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# Double Series and Sums[1]

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**Summary.** In this paper the author constructs several properties for double series and its convergence. The notions of convergence of double sequence have already been introduced in our previous paper [18]. In section 1 we introduce double series and their convergence. Then we show the relationship between Pringsheim-type convergence and iterated convergence. In section 2 we study double series having non-negative terms. As a result, we have equality of three type sums of non-negative double sequence. In section 3 we show that if a non-negative sequence is summable, then the squence of rearrangement of terms is summable and it has the same sums. In the last section two basic relations between double sequences and matrices are introduced.

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The notation and terminology used in this paper have been introduced in the following articles: [7], [1], [2], [18], [6], [9], [16], [11], [12], [23], [25], [30], [17], [3], [4], [13], [21], [20], [28], [29], [14], [22], [24], [27], and [15].

## 1. Double Series and their Convergence

From now on $R_1$, $R_2$, $R_3$ denote functions from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$.

Let $f$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Let us note that $f$ is non-negative yielding if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let us consider natural numbers $i$, $j$. Then $f(i, j) \geqslant 0$.

Now we state the propositions:

(1) Suppose $R_1$ is non-decreasing. Then

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(i) for every element $m$ of $\mathbb{N}$, $\mathrm{curry}(R_1, m)$ is non-decreasing, and

(ii) for every element $n$ of $\mathbb{N}$, $\mathrm{curry}'(R_1, n)$ is non-decreasing.

(2) If $R_1$ is non-decreasing and convergent in the first coordinate, then the lim in the first coordinate of $R_1$ is non-decreasing.

(3) If $R_1$ is non-decreasing and convergent in the second coordinate, then the lim in the second coordinate of $R_1$ is non-decreasing.

(4) If $R_1$ is non-decreasing and p-convergent, then for every natural numbers $n$, $m$, $R_1(n, m) \leqslant \text{P-lim}\, R_1$.

(5) (i) $\mathrm{dom}(R_2 + R_3) = \mathbb{N} \times \mathbb{N}$, and

(ii) $\mathrm{dom}(R_2 - R_3) = \mathbb{N} \times \mathbb{N}$, and

(iii) for every natural numbers $n$, $m$, $(R_2 + R_3)(n, m) = R_2(n, m) + R_3(n, m)$, and

(iv) for every natural numbers $n$, $m$, $(R_2 - R_3)(n, m) = R_2(n, m) - R_3(n, m)$.

(6) Let us consider non empty sets $C$, $D$, $E$ and a function $f$ from $C \times D$ into $E$. Then there exists a function $g$ from $D \times C$ into $E$ such that for every element $d$ of $D$ for every element $c$ of $C$, $g(d, c) = f(c, d)$. PROOF: Define $\mathcal{F}(\text{element of } D, \text{element of } C) = f(\$_2, \$_1)$. Consider $I$ being a function from $D \times C$ into $E$ such that for every element $d$ of $D$ and for every element $c$ of $C$, $I(d, c) = \mathcal{F}(d, c)$ from [5, Sch. 2]. □

Let $C$, $D$, $E$ be non empty sets and $f$ be a function from $C \times D$ into $E$. The functor $f^{\mathrm{T}}$ yielding a function from $D \times C$ into $E$ is defined by

(Def. 2) Let us consider an element $d$ of $D$ and an element $c$ of $C$. Then $it(d, c) = f(c, d)$.

Now we state the proposition:

(7) Let us consider non empty sets $C$, $D$, $E$ and a function $f$ from $C \times D$ into $E$. Then $f = (f^{\mathrm{T}})^{\mathrm{T}}$.

The scheme *RecEx2D1* deals with a non empty set $\mathcal{C}$ and a non empty set $\mathcal{D}$ and a function $\mathcal{H}$ from $\mathcal{C}$ into $\mathcal{D}$ and a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$ and states that

(Sch. 1) There exists a function $g$ from $\mathcal{C} \times \mathbb{N}$ into $\mathcal{D}$ such that for every element $a$ of $\mathcal{C}$, $g(a, 0) = \mathcal{H}(a)$ and for every natural number $n$, $g(a, n+1) = \mathcal{F}(g(a, n), a, n)$.

The scheme *RecEx2D1R* deals with a non empty set $\mathcal{C}$ and a function $\mathcal{H}$ from $\mathcal{C}$ into $\mathbb{R}$ and a ternary functor $\mathcal{F}$ yielding a real number and states that

(Sch. 2) There exists a function $g$ from $\mathcal{C} \times \mathbb{N}$ into $\mathbb{R}$ such that for every element $a$ of $\mathcal{C}$, $g(a, 0) = \mathcal{H}(a)$ and for every natural number $n$, $g(a, n+1) = \mathcal{F}(g(a, n), a, n)$.

The scheme *RecEx2D2* deals with a non empty set $\mathcal{C}$ and a non empty set $\mathcal{D}$ and a function $\mathcal{H}$ from $\mathcal{C}$ into $\mathcal{D}$ and a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{D}$ and states that

(Sch. 3) There exists a function $g$ from $\mathbb{N} \times \mathcal{C}$ into $\mathcal{D}$ such that for every element $a$ of $\mathcal{C}$, $g(0, a) = \mathcal{H}(a)$ and for every natural number $n$, $g(n+1, a) = \mathcal{F}(g(n, a), a, n)$.

The scheme *RecEx2D2R* deals with a non empty set $\mathcal{C}$ and a function $\mathcal{H}$ from $\mathcal{C}$ into $\mathbb{R}$ and a ternary functor $\mathcal{F}$ yielding a real number and states that

(Sch. 4) There exists a function $g$ from $\mathbb{N} \times \mathcal{C}$ into $\mathbb{R}$ such that for every element $a$ of $\mathcal{C}$, $g(0, a) = \mathcal{H}(a)$ and for every natural number $n$, $g(n+1, a) = \mathcal{F}(g(n, a), a, n)$.

Let $R_1$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. The partial sums in the second coordinate of $R_1$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 3) Let us consider natural numbers $n$, $m$. Then

(i) $it(n, 0) = R_1(n, 0)$, and

(ii) $it(n, m+1) = it(n, m) + R_1(n, m+1)$.

The partial sums in the first coordinate of $R_1$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 4) Let us consider natural numbers $n$, $m$. Then

(i) $it(0, m) = R_1(0, m)$, and

(ii) $it(n+1, m) = it(n, m) + R_1(n+1, m)$.

Now we state the propositions:

(8) (i) the partial sums in the second coordinate of $R_2 + R_3 =$ (the partial sums in the second coordinate of $R_2$)+(the partial sums in the second coordinate of $R_3$), and

(ii) the partial sums in the first coordinate of $R_2 + R_3 =$ (the partial sums in the first coordinate of $R_2$) + (the partial sums in the first coordinate of $R_3$).

The theorem is a consequence of (5).

(9) Let us consider natural numbers $n$, $m$. Then

(i) (the partial sums in the second coordinate of $R_1$)$(n, m) =$ (the partial sums in the first coordinate of ${R_1}^{\mathrm{T}}$)$(m, n)$, and

(ii) (the partial sums in the first coordinate of $R_1$)$(n, m) =$ (the partial sums in the second coordinate of ${R_1}^{\mathrm{T}}$)$(m, n)$.

(10) (i) the partial sums in the second coordinate of $R_1 =$ (the partial sums in the first coordinate of ${R_1}^{\mathrm{T}})^{\mathrm{T}}$, and

(ii) the partial sums in the second coordinate of ${R_1}^{\mathrm{T}} =$ (the partial sums in the first coordinate of $R_1)^{\mathrm{T}}$, and

(iii) (the partial sums in the second coordinate of $R_1$)$^{\mathrm{T}}$ = the partial sums in the first coordinate of $R_1{}^{\mathrm{T}}$, and

(iv) (the partial sums in the second coordinate of $R_1{}^{\mathrm{T}}$)$^{\mathrm{T}}$ = the partial sums in the first coordinate of $R_1$.

The theorem is a consequence of (9).

Let $R_1$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. The functor $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by the term

(Def. 5) The partial sums in the second coordinate of the partial sums in the first coordinate of $R_1$.

Now we state the propositions:

(11) Let us consider natural numbers $n$, $m$. Then

(i) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) =$ (the partial sums in the second coordinate of $R_1$)$(n+1, m) + (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1$)$(n, m+1) =$ (the partial sums in the first coordinate of $R_1$)$(n, m+1) +$ (the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1$)$(n, m)$.

Proof: Set $R_4 = (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$. Set $C_5$ = the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1$. Set $R_5$ = the partial sums in the first coordinate of $R_1$. Set $C_6$ = the partial sums in the second coordinate of $R_1$. Define $\mathcal{P}[\text{natural number}] \equiv R_4(n+1, \$_1) = C_6(n+1, \$_1) + R_4(n, \$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [3, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv C_5(\$_1, m+1) = R_5(\$_1, m+1) + C_5(\$_1, m)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number $k$, $\mathcal{Q}[k]$ from [3, Sch. 2]. $\square$

(12) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}} =$ the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1$.

Let us consider natural numbers $n$, $m$. Now we state the propositions:

(13) $R_1(n+1, m+1) = (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n+1, m+1) - (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n, m+1) - (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) + (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n, m)$.

(14) $R_1(n+1, m+1) =$ (the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1$)$(n+1, m+1) -$ (the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1$)$(n+1, m) -$ (the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1$)$(n, m+1) +$ (the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1$)$(n, m)$.

Now we state the propositions:

(15) If $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent, then $R_1$ is p-convergent and P-lim $R_1$

$= 0$. Proof: For every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|R_1(n, m) - 0| < e$ by [3, (13), (20)], (13), [8, (57)]. $\square$

(16) $(\sum_{\alpha=0}^{\kappa}(R_2 + R_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (8).

(17) Suppose $(\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa\in\mathbb{N}}$ is p-convergent and $(\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa\in\mathbb{N}}$ is p-convergent. Then $(\sum_{\alpha=0}^{\kappa}(R_2 + R_3)(\alpha))_{\kappa\in\mathbb{N}}$ is p-convergent. The theorem is a consequence of (16).

(18) Let us consider elements $m$, $n$ of $\mathbb{N}$. Then

  (i) (the partial sums in the first coordinate of $R_1)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(R_1, n))(\alpha))_{\kappa\in\mathbb{N}}(m)$, and

  (ii) (the partial sums in the second coordinate of $R_1)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(R_1, m))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the first coordinate of $R_1)(\$_1, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(R_1, n))(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [3, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ (the partial sums in the second coordinate of $R_1)(m, \$_1) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(R_1, m))(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k + 1]$. For every natural number $k$, $\mathcal{Q}[k]$ from [3, Sch. 2]. $\square$

(19) (i) $\mathrm{curry}((\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}, 0) = \mathrm{curry}$(the partial sums in the second coordinate of $R_1, 0)$, and

  (ii) $\mathrm{curry}'((\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}, 0) = \mathrm{curry}'$(the partial sums in the first coordinate of $R_1, 0)$.

The theorem is a consequence of (12).

(20) Let us consider non empty sets $C$, $D$, functions $F_1$, $F_2$ from $C\times D$ into $\mathbb{R}$, and an element $c$ of $C$. Then $\mathrm{curry}(F_1+F_2, c) = \mathrm{curry}(F_1, c)+\mathrm{curry}(F_2, c)$.

(21) Let us consider non empty sets $C$, $D$, functions $F_1$, $F_2$ from $C \times D$ into $\mathbb{R}$, and an element $d$ of $D$. Then $\mathrm{curry}'(F_1 + F_2, d) = \mathrm{curry}'(F_1, d) + \mathrm{curry}'(F_2, d)$.

(22) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the first coordinate if and only if the partial sums in the first coordinate of $R_1$ is convergent in the first coordinate. The theorem is a consequence of (19), (12), and (11).

(23) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the second coordinate if and only if the partial sums in the second coordinate of $R_1$ is convergent in the second coordinate. The theorem is a consequence of (19), (12), and (11).

Let us consider a natural number $k$. Now we state the propositions:

(24) Suppose $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the first coordinate. Then (the lim in the first coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}})(k + 1) =$ (the lim

in the first coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}})(k)$ + (the lim in the first coordinate of the partial sums in the first coordinate of $R_1)(k+1)$. The theorem is a consequence of (22).

(25) Suppose $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the second coordinate. Then (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}})(k+1)$ = (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}})(k)$ + (the lim in the second coordinate of the partial sums in the second coordinate of $R_1)(k+1)$. The theorem is a consequence of (23) and (12).

Now we state the propositions:

(26) Suppose $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the first coordinate. Then the lim in the first coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $R_1)(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (19) and (24).

(27) Suppose $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the second coordinate. Then the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}$(the lim in the second coordinate of the partial sums in the second coordinate of $R_1)(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (19) and (25).

## 2. Double Series of Non-Negative Double Sequence

Let us assume that $R_1$ is non-negative yielding. Now we state the propositions:

(28) (i) the partial sums in the second coordinate of $R_1$ is non-negative yielding, and

(ii) the partial sums in the first coordinate of $R_1$ is non-negative yielding.

(29) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing. The theorem is a consequence of (11) and (28).

(30) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is p-convergent if and only if $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is lower bounded and upper bounded. The theorem is a consequence of (29).

Let us consider natural numbers $i$, $j$. Now we state the propositions:

(31) Suppose for every natural numbers $n$, $m$, $R_2(n,m) \leqslant R_3(n,m)$. Then

(i) (the partial sums in the first coordinate of $R_2)(i,j) \leqslant$ (the partial sums in the first coordinate of $R_3)(i,j)$, and

(ii) (the partial sums in the second coordinate of $R_2)(i,j) \leqslant$ (the partial sums in the second coordinate of $R_3)(i,j)$.

Proof: Set $R_4$ = the partial sums in the first coordinate of $R_2$. Set $R_5$ = the partial sums in the first coordinate of $R_3$. Set $C_1$ = the partial sums in the second coordinate of $R_2$. Set $C_2$ = the partial sums in the second coordinate of $R_3$. Define $\mathcal{R}$[natural number] $\equiv R_4(\$_1, j) \leqslant R_5(\$_1, j)$. For every natural number $k$ such that $\mathcal{R}[k]$ holds $\mathcal{R}[k+1]$. For every natural number $k$, $\mathcal{R}[k]$ from [3, Sch. 2]. Define $\mathcal{C}[\text{natural number}] \equiv C_1(i, \$_1) \leqslant C_2(i, \$_1)$. For every natural number $k$ such that $\mathcal{C}[k]$ holds $\mathcal{C}[k+1]$. For every natural number $k$, $\mathcal{C}[k]$ from [3, Sch. 2]. □

(32) Suppose $R_2$ is non-negative yielding and for every natural numbers $n$, $m$, $R_2(n, m) \leqslant R_3(n, m)$. Then $(\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa \in \mathbb{N}}(i, j) \leqslant (\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa \in \mathbb{N}}(i, j)$. Proof: Set $R_4 = (\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa \in \mathbb{N}}$. Set $R_5 = (\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa \in \mathbb{N}}$. Define $\mathcal{P}[\text{natural number}] \equiv R_4(i, \$_1) \leqslant R_5(i, \$_1)$. $\mathcal{P}[0]$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $k$, $\mathcal{P}[k]$ from [3, Sch. 2]. □

Now we state the propositions:

(33) Suppose $R_2$ is non-negative yielding and for every natural numbers $n$, $m$, $R_2(n, m) \leqslant R_3(n, m)$ and $(\sum_{\alpha=0}^{\kappa} R_3(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent. Then $(\sum_{\alpha=0}^{\kappa} R_2(\alpha))_{\kappa \in \mathbb{N}}$ is p-convergent. The theorem is a consequence of (29) and (32).

(34) Let us consider a sequence $r_1$ of real numbers and a natural number $m$. Suppose $r_1$ is non-negative. Then $r_1(m) \leqslant (\sum_{\alpha=0}^{\kappa} r_1(\alpha))_{\kappa \in \mathbb{N}}(m)$. Proof: Define $\mathcal{P}[\text{natural number}] \equiv r_1(\$_1) \leqslant (\sum_{\alpha=0}^{\kappa} r_1(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [19, (34)]. For every natural number $k$, $\mathcal{P}[k]$ from [3, Sch. 2]. □

Let us assume that $R_1$ is non-negative yielding. Now we state the propositions:

(35) Let us consider natural numbers $m$, $n$. Then

   (i) $R_1(m, n) \leqslant$ (the partial sums in the first coordinate of $R_1$)$(m, n)$, and

   (ii) $R_1(m, n) \leqslant$ (the partial sums in the second coordinate of $R_1$)$(m, n)$.

   The theorem is a consequence of (34) and (18).

(36) (i) for every natural numbers $i_1, i_2, j$ such that $i_1 \leqslant i_2$ holds (the partial sums in the first coordinate of $R_1$)$(i_1, j) \leqslant$ (the partial sums in the first coordinate of $R_1$)$(i_2, j)$, and

   (ii) for every natural numbers $i, j_1, j_2$ such that $j_1 \leqslant j_2$ holds (the partial sums in the second coordinate of $R_1$)$(i, j_1) \leqslant$ (the partial sums in the second coordinate of $R_1$)$(i, j_2)$.

(37) (i) for every natural numbers $i_1$, $i_2$, $j$ such that $i_1 \leqslant i_2$ holds $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i_1, j) \leqslant (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i_2, j)$, and

   (ii) for every natural numbers $i$, $j_1$, $j_2$ such that $j_1 \leqslant j_2$ holds $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i, j_1) \leqslant (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(i, j_2)$.

   The theorem is a consequence of (36).

(38) Let us consider natural numbers $i_1$, $i_2$, $j_1$, $j_2$. Suppose

(i) $i_1 \leqslant i_2$, and

(ii) $j_1 \leqslant j_2$.

Then $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}(i_1, j_1) \leqslant (\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}(i_2, j_2)$. The theorem is a consequence of (37).

(39) Let us consider an element $k$ of $\mathbb{N}$. Then

(i) curry′(the partial sums in the first coordinate of $R_1, k$) is non-decreasing, and

(ii) curry(the partial sums in the second coordinate of $R_1, k$) is non-decreasing, and

(iii) curry′(the partial sums in the first coordinate of $R_1, k$) is non-negative, and

(iv) curry(the partial sums in the second coordinate of $R_1, k$) is non-negative, and

(v) curry′(the partial sums in the second coordinate of $R_1, k$) is non-negative, and

(vi) curry(the partial sums in the first coordinate of $R_1, k$) is non-negative.

The theorem is a consequence of (18) and (34).

Let us assume that $R_1$ is non-negative yielding and $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is p-convergent. Now we state the propositions:

(40) (i) the partial sums in the first coordinate of $R_1$ is convergent in the first coordinate, and

(ii) the partial sums in the second coordinate of $R_1$ is convergent in the second coordinate.

The theorem is a consequence of (39), (18), (34), and (29).

(41) $(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}}$ is convergent in the first coordinate and convergent in the second coordinate. The theorem is a consequence of (40), (22), and (23).

(42) (i) the lim in the first coordinate of the partial sums in the first coordinate of $R_1$ is summable, and

(ii) the lim in the second coordinate of the partial sums in the second coordinate of $R_1$ is summable.

The theorem is a consequence of (41), (26), and (27).

(43) (i) P-lim$(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}} = \sum$(the lim in the first coordinate of the partial sums in the first coordinate of $R_1$), and

(ii) P-lim$(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa\in\mathbb{N}} = \sum$(the lim in the second coordinate of the partial sums in the second coordinate of $R_1$).

The theorem is a consequence of (41), (26), and (27).

## 3. Summability for Rearrangements of Non-Negative Real Sequence

Now we state the propositions:

(44) Let us consider sequences $s_1$, $s_2$ of real numbers. Suppose

(i) $s_1$ is non-negative, and

(ii) $s_1$ and $s_2$ are fiberwise equipotent.

Then $s_2$ is non-negative.

(45) Let us consider a non empty set $X$, a sequence $s$ of $X$, and a natural number $n$. Then $\mathrm{dom}\,\mathrm{Shift}(s{\restriction}\mathbb{Z}_n, 1) = \mathrm{Seg}\,n$.

Let $X$ be a non empty set, $s$ be a sequence of $X$, and $n$ be a natural number. Note that $\mathrm{Shift}(s{\restriction}\mathbb{Z}_n, 1)$ is finite sequence-like.

Now we state the propositions:

(46) Let us consider a non empty set $X$, a sequence $s$ of $X$, and a natural number $n$. Then $\mathrm{Shift}(s{\restriction}\mathbb{Z}_n, 1)$ is a finite sequence of elements of $X$.

(47) Let us consider a non empty set $X$, a sequence $s$ of $X$, and natural numbers $n$, $m$. Suppose $m+1 \in \mathrm{dom}\,\mathrm{Shift}(s{\restriction}\mathbb{Z}_n, 1)$. Then $(\mathrm{Shift}(s{\restriction}\mathbb{Z}_n, 1))(m+1) = s(m)$.

(48) Let us consider a non empty set $X$ and a sequence $s$ of $X$. Then

(i) $\mathrm{Shift}(s{\restriction}\mathbb{Z}_0, 1) = \emptyset$, and

(ii) $\mathrm{Shift}(s{\restriction}\mathbb{Z}_1, 1) = \langle s(0)\rangle$, and

(iii) $\mathrm{Shift}(s{\restriction}\mathbb{Z}_2, 1) = \langle s(0), s(1)\rangle$, and

(iv) for every natural number $n$, $\mathrm{Shift}(s{\restriction}\mathbb{Z}_{n+1}, 1) = \mathrm{Shift}(s{\restriction}\mathbb{Z}_n, 1) ^\frown \langle s(n)\rangle$.

The theorem is a consequence of (45) and (47).

(49) Let us consider a sequence $s$ of real numbers and a natural number $n$. Then $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) = \sum \mathrm{Shift}(s{\restriction}\mathbb{Z}_{n+1}, 1)$. Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1) = \sum \mathrm{Shift}(s{\restriction}\mathbb{Z}_{\$_1+1}, 1)$. $\mathrm{Shift}(s{\restriction}\mathbb{Z}_{0+1}, 1) = \langle s(0)\rangle$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (48), [14, (74)]. For every natural number $k$, $\mathcal{P}[k]$ from [3, Sch. 2]. □

(50) Let us consider a non empty set $X$, sequences $s_1$, $s_2$ of $X$, and a natural number $n$. Suppose $s_1$ and $s_2$ are fiberwise equipotent. Then there exists a natural number $m$ and there exists a subset $f_2$ of $\mathrm{Shift}(s_2{\restriction}\mathbb{Z}_m, 1)$ such that $\mathrm{Shift}(s_1{\restriction}\mathbb{Z}_{n+1}, 1)$ and $f_2$ are fiberwise equipotent. Proof: Consider $P$ being a permutation of $\mathrm{dom}\,s_1$ such that $s_1 = s_2 \cdot P$. Define $\mathcal{F}(\text{set}) = P(\$_1) + 1$. Define $\mathcal{G}[\text{set}] \equiv \$_1$ is a natural number. $\{\mathcal{F}(i)$, where $i$ is a natural number : $i \leqslant n$ and $\mathcal{G}[i]\}$ is finite from [6, Sch. 6]. Reconsider $D = \{\mathcal{F}(i)$, where $i$ is a natural number : $i \leqslant n$ and $\mathcal{G}[i]\}$ as a finite set. Set $f_2 = \{\langle j+1, s_2(j)\rangle$, where $j$ is a natural number : $j+1 \in D\}$. Define $\mathcal{G}[\text{object}, \text{object}] \equiv$ there exists a natural number $i$ such that $\$_1 = i+1$

and $\$_2 = P(i) + 1$. For every object $x$ such that $x \in \mathrm{Seg}(n + 1)$ there exists an object $y$ such that $\mathcal{G}[x, y]$ by [6, (1)], [3, (21)]. Consider $G$ being a function such that $\mathrm{dom}\, G = \mathrm{Seg}(n + 1)$ and for every object $x$ such that $x \in \mathrm{Seg}(n + 1)$ holds $\mathcal{G}[x, G(x)]$ from [11, Sch. 2]. $\mathrm{dom}\, G = \mathrm{dom}\,\mathrm{Shift}(s_1 \restriction \mathbb{Z}_{n+1}, 1)$. $\mathrm{dom}(f_2 \cdot G) = \mathrm{dom}\,\mathrm{Shift}(s_1 \restriction \mathbb{Z}_{n+1}, 1)$. For every object $x$ such that $x \in \mathrm{dom}\,\mathrm{Shift}(s_1 \restriction \mathbb{Z}_{n+1}, 1)$ holds $(\mathrm{Shift}(s_1 \restriction \mathbb{Z}_{n+1}, 1))(x) = (f_2 \cdot G)(x)$ by (45), [6, (1)], [3, (21)], (47). $\square$

(51) Let us consider a non empty set $X$, a finite sequence $f_1$ of elements of $X$, and a subset $f_3$ of $f_1$. Then $\mathrm{Seq}\, f_3$ and $f_3$ are fiberwise equipotent.

(52) Let us consider sequences $s_1$, $s_2$ of real numbers and a natural number $n$. Suppose

(i) $s_1$ and $s_2$ are fiberwise equipotent, and

(ii) $s_1$ is non-negative.

Then there exists a natural number $m$ such that $(\sum_{\alpha=0}^{\kappa} s_1(\alpha))_{\kappa \in \mathbb{N}}(n) \leqslant (\sum_{\alpha=0}^{\kappa} s_2(\alpha))_{\kappa \in \mathbb{N}}(m)$. The theorem is a consequence of (44), (50), (46), (51), (47), (49), and (48).

(53) Let us consider sequences $s_1$, $s_2$ of real numbers. Suppose

(i) $s_1$ and $s_2$ are fiberwise equipotent, and

(ii) $s_1$ is non-negative and summable.

Then

(iii) $s_2$ is summable, and

(iv) $\sum s_1 = \sum s_2$.

The theorem is a consequence of (44) and (52).

## 4. Basic Relations between Double Sequences and Matrices

Now we state the propositions:

(54) Let us consider a non empty set $D$, a function $R_1$ from $\mathbb{N} \times \mathbb{N}$ into $D$, and natural numbers $n$, $m$. Then there exists a matrix $M$ over $D$ of dimension $n+1 \times m+1$ such that for every natural numbers $i$, $j$ such that $i \leqslant n$ and $j \leqslant m$ holds $R_1(i, j) = M_{i+1,j+1}$. Proof: Define $\mathcal{P}[$natural number, natural number, object$] \equiv$ there exist natural numbers $i_1$, $j_1$ such that $i_1 = \$_1 - 1$ and $j_1 = \$_2 - 1$ and $\$_3 = R_1(i_1, j_1)$. Consider $M$ being a matrix over $D$ of dimension $n + 1 \times m + 1$ such that for every natural numbers $i$, $j$ such that $\langle i, j \rangle \in$ the indices of $M$ holds $\mathcal{P}[i, j, M_{i,j}]$. $\square$

(55) Let us consider natural numbers $n$, $m$, a function $R_1$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, and a matrix $M$ over $\mathbb{R}$ of dimension $n + 1 \times m + 1$. Suppose natural numbers $i$, $j$. If $i \leqslant n$ and $j \leqslant m$, then $R_1(i, j) = M_{i+1,j+1}$. Then

$(\sum_{\alpha=0}^{\kappa} R_1(\alpha))_{\kappa \in \mathbb{N}}(n,m) = \mathrm{SumAll}\, M$. Proof: For every natural number $i$ such that $i \leqslant n$ holds (the partial sums in the second coordinate of $R_1)(i,m) = (\mathrm{LineSum}\, M)(i+1)$ by [3, (11)], [6, (1), (59)], [26, (112)]. Define $\mathcal{G}[\text{natural number}] \equiv$ if $\$_1 \leqslant n$, then (the partial sums in the first coordinate of the partial sums in the second coordinate of $R_1)(\$_1, m) = \sum(\mathrm{LineSum}\, M {\restriction} (\$_1 + 1))$. For every natural number $k$ such that $\mathcal{G}[k]$ holds $\mathcal{G}[k+1]$ by [3, (11)], [30, (20)], [6, (59)], [10, (21)]. For every natural number $k$, $\mathcal{G}[k]$ from [3, Sch. 2]. □

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