

Dual Spaces and Hahn-Banach Theorem¹

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Summary. In this article, we deal with dual spaces and the Hahn-Banach Theorem. At the first, we defined dual spaces of real linear spaces and proved related basic properties. Next, we defined dual spaces of real normed spaces. We formed the definitions based on dual spaces of real linear spaces. In addition, we proved properties of the norm about elements of dual spaces. For the proof we referred to descriptions in the article [21]. Finally, applying theorems of the second section, we proved the Hahn-Banach extension theorem in real normed spaces. We have used extensively used [17].

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The notation and terminology used in this paper have been introduced in the following articles: [5], [16], [23], [18], [6], [7], [17], [15], [21], [24], [1], [2], [20], [3], [8], [4], [28], [25], [26], [10], [22], [12], [13], [27], [14], and [9].

1. DUAL SPACES OF REAL LINEAR SPACES

From now on V denotes a non empty real linear space.

Let X be a real linear space. The functor $\text{MultFReal}^* X$ yielding a function from $(\text{the carrier of } \mathbb{R}_F) \times (\text{the carrier of } X)$ into the carrier of X is defined by the term

(Def. 1) The external multiplication of X .

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Now we state the proposition:

- (1) Let us consider a real linear space X . Then \langle the carrier of X , the addition of X , the zero of X , $\text{MultFReal} * X \rangle$ is a vector space over \mathbb{R}_F .

Let X be a real linear space. The functor $\text{RLSp2RVSp } X$ yielding a vector space over \mathbb{R}_F is defined by the term

- (Def. 2) \langle the carrier of X , the addition of X , the zero of X , $\text{MultFReal} * X \rangle$.

Let X be a vector space structure over \mathbb{R}_F . The functor $\text{MultReal} * X$ yielding a function from $\mathbb{R} \times (\text{the carrier of } X)$ into the carrier of X is defined by the term

- (Def. 3) The left multiplication of X .

Now we state the proposition:

- (2) Let us consider a vector space X over \mathbb{R}_F . Then \langle the carrier of X , the zero of X , the addition of X , $\text{MultReal} * X \rangle$ is a real linear space.

Let X be a vector space over \mathbb{R}_F . The functor $\text{RVSp2RLSp } X$ yielding a real linear space is defined by the term

- (Def. 4) \langle the carrier of X , the zero of X , the addition of X , $\text{MultReal} * X \rangle$.

Now we state the propositions:

- (3) Let us consider a real linear space X , elements v, w of X , and elements v_1, w_1 of $\text{RLSp2RVSp } X$. If $v = v_1$ and $w = w_1$, then $v + w = v_1 + w_1$ and $v - w = v_1 - w_1$.
- (4) Let us consider a vector space X over \mathbb{R}_F , elements v, w of X , and elements v_1, w_1 of $\text{RVSp2RLSp } X$. If $v = v_1$ and $w = w_1$, then $v + w = v_1 + w_1$ and $v - w = v_1 - w_1$.

Let V be a non empty real linear space. The functor \overline{V} yielding a strict non empty real linear space is defined by

- (Def. 5) There exists a non empty vector space X over \mathbb{R}_F such that

- (i) $X = \text{RLSp2RVSp } V$, and
- (ii) $it = \text{RVSp2RLSp } \overline{X}$.

Now we state the proposition:

- (5) Let us consider an object x . Then $x \in$ the carrier of \overline{V} if and only if x is a linear functional in V .

Let V be a non empty real linear space. One can check that \overline{V} is constituted functions.

Let f be an element of \overline{V} and v be a vector of V . Note that the functor $f(v)$ yields an element of \mathbb{R} . Now we state the propositions:

- (6) Let us consider a non empty real linear space V and vectors f, g, h of \overline{V} . Then $h = f + g$ if and only if for every vector x of V , $h(x) = f(x) + g(x)$.

- (7) Let us consider a non empty real linear space V , vectors f, h of \overline{V} , and a real number a . Then $h = a \cdot f$ if and only if for every vector x of V , $h(x) = a \cdot f(x)$.
- (8) Let us consider a non empty real linear space V . Then $0_{\overline{V}} = (\text{the carrier of } V) \mapsto 0$.
- (9) Let us consider a real linear space X . Then $(\text{the carrier of } X) \mapsto 0$ is a linear functional in X . PROOF: Set $f = (\text{the carrier of } X) \mapsto 0$. f is additive by [23, (7)]. f is homogeneous by [23, (7)]. \square

Let X be a real linear space. The linear functionals of X yielding a subset of $\mathbb{R}_{\mathbb{R}}^{(\text{the carrier of } X)}$ is defined by

- (Def. 6) Let us consider an object x . Then $x \in it$ if and only if x is a linear functional in X .

Let X be a real normed space. One can verify that the linear functionals of X is non empty.

Let X be a real linear space. One can verify that the linear functionals of X is non empty and functional.

Let us consider a real linear space X . Now we state the propositions:

- (10) The linear functionals of X is linearly closed. PROOF: Set $W =$ the linear functionals of X . For every vectors v, u of $\mathbb{R}_{\mathbb{R}}^{\alpha}$ such that $v, u \in$ the linear functionals of X holds $v + u \in$ the linear functionals of X , where α is the carrier of X by [7, (66)], [18, (1)]. For every real number a and for every vector v of $\mathbb{R}_{\mathbb{R}}^{\alpha}$ such that $v \in W$ holds $a \cdot v \in W$, where α is the carrier of X by [7, (66)], [18, (4)]. \square
- (11) $\langle \text{the linear functionals of } X, \text{Zero}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{\alpha}), \text{Add}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{\alpha}), \text{Mult}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{\alpha}) \rangle$ is a subspace of $\mathbb{R}_{\mathbb{R}}^{\alpha}$, where α is the carrier of X .

Let X be a real linear space. Note that $\langle \text{the linear functionals of } X, \text{Zero}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{(\text{the carrier of } X)}), \text{Add}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{(\text{the carrier of } X)}), \text{Mult}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{(\text{the carrier of } X)}) \rangle$ is Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative and scalar unital.

The functor \overline{X} yielding a strict real linear space is defined by the term

- (Def. 7) $\langle \text{the linear functionals of } X, \text{Zero}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{\alpha}), \text{Add}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{\alpha}), \text{Mult}(\text{the linear functionals of } X, \mathbb{R}_{\mathbb{R}}^{\alpha}) \rangle$, where α is the carrier of X .

Observe that \overline{X} is constituted functions.

Let f be an element of \overline{X} and v be a vector of X . One can verify that the functor $f(v)$ yields an element of \mathbb{R} . Now we state the propositions:

- (12) Let us consider a real linear space X and vectors f, g, h of \overline{X} . Then $h = f + g$ if and only if for every vector x of X , $h(x) = f(x) + g(x)$. The

theorem is a consequence of (10).

- (13) Let us consider a real linear space X , vectors f, h of \overline{X} , and a real number a . Then $h = a \cdot f$ if and only if for every vector x of X , $h(x) = a \cdot f(x)$. The theorem is a consequence of (10).
- (14) Let us consider a real linear space X . Then $0_{\overline{X}} = (\text{the carrier of } X) \mapsto 0$. The theorem is a consequence of (10).

2. DUAL SPACES OF REAL NORMED SPACES

In the sequel S denotes a sequence of real numbers, k, n, m, m_1 denote natural numbers, and g, h, r, x denote real numbers.

Let S be a sequence of real numbers and x be a real number. The functor $S - x$ yielding a sequence of real numbers is defined by

(Def. 8) $it(n) = S(n) - x$.

Now we state the proposition:

- (15) If S is convergent, then $S - x$ is convergent and $\lim(S - x) = \lim S - x$.

Let X be a real normed space and I be a functional in X . We say that I is Lipschitzian if and only if

(Def. 9) There exists a real number K such that

- (i) $0 \leq K$, and
- (ii) for every vector x of X , $|I(x)| \leq K \cdot \|x\|$.

Now we state the proposition:

- (16) Let us consider a real normed space X and a functional f in X . If for every vector x of X , $f(x) = 0$, then f is Lipschitzian.

Let X be a real normed space. One can check that there exists a linear functional in X which is Lipschitzian.

The bounded linear functionals X yielding a subset of \overline{X} is defined by

(Def. 10) Let us consider a set x . Then $x \in it$ if and only if x is a Lipschitzian linear functional in X .

One can check that the bounded linear functionals X is non empty.

Let us consider a real normed space X . Now we state the propositions:

- (17) The bounded linear functionals X is linearly closed. PROOF: Set $W =$ the bounded linear functionals X . For every vectors v, u of \overline{X} such that $v, u \in W$ holds $v + u \in W$ by [5, (56)], (12). For every real number a and for every vector v of \overline{X} such that $v \in W$ holds $a \cdot v \in W$ by [5, (46), (65)], (13). \square
- (18) $\langle \text{the bounded linear functionals } X, \text{Zero}(\text{the bounded linear functionals } X, \overline{X}), \text{Add}(\text{the bounded linear functionals } X, \overline{X}), \text{Mult}(\text{the bounded linear functionals } X, \overline{X}) \rangle$ is a subspace of \overline{X} .

Let X be a real normed space. Let us observe that $\langle \text{the bounded linear functionals } X, \text{Zero}(\text{the bounded linear functionals } X, \overline{X}), \text{Add}(\text{the bounded linear functionals } X, \overline{X}), \text{Mult}(\text{the bounded linear functionals } X, \overline{X}) \rangle$ is Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative and scalar unital.

The \mathbb{R} -vector space of bounded linear functionals of X yielding a strict real linear space is defined by the term

(Def. 11) $\langle \text{the bounded linear functionals } X, \text{Zero}(\text{the bounded linear functionals } X, \overline{X}), \text{Add}(\text{the bounded linear functionals } X, \overline{X}), \text{Mult}(\text{the bounded linear functionals } X, \overline{X}) \rangle$.

One can check that every element of the \mathbb{R} -vector space of bounded linear functionals of X is function-like and relation-like.

Let f be an element of the \mathbb{R} -vector space of bounded linear functionals of X and v be a vector of X . Note that the functor $f(v)$ yields an element of \mathbb{R} . Now we state the propositions:

- (19) Let us consider a real normed space X and vectors f, g, h of the \mathbb{R} -vector space of bounded linear functionals of X . Then $h = f + g$ if and only if for every vector x of X , $h(x) = f(x) + g(x)$. The theorem is a consequence of (17) and (12).
- (20) Let us consider a real normed space X , vectors f, h of the \mathbb{R} -vector space of bounded linear functionals of X , and a real number a . Then $h = a \cdot f$ if and only if for every vector x of X , $h(x) = a \cdot f(x)$. The theorem is a consequence of (17) and (13).
- (21) Let us consider a real normed space X . Then $0_\alpha = (\text{the carrier of } X) \mapsto 0$, where α is the \mathbb{R} -vector space of bounded linear functionals of X . The theorem is a consequence of (14) and (17).

Let X be a real normed space and f be an object.

The functor $\text{Bound2Lipschitz}(f, X)$ yielding a Lipschitzian linear functional in X is defined by the term

(Def. 12) $f(\in \text{the bounded linear functionals } X)$.

Let u be a linear functional in X . The functor $\text{PreNorms}(u)$ yielding a non empty subset of \mathbb{R} is defined by the term

(Def. 13) $\{|u(t)|, \text{ where } t \text{ is a vector of } X : \|t\| \leq 1\}$.

Let g be a Lipschitzian linear functional in X . Observe that $\text{PreNorms}(g)$ is upper bounded.

Now we state the proposition:

- (22) Let us consider a real normed space X and a linear functional g in X . Then g is Lipschitzian if and only if $\text{PreNorms}(g)$ is upper bounded.

Let X be a real normed space. The bounded linear functionals norm X yielding a function from the bounded linear functionals X into \mathbb{R} is defined by

(Def. 14) Let us consider an object x . Suppose $x \in$ the bounded linear functionals X . Then $it(x) = \sup \text{PreNorms}(\text{Bound2Lipschitz}(x, X))$.

Let us consider a real normed space X and a Lipschitzian linear functional f in X . Now we state the propositions:

$$(23) \quad \text{Bound2Lipschitz}(f, X) = f.$$

(24) (The bounded linear functionals norm X)(f) = $\sup \text{PreNorms}(f)$. The theorem is a consequence of (23).

Let X be a real normed space. The functor $\text{DualSp } X$ yielding a non empty normed structure is defined by the term

(Def. 15) $\langle \text{the bounded linear functionals } X, \text{Zero}(\text{the bounded linear functionals } X, \overline{X}), \text{Add}(\text{the bounded linear functionals } X, \overline{X}), \text{Mult}(\text{the bounded linear functionals } X, \overline{X}), \text{the bounded linear functionals norm } X \rangle$.

Now we state the propositions:

(25) Let us consider a real normed space X . Then (the carrier of X) $\mapsto 0 = 0_{\text{DualSp } X}$. The theorem is a consequence of (21).

(26) Let us consider a real normed space X , a point f of $\text{DualSp } X$, and a Lipschitzian linear functional g in X . Suppose $g = f$. Let us consider a vector t of X . Then $|g(t)| \leq \|f\| \cdot \|t\|$. The theorem is a consequence of (24).

(27) Let us consider a real normed space X and a point f of $\text{DualSp } X$. Then $0 \leq \|f\|$. The theorem is a consequence of (24).

(28) Let us consider real normed spaces X, Y and a point f of $\text{DualSp } X$. If $f = 0_{\text{DualSp } X}$, then $0 = \|f\|$. PROOF: $\|f\| = 0$ by [23, (45)], [13, (45)], (25), [23, (7)]. \square

Let X be a real normed space. Note that every element of $\text{DualSp } X$ is function-like and relation-like.

Let f be an element of $\text{DualSp } X$ and v be a vector of X . Let us note that the functor $f(v)$ yields an element of \mathbb{R} . Now we state the propositions:

(29) Let us consider a real normed space X and points f, g, h of $\text{DualSp } X$. Then $h = f + g$ if and only if for every vector x of X , $h(x) = f(x) + g(x)$. The theorem is a consequence of (19).

(30) Let us consider a real normed space X , points f, h of $\text{DualSp } X$, and a real number a . Then $h = a \cdot f$ if and only if for every vector x of X , $h(x) = a \cdot f(x)$. The theorem is a consequence of (20).

(31) Let us consider a real normed space X , points f, g of $\text{DualSp } X$, and a real number a . Then

$$(i) \quad \|f\| = 0 \text{ iff } f = 0_{\text{DualSp } X}, \text{ and}$$

$$(ii) \quad \|a \cdot f\| = |a| \cdot \|f\|, \text{ and}$$

$$(iii) \quad \|f + g\| \leq \|f\| + \|g\|.$$

PROOF: $\|f + g\| \leq \|f\| + \|g\|$ by [13, (45)], (27), [5, (56)], (26). $\|a \cdot f\| = |a| \cdot \|f\|$ by (27), (26), [5, (65), (46)]. \square

Let X be a real normed space. Note that $\text{DualSp } X$ is reflexive discernible and real normed space-like.

Now we state the proposition:

- (32) Let us consider a real normed space X . Then $\text{DualSp } X$ is a real normed space.

Let X be a real normed space. Let us note that $\text{DualSp } X$ is reflexive discernible real normed space-like vector distributive scalar distributive scalar associative scalar unital Abelian add-associative right zeroed and right complementable.

Now we state the proposition:

- (33) Let us consider a real normed space X and points f, g, h of $\text{DualSp } X$. Then $h = f - g$ if and only if for every vector x of X , $h(x) = f(x) - g(x)$. The theorem is a consequence of (29).

Let X be a real normed space, s be a sequence of $\text{DualSp } X$, and n be a natural number. Let us note that the functor $s(n)$ yields an element of $\text{DualSp } X$. Now we state the propositions:

- (34) Let us consider a real normed space X and a sequence s_1 of $\text{DualSp } X$. If s_1 is Cauchy sequence by norm, then s_1 is convergent. PROOF: Define $\mathcal{P}[\text{set}, \text{set}] \equiv$ there exists a sequence x_1 of \mathbb{R} such that for every natural number n , $x_1(n) = (\text{Bound2Lipschitz}(vseq(n), X))(\$1)$ and x_1 is convergent and $\$2 = \lim x_1$. For every element x of X , there exists an element y of \mathbb{R} such that $\mathcal{P}[x, y]$ by (23), (33), (26), [5, (44)]. Consider f being a function from the carrier of X into \mathbb{R} such that for every element x of X , $\mathcal{P}[x, f(x)]$ from [7, Sch. 3]. Reconsider $t_1 = f$ as a function from the carrier of X into \mathbb{R} . t_1 is Lipschitzian by [13, (14)], [11, (12)], (23), (26). For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geq k$ for every vector x of X , $|(\text{Bound2Lipschitz}(vseq(n), X))(x) - t_1(x)| \leq e \cdot \|x\|$ by [22, (8)], (23), (33), (26). Reconsider $t_2 = t_1$ as a point of $\text{DualSp } X$. For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geq k$ holds $\|vseq(n) - t_2\| \leq e$ by (23), (33), [13, (45)], (24). For every real number e such that $e > 0$ there exists a natural number m such that for every natural number n such that $n \geq m$ holds $\|vseq(n) - t_2\| < e$. \square
- (35) Let us consider a real normed space X . Then $\text{DualSp } X$ is a real Banach space. The theorem is a consequence of (34).

Let X be a real normed space. One can verify that $\text{DualSp } X$ is complete.

3. HAHN-BANACH EXTENSION THEOREM

Let V be a real normed space.

A subreal normal space of V is a real normed space and is defined by

- (Def. 16) (i) the carrier of $it \subseteq$ the carrier of V , and
(ii) $0_{it} = 0_V$, and
(iii) the addition of $it =$ (the addition of V) \upharpoonright (the carrier of it), and
(iv) the external multiplication of $it =$ (the external multiplication of V) \upharpoonright ($\mathbb{R} \times$ (the carrier of it)), and
(v) the normed of $it =$ (the normed of V) \upharpoonright (the carrier of it).

- (36) Let us consider a real normed space V , a subreal normal space X of V , a Lipschitzian linear functional f in X , and a point F of $\text{DualSp } X$. Suppose $f = F$. Then there exists a Lipschitzian linear functional g in V and there exists a point G of $\text{DualSp } V$ such that $g = G$ and $g \upharpoonright (\text{the carrier of } X) = f$ and $\|G\| = \|F\|$. PROOF: Reconsider $X_0 = X$ as a real linear space. Reconsider $f_3 = f$ as a linear functional in X_0 . Define \mathcal{F} (element of the carrier of V) $= \|F\| \cdot \|\cdot\|_1$. Consider q being a function from the carrier of V into \mathbb{R} such that for every element v of the carrier of V , $q(v) = \mathcal{F}(v)$ from [7, Sch. 8]. q is a Banach functional in V . For every vector x of X_0 and for every vector v of V such that $x = v$ holds $f_3(x) \leq q(v)$ by [19, (4)], (26), [6, (49)]. Consider g being a linear functional in V such that $g \upharpoonright (\text{the carrier of } X_0) = f_3$ and for every vector x of V , $g(x) \leq q(x)$. For every vector x of V , $|g(x)| \leq \|F\| \cdot \|x\|$ by [26, (16)], [20, (2)], [19, (5)]. (The bounded linear functionals norm V)(g) \leq (the bounded linear functionals norm X)(f). (The bounded linear functionals norm X)(f) $\leq \sup \text{PreNorms}(g)$. (The bounded linear functionals norm X)(f) \leq (the bounded linear functionals norm V)(g). \square

- (37) HAHN-BANACH EXTENSION THEOREM (REAL NORMED SPACES):

Let us consider a real normed space V , a subreal normal space X of V , a Lipschitzian linear functional f in X , and a point F of $\text{DualSp } X$. Suppose

- (i) $f = F$, and
(ii) for every vector x of X and for every vector v of V such that $x = v$ holds $f(x) \leq \|v\|$.

Then there exists a Lipschitzian linear functional g in V and there exists a point G of $\text{DualSp } V$ such that $g = G$ and $g \upharpoonright (\text{the carrier of } X) = f$ and for every vector x of V , $g(x) \leq \|x\|$ and $\|G\| = \|F\|$. PROOF: Consider g being a Lipschitzian linear functional in V , G being a point of $\text{DualSp } V$ such that $g = G$ and $g \upharpoonright (\text{the carrier of } X) = f$ and $\|G\| = \|F\|$. $\|G\| \leq 1$. For every vector x of V , $g(x) \leq \|x\|$ by [19, (4)], (26). \square

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