm{dom}\,\mathcal{C}$ such that $\mathcal{C}(\xi) = \langle x, \sigma\rangle$ holds $\mathcal{C}[\tau] = \mathcal{C}$ with-replacement(ξ, τ). $\mathcal{P}[x\text{-term}]$ by [17, (29)], [20, (3)], (84). For every operation symbol o of Σ and for every element w of $\mathrm{Args}(o, \mathfrak{F}_\Sigma(X))$ such that w is x-context including once only holds if $\mathcal{P}$[the x-context in w], then for every context $\mathcal{C}$ of x such that $\mathcal{C} = o\text{-term}\, w$ holds $\mathcal{P}[\mathcal{C}]$ by [20, (10)], [19, (38)], [13, (12), (13)]. $\mathcal{P}[\mathcal{C}]$ from *ContextInd*. □

(133) Let us consider finite sequences ξ_1, ξ_2. Suppose

(i) $\xi_1 \neq \xi_2$, and

(ii) $\xi_1, \xi_2 \in \mathrm{dom}\,\tau$.

Let us consider sort symbols σ_1, σ_2 of Σ, an element x_1 of $X(\sigma_1)$, and an element x_2 of $X(\sigma_2)$. Suppose $\tau(\xi_1) = \langle x_1, \sigma_1\rangle$. Then $\xi_1 \not\preceq \xi_2$. The theorem is a consequence of (36).

Let us consider τ, τ_1, and an element ξ of $\mathrm{dom}\,\tau$. Now we state the propositions:

(134) If $\tau_1 = \tau$ with-replacement$(\xi, x\text{-term})$ and τ is x-omitting, then τ_1 is a context of x. PROOF: $\mathrm{Coim}(\tau_1, \langle x, \sigma\rangle) = \{\xi\}$ by [17, (1), (29)], [20, (3)], [22, (87)]. □

(135) If $\tau(\xi) = \langle x, \sigma\rangle$, then $\mathrm{dom}\,\tau \subseteq \mathrm{dom}(\tau$ with-replacement$(\xi, \tau_1))$. The theorem is a consequence of (89).

Now we state the propositions:

(136) Let us consider an element ξ of $\operatorname{dom}\tau$. Suppose $\tau(\xi) = \langle x, \sigma\rangle$. Then $\operatorname{dom}\tau = \operatorname{dom}(\tau \text{ with-replacement}(\xi, x_1\text{-term}))$. PROOF: $\operatorname{dom}\tau \subseteq \operatorname{dom}(\tau \text{ with-replacement}(\xi, x_1\text{-term}))$. $\operatorname{dom}(\tau \text{ with-replacement}(\xi, x_1\text{-term})) \subseteq \operatorname{dom}\tau$ by [17, (29)], [20, (3)]. $\square$

(137) Let us consider trees τ, τ_1 and an element ξ of τ. Then $(\tau \text{ with-replacement}(\xi, \tau_1)){\restriction}\xi = \tau_1$. The theorem is a consequence of (1).

(138) Let us consider decorated trees τ, τ_1 and a node ξ of τ. Then $(\tau \text{ with-replacement}(\xi, \tau_1)){\restriction}\xi = \tau_1$. The theorem is a consequence of (137).

Let us consider a node ξ of τ. Now we state the propositions:

(139) If $\tau_1 = \tau{\restriction}\xi$, then $h(\tau){\restriction}\xi = h(\tau_1)$. PROOF: Define $\mathcal{P}[\text{element of } \mathfrak{F}_\Sigma(X)] \equiv$ for every node ξ of $\$_1$ for every τ_1 such that $\tau_1 = \$_1{\restriction}\xi$ holds $h(\$_1){\restriction}\xi = h(\tau_1)$ and $\xi \in \operatorname{dom}(h(\$_1))$. $\mathcal{P}[x\text{-term}]$ by [17, (29)], [20, (3)], [21, (1)], [17, (22)]. For every o and p such that for every τ such that $\tau \in \operatorname{rng} p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by [20, (11)], [21, (1)], [17, (22)], [21, (3)]. $\mathcal{P}[\tau]$ from *TermInd*. $\square$

(140) If $\tau(\xi) = \langle x, \sigma\rangle$, then $\tau{\restriction}\xi = x\text{-term}$. The theorem is a consequence of (36).

Now we state the propositions:

(141) Let us consider trees τ, τ_1 and elements ξ, ν of τ. Suppose

(i) $\xi \not\subseteq \nu$, and

(ii) $\nu \not\subseteq \xi$.

Then $(\tau \text{ with-replacement}(\xi, \tau_1)){\restriction}\nu = \tau{\restriction}\nu$. The theorem is a consequence of (2) and (5).

(142) Let us consider decorated trees τ, τ_1 and nodes ξ, ν of τ. Suppose

(i) $\xi \not\subseteq \nu$, and

(ii) $\nu \not\subseteq \xi$.

Then $(\tau \text{ with-replacement}(\xi, \tau_1)){\restriction}\nu = \tau{\restriction}\nu$. The theorem is a consequence of (141) and (5).

(143) If $\tau \subseteq \tau_1$, then $\tau = \tau_1$. PROOF: Define $\mathcal{P}[\text{element of } \mathfrak{F}_\Sigma(X)] \equiv$ for every τ_1 such that $\$_1 \subseteq \tau_1$ holds $\$_1 = \tau_1$. $\mathcal{P}[x\text{-term}]$ by [17, (22)], [30, (2)], [20, (3)], (36). For every o and p such that for every τ such that $\tau \in \operatorname{rng} p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by [17, (22)], [30, (2)], (36), [20, (3)]. $\mathcal{P}[\tau]$ from *TermInd*. $\square$

(144) Let us consider an endomorphism h of $\mathfrak{F}_\Sigma(X)$. Then

(i) $\operatorname{dom}\tau \subseteq \operatorname{dom}(h(\tau))$, and

(ii) for every I such that $I = \{\xi$, where ξ is an element of $\operatorname{dom}\tau$: there exists σ and there exists x such that $\tau(\xi) = \langle x, \sigma\rangle\}$ holds $\tau\restriction(\operatorname{dom}\tau \setminus I) = h(\tau)\restriction(\operatorname{dom}\tau \setminus I)$.

PROOF: Define $\mathcal{P}$[element of $\mathfrak{F}_\Sigma(X)$] $\equiv \operatorname{dom}\$_1 \subseteq \operatorname{dom}(h(\$_1))$ and for every I such that $I = \{\xi$, where ξ is an element of $\operatorname{dom}\$_1$: there exists σ and there exists x such that $\$_1(\xi) = \langle x, \sigma\rangle\}$ holds $\$_1\restriction(\operatorname{dom}\$_1 \setminus I) = h(\$_1)\restriction(\operatorname{dom}\$_1 \setminus I)$. $\mathcal{P}[x\text{-term}]$ by [17, (22)], [20, (3)], [17, (29)]. For every o and p such that for every τ such that $\tau \in \operatorname{rng} p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by (34), [10, (13)], [20, (11)], [17, (22)]. $\mathcal{P}[\tau]$ from *TermInd*. □

(145) Suppose $I = \{\xi$, where ξ is an element of $\operatorname{dom}\tau$: there exists σ and there exists x such that $\tau(\xi) = \langle x, \sigma\rangle\}$. Let us consider a node ξ of $h(\tau)$. Then

(i) $\xi \in \operatorname{dom}\tau \setminus I$, or

(ii) there exists an element ν of $\operatorname{dom}\tau$ such that $\nu \in I$ and there exists a node μ of $h(\tau)\restriction\nu$ such that $\xi = \nu \frown \mu$.

PROOF: Define $\mathcal{P}$[element of $\mathfrak{F}_\Sigma(X)$] $\equiv$ for every I such that $I = \{\xi$, where ξ is an element of $\operatorname{dom}\$_1$: there exists σ and there exists x such that $\$_1(\xi) = \langle x, \sigma\rangle\}$ for every node ξ of $h(\$_1)$, $\xi \in \operatorname{dom}\$_1 \setminus I$ or there exists an element ν of $\operatorname{dom}\$_1$ such that $\nu \in I$ and there exists a node μ of $h(\$_1)\restriction\nu$ such that $\xi = \nu\frown\mu$. $\mathcal{P}[x\text{-term}]$ by [17, (22)], [20, (3)], [21, (1)]. For every o and p such that for every τ such that $\tau \in \operatorname{rng} p$ holds $\mathcal{P}[\tau]$ holds $\mathcal{P}[o\text{-term}\, p]$ by (34), [10, (13)], [20, (11)], [17, (22)]. $\mathcal{P}[\tau]$ from *TermInd*. □

(146) Let us consider an endomorphism h of $\mathfrak{F}_\Sigma(Y)$a one-to-one finite sequence g of elements of $\operatorname{dom} v$. Suppose

(i) $\operatorname{rng} g = \{\xi$, where ξ is an element of $\operatorname{dom} v$: there exists σ and there exists y such that $v(\xi) = \langle y, \sigma\rangle\}$, and

(ii) $\operatorname{dom} v \subseteq \operatorname{dom} v_1$, and

(iii) $v\restriction(\operatorname{dom} v \setminus \operatorname{rng} g) = v_1\restriction(\operatorname{dom} v \setminus \operatorname{rng} g)$, and

(iv) for every i such that $i \in \operatorname{dom} g$ holds $h(v)\restriction(g_i$ **qua** node of $v) = v_1\restriction(g_i$ **qua** node of $v)$.

Then $h(v) = v_1$. PROOF: $h(v)\restriction(\operatorname{dom} v \setminus \operatorname{rng} g) = v_1\restriction(\operatorname{dom} v \setminus \operatorname{rng} g)$. $h(v) \subseteq v_1$ by [27, (1)], (145), [27, (49)], (144). □

(147) Let us consider an endomorphism h of $\mathfrak{F}_\Sigma(Y)$ and a vf-sequence f of v. Suppose $f \neq \emptyset$. Then there exists a non empty finite sequence $\mathcal{B}$ of elements of the carrier of Σ and there exists a $\mathcal{B}$-sorting finite sequence V_1 of elements of $\bigcup Y$ such that $\operatorname{dom}\mathcal{B} = \operatorname{dom} f$ and $\mathcal{B} = \operatorname{pr2}(f)$ and $V_1 = \operatorname{pr1}(f)$ and there exists a $\mathcal{B}$-sorting finite sequence D of elements

of $\mathfrak{F}_\Sigma(Y)$ and there exists a V_1-omitting D-omitting $\mathcal{B}$-sorting finite sequence V_2 of elements of $\bigcup Y$ such that for every element i of $\operatorname{dom}\mathcal{B}$, $D(i) = h(V_1(i)\text{-term})$ and there exists a V_2-context sequence finite sequence F of elements of $\mathfrak{F}_\Sigma(Y)$ such that F is (V_1, V_2, D)-consequent context sequence and $F(1(\in \operatorname{dom}\mathcal{B}))[V_1(1(\in \operatorname{dom}\mathcal{B}))\text{-term}] = v$ and $h(v) = F((\operatorname{len}\mathcal{B})(\in \operatorname{dom}\mathcal{B}))[D((\operatorname{len}\mathcal{B})(\in \operatorname{dom}\mathcal{B}))]$. PROOF: Reconsider $\mathcal{B} = \operatorname{pr2}(f)$ as a non empty finite sequence of elements of the carrier of Σ. Consider g being a one-to-one finite sequence such that $\operatorname{rng} g = \{\xi$, where ξ is an element of $\operatorname{dom} v$: there exists σ and there exists y such that $v(\xi) = \langle y, \sigma\rangle\}$ and $\operatorname{dom} f = \operatorname{dom} g$ and for every i such that $i \in \operatorname{dom} f$ holds $f(i) = v(g(i))$. $\operatorname{rng} g \subseteq \operatorname{dom} v$. Reconsider $V_1 = \operatorname{pr1}(f)$ as a $\mathcal{B}$-sorting finite sequence of elements of $\bigcup Y$. Define $\mathcal{F}(\text{element of } \operatorname{dom}\mathcal{B}) = h(V_1(\$_1)\text{-term})$. Consider D being a non empty finite sequence such that $\operatorname{dom} D = \operatorname{dom}\mathcal{B}$ and for every element i of $\operatorname{dom}\mathcal{B}$, $D(i) = \mathcal{F}(i)$ from *FinSeqLambda*. D is a finite sequence of elements of $\mathfrak{F}_\Sigma(Y)$. D is $\mathcal{B}$-sorting. Set $V_2 =$ the one-to-one V_1-omitting D-omitting $\mathcal{B}$-sorting finite sequence of elements of $\bigcup Y$. Define $\mathcal{H}(\text{element of } \operatorname{dom}\mathcal{B}, \text{decorated tree}) = (\$_2$ with-replacement$(((g_{\$_1}$ **qua** element of $\operatorname{dom} v)$ **qua** finite sequence of elements of $\mathbb{N}), D(\$_1)))$ with-replacement$(((g_{\$_1+1}$ **qua** element of $\operatorname{dom} v)$ **qua** finite sequence of elements of $\mathbb{N})$, the root tree of $\langle V_2(\$_1+1), \mathcal{B}(\$_1+1)\rangle)$. Consider F being a non empty decorated tree yielding finite sequence such that $\operatorname{dom} F = \operatorname{dom}\mathcal{B}$ and $F(1) = v$ with-replacement$(((g_1$ **qua** element of $\operatorname{dom} v)$ **qua** finite sequence of elements of $\mathbb{N})$, the root tree of $\langle V_2(1), \mathcal{B}(1)\rangle)$ and for every elements i, j of $\operatorname{dom}\mathcal{B}$ such that $j = i+1$ for every decorated tree d such that $d = F(i)$ holds $F(j) = \mathcal{H}(i, d)$ from *FinSeqRec2Lambda*. $\operatorname{rng} F \subseteq \bigcup(\text{the sorts of } \mathfrak{F}_\Sigma(Y))$ by (131), [22, (87)], [20, (3)], (133). Define $\mathcal{Q}[\text{natural number}] \equiv$ for every element b of $\operatorname{dom}\mathcal{B}$ such that $\$_1 = b$ holds $F(b)$ is a context of $V_2(b)$ and $\operatorname{dom} v \subseteq \operatorname{dom}(F(b))$ and $F(b)(g_b) = \langle V_2(b), \mathcal{B}(b)\rangle$ and for every element b_1 of $\operatorname{dom}\mathcal{B}$ such that $b_1 > b$ holds F_b is $(V_2(b_1))$-omitting and $F(b)(g_{b_1}) = \langle V_1(b_1), \mathcal{B}(b_1)\rangle$. $\mathcal{Q}[1]$ by [27, (102)], (134), (135), [22, (87)]. For every i such that $1 \leqslant i$ and $\mathcal{Q}[i]$ holds $\mathcal{Q}[i+1]$ by [52, (25)], [13, (13)], [27, (102)], (132). For every i such that $i \geqslant 1$ holds $\mathcal{Q}[i]$ from [13, Sch. 8]. F is V_2-context sequence by [52, (25)]. F is (V_1, V_2, D)-consequent context sequence by [52, (25)], [13, (12), (13)], (132). Set $b = 1(\in \operatorname{dom}\mathcal{B})$. Reconsider $\nu = g_b$, $\xi = g_{\operatorname{len}\mathcal{B}}$ as a node of v. Consider μ being a node of v such that $\nu = \mu$ and there exists σ and there exists y such that $v(\mu) = \langle y, \sigma\rangle$. $\operatorname{dom}(F(b)) = \operatorname{dom} v$. Reconsider $\tau = V_1(b)\text{-term}$ as an element of $\mathfrak{F}_\Sigma(Y)$. Consider μ being a finite sequence of elements of $\mathbb{N}$ such that $\mu \in \operatorname{dom}(V_2(b)\text{-term})$ and $\nu = \nu \mathbin{^\frown} \mu$ and $F(b)(\nu) = V_2(b)\text{-term}(\mu)$. $F(b)[\tau] = F(b)$ with-replacement(ν, τ). Define $\Sigma[\text{natural number}] \equiv$ for every elements b, b_1 of $\operatorname{dom}\mathcal{B}$ such that $\$_1 = b$ and $b_1 \leqslant b$ holds $(F(b)[D(b)]){\restriction}(g_{b_1}$ **qua** node of $v) = h(v){\restriction}(g_{b_1}$ **qua** node of

v) and $(F(b)[D(b)])\restriction(\operatorname{dom} v \setminus \operatorname{rng} g) = v\restriction(\operatorname{dom} v \setminus \operatorname{rng} g)$. $\Sigma[1]$ by [52, (25)], (132), (138), (140). For every i such that $i \geqslant 1$ and $\Sigma[i]$ holds $\Sigma[i+1]$ by [52, (25)], [13, (13)], (132), (135). Set $b = (\operatorname{len} \mathcal{B})(\in \operatorname{dom} \mathcal{B})$. Set $v_1 = F(b)[D(b)]$. For every i such that $i \geqslant 1$ holds $\Sigma[i]$ from [13, Sch. 8]. $v_1 = F(b)$ with-replacement$((g_b$ **qua** node of $v), D(b))$. $\operatorname{dom}(F(b)) \subseteq \operatorname{dom} v_1$. $\square$

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