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Brouwer Invariance of Domain Theorem[1]

Karol Pąk
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok
Poland

Summary. In this article we focus on a special case of the Brouwer invariance of domain theorem. Let us A, B be a subsets of $\mathcal{E}^n$, and $f : A \to B$ be a homeomorphic. We prove that, if A is closed then f transform the boundary of A to the boundary of B; and if B is closed then f transform the interior of A to the interior of B. These two cases are sufficient to prove the topological invariance of dimension, which is used to prove basic properties of the n-dimensional manifolds, and also to prove basic properties of the boundary and the interior of manifolds, e.g. the boundary of an n-dimension manifold with boundary is an $(n-1)$-dimension manifold. This article is based on [18]; [21] and [20] can also serve as reference books.

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The notation and terminology used in this paper have been introduced in the following articles: [27], [1], [14], [4], [6], [15], [37], [7], [8], [40], [31], [34], [38], [2], [3], [9], [5], [33], [13], [44], [45], [10], [42], [43], [35], [17], [28], [29], [25], [46], [16], [47], [26], [30], [32], and [12].

1. Preliminaries

From now on x, X denote sets, n, m, i denote natural numbers, p, q denote points of $\mathcal{E}^n_{\mathrm{T}}$, A, B denote subsets of $\mathcal{E}^n_{\mathrm{T}}$, and r, s denote real numbers.

Let us consider X and n. One can verify that every function from X into $\mathcal{E}^n_{\mathrm{T}}$ is finite sequence-yielding.

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Let us consider m. Let f be a function from X into $\mathcal{E}^{n}_{\mathrm{T}}$ and g be a function from X into $\mathcal{E}^{m}_{\mathrm{T}}$. Let us observe that the functor $f \frown g$ yields a function from X into $\mathcal{E}^{n+m}_{\mathrm{T}}$. Let T be a topological space. Let f be a continuous function from T into $\mathcal{E}^{n}_{\mathrm{T}}$ and g be a continuous function from T into $\mathcal{E}^{m}_{\mathrm{T}}$. Note that $f \frown g$ is continuous as a function from T into $\mathcal{E}^{n+m}_{\mathrm{T}}$.

Let f be a real-valued function. The functor $|[f]|$ yielding a function is defined by

(Def. 1) (i) $\mathrm{dom}\, it = \mathrm{dom}\, f$, and

(ii) for every object x such that $x \in \mathrm{dom}\, it$ holds $it(x) = |[f(x)]|$.

One can verify that $|[f]|$ is (the carrier of $\mathcal{E}^{1}_{\mathrm{T}}$)-valued.

Let us consider X. Let Y be a non empty real-membered set and f be a function from X into Y. One can verify that the functor $|[f]|$ yields a function from X into $\mathcal{E}^{1}_{\mathrm{T}}$. Let T be a non empty topological space and f be a continuous function from T into $\mathbb{R}^{\mathbf{1}}$. Note that $|[f]|$ is continuous as a function from T into $\mathcal{E}^{1}_{\mathrm{T}}$.

Let f be a continuous real map of T. Observe that $|[f]|$ is continuous as a function from T into $\mathcal{E}^{1}_{\mathrm{T}}$.

2. A Distribution of Sphere

In the sequel N denotes a non zero natural number and u, t denote points of $\mathcal{E}^{N+1}_{\mathrm{T}}$.

Now we state the propositions:

(1) Let us consider an element F of ((the carrier of $\mathbb{R}^{\mathbf{1}})^{\alpha})^{N}$. Suppose If $i \in \mathrm{dom}\, F$, then $F(i) = \mathrm{PROJ}(N+1, i)$. Then

(i) for every t, $(\prod^{*} F)(t) = t{\restriction}N$, and

(ii) for every subsets S_3, S_2 of $\mathcal{E}^{N+1}_{\mathrm{T}}$ such that $S_3 = \{u : u(N+1) \geqslant 0 \text{ and } |u| = 1\}$ and $S_2 = \{t : t(N+1) \leqslant 0 \text{ and } |t| = 1\}$ holds $(\prod^{*} F)^{\circ} S_3 = \overline{\mathrm{Ball}}(0_{\mathcal{E}^{N}_{\mathrm{T}}}, 1)$ and $(\prod^{*} F)^{\circ} S_2 = \overline{\mathrm{Ball}}(0_{\mathcal{E}^{N}_{\mathrm{T}}}, 1)$ and $(\prod^{*} F)^{\circ}(S_3 \cap S_2) = \mathrm{Sphere}(0_{\mathcal{E}^{N}_{\mathrm{T}}}, 1)$ and for every function H from $\mathcal{E}^{N+1}_{\mathrm{T}}{\restriction}S_3$ into $\mathrm{Tdisk}(0_{\mathcal{E}^{N}_{\mathrm{T}}}, 1)$ such that $H = \prod^{*} F{\restriction}S_3$ holds H is a homeomorphism and for every function H from $\mathcal{E}^{N+1}_{\mathrm{T}}{\restriction}S_2$ into $\mathrm{Tdisk}(0_{\mathcal{E}^{N}_{\mathrm{T}}}, 1)$ such that $H = \prod^{*} F{\restriction}S_2$ holds H is a homeomorphism,

where α is the carrier of $\mathcal{E}^{N+1}_{\mathrm{T}}$. Proof: Set $N_2 = N+1$. Set $T_{10} = \mathcal{E}^{N_2}_{\mathrm{T}}$. Set $T_4 = \mathcal{E}^{N}_{\mathrm{T}}$. Set $N_3 = N\,\mathrm{NormF}$. Set $N_4 = N_3 \cdot N_3$. Reconsider $O = 1$ as an element of $\mathbb{N}$. Set $T_3 = \mathrm{Tdisk}(0_{\mathcal{E}^{N}_{\mathrm{T}}}, 1)$. Reconsider $m_2 = -N_4$ as a function from T_4 into $\mathbb{R}^{\mathbf{1}}$. Reconsider $m_1 = 1 + m_2$ as a function from T_4 into $\mathbb{R}^{\mathbf{1}}$. Set $F_1 = \prod^{*} F$. For every t, $(\prod^{*} F)(t) = t{\restriction}N$ by [2, (13)], [41, (25)], [4, (1)]. $\overline{\mathrm{Ball}}(0_{T_4}, 1) \subseteq F_1{}^{\circ} S_3$ by [14, (22)], [28, (11)], [6, (16)], [11, (145)]. $\overline{\mathrm{Ball}}(0_{T_4}, 1) \subseteq F_1{}^\circ S_2$ by [14, (22)], [28, (11)], [6, (16)], [11, (145)]. $\mathrm{Sphere}(0_{T_4}, 1) \subseteq F_1{}^\circ(S_2 \cap S_3)$ by [14, (22)], [28, (12)], [6, (16), (92)]. $F_1{}^\circ S_3 \subseteq \overline{\mathrm{Ball}}(0_{T_4}, 1)$ by [14, (22)], [4, (59)], [24, (17)], [19, (10)]. $F_1{}^\circ S_2 \subseteq \overline{\mathrm{Ball}}(0_{T_4}, 1)$ by [14, (22)], [4, (59)], [24, (17)], [19, (10)]. $F_1{}^\circ(S_2 \cap S_3) \subseteq \mathrm{Sphere}(0_{T_4}, 1)$ by [14, (22)], [4, (59)], [24, (17)], [19, (10)]. For every function H from $\mathcal{E}^{N+1}_{\mathrm{T}} \restriction S_3$ into $\mathrm{Tdisk}(0_{\mathcal{E}^N_{\mathrm{T}}}, 1)$ such that $H = \prod^* F \restriction S_3$ holds H is a homeomorphism by [24, (17)], [17, (17)], [2, (11)], [25, (13)]. For every objects x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} H$ and $H(x_1) = H(x_2)$ holds $x_1 = x_2$ by [14, (22)], [19, (10)], [7, (47)], [39, (40)]. Set $T_3 = \mathrm{Tdisk}(0_{T_4}, 1)$. Set $M = m_1 \restriction T_3$. Reconsider $M_1 = M$ as a continuous function from T_3 into $\mathbb{R}$. Reconsider $M_2 = -\sqrt{M_1}$ as a function from T_3 into $\mathbb{R}$. For every point p of T_4 such that $p \in$ the carrier of T_3 holds $M_1(p) = 1 - |p| \cdot |p|$ by [7, (49)]. Reconsider $S_1 = |[M_2]|$ as a continuous function from T_3 into $\mathcal{E}^1_{\mathrm{T}}$. Reconsider $I_3 = \mathrm{id}_{T_3}$ as a continuous function from T_3 into T_4. Reconsider $I_4 = I_3 {}^\frown S_1$ as a continuous function from T_3 into $\mathcal{E}^{N+O}_{\mathrm{T}}$. For every objects y, x, $y \in \operatorname{rng} H$ and $x = I_4(y)$ iff $x \in \operatorname{dom} H$ and $y = H(x)$ by [7, (17)], [11, (145), (144), (55)]. For every subset P of $T_{10} \restriction S_2$, P is open iff $H^\circ P$ is open by [4, (1)], [2, (13)], [25, (57)]. □

(2) Let us consider subsets S_3, S_2 of $\mathcal{E}^n_{\mathrm{T}}$. Suppose

(i) $S_3 = \{s$, where s is a point of $\mathcal{E}^n_{\mathrm{T}} : s(n) \geqslant 0$ and $|s| = 1\}$, and

(ii) $S_2 = \{t$, where t is a point of $\mathcal{E}^n_{\mathrm{T}} : t(n) \leqslant 0$ and $|t| = 1\}$.

Then

(iii) S_3 is closed, and

(iv) S_2 is closed.

(3) Let us consider a metrizable topological space T_2. Suppose T_2 is finite-ind and second-countable. Let us consider a closed subset F of T_2. Suppose $\operatorname{ind} F^{\mathrm{c}} \leqslant n$. Let us consider a continuous function f from $T_2 \restriction F$ into $\mathrm{TopUnitCircle}(n + 1)$. Then there exists a continuous function g from T_2 into $\mathrm{TopUnitCircle}(n + 1)$ such that $g \restriction F = f$. Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every metrizable topological space T_2 such that T_2 is finite-ind and second-countable for every closed subset F of T_2 such that $\operatorname{ind} F^{\mathrm{c}} \leqslant \$_1$ for every continuous function f from $T_2 \restriction F$ into $\mathrm{TopUnitCircle}(\$_1 + 1)$, there exists a function g from T_2 into TopUnitCircle $(\$_1 + 1)$ such that g is continuous and $g \restriction F = f$. For every n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$ by (2), [29, (9)], [42, (13)], [44, (121)]. $\mathcal{P}[(0$ **qua** natural number$)]$ by [44, (143), (135)], [29, (9)], [14, (70)]. For every n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

(4) Suppose $p \notin A$ and $r > 0$. Then there exists a function h from $\mathcal{E}^n_{\mathrm{T}} \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction \mathrm{Sphere}(p, r)$ such that

(i) h is continuous, and

(ii) $h \restriction \mathrm{Sphere}(p, r) = \mathrm{id}_{A \cap \mathrm{Sphere}(p,r)}$.

(5) If $r + |p - q| \leqslant s$, then $\mathrm{Ball}(p, r) \subseteq \mathrm{Ball}(q, s)$.

(6) If A is not boundary, then $\mathrm{ind}\, A = n$.

Now we state the proposition:

(7) The Small Inductive Dimension of the Sphere:
If $r > 0$, then $\mathrm{ind}\, \mathrm{Sphere}(p, r) = n - 1$. Proof: If $\mathrm{ind}\, A \leqslant i$ and $\mathrm{ind}\, B \leqslant i$ and A is closed, then $\mathrm{ind}(A \cup B) \leqslant i$ by [33, (31)], [23, (93)], [35, (22)], [36, (5)]. □

3. A Characterization of Open Sets in Euclidean Space in Terms of Continuous Transformations

Now we state the propositions:

(8) Suppose $n > 0$ and $p \in A$ and for every r such that $r > 0$ there exists an open subset U of $\mathcal{E}^n_{\mathrm{T}} \restriction A$ such that $p \in U$ and $U \subseteq \mathrm{Ball}(p, r)$ and for every function f from $\mathcal{E}^n_{\mathrm{T}} \restriction (A \setminus U)$ into $\mathrm{TopUnitCircle}\, n$ such that f is continuous there exists a function h from $\mathcal{E}^n_{\mathrm{T}} \restriction A$ into $\mathrm{TopUnitCircle}\, n$ such that h is continuous and $h \restriction (A \setminus U) = f$. Then $p \in \mathrm{Fr}\, A$. Proof: Set $T_7 = \mathcal{E}^n_{\mathrm{T}}$. Set $c_1 =$ the carrier of T_7. Set $S = \mathrm{Sphere}(0_{T_7}, 1)$. Set $T_9 = \mathrm{TopUnitCircle}\, n$. Reconsider $c = c_1 \setminus \{0_{T_7}\}$ as a non empty open subset of T_7. Set $n_3 = n\, \mathrm{NormF}$. Set $T_8 = T_7 \restriction c$. Set $G = \mathrm{transl}(p, T_7)$. Reconsider $I = \overset{T_8}{\hookrightarrow}$ as a continuous function from T_8 into T_7. $0 \notin \mathrm{rng}(n_3 \restriction T_8)$ by [44, (57)], [14, (22)], [7, (47)], [14, (8), (70)]. Reconsider $n_2 = n_3 \restriction T_8$ as a non-empty continuous function from T_8 into $\mathbb{R}^{\mathbf{1}}$. Reconsider $b = I / n_2$ as a function from T_8 into T_7. Set $E_1 = \mathcal{E}^n$. Set $T_2 = E_{1\,\mathrm{top}}$. Reconsider $e = p$ as a point of E_1. Reconsider $I_1 = \mathrm{Int}\, A$ as a subset of T_2. Consider r being a real number such that $r > 0$ and $\mathrm{Ball}(e, r) \subseteq I_1$. Set $r_2 = \frac{r}{2}$. Consider U being an open subset of $T_7 \restriction A$ such that $p \in U$ and $U \subseteq \mathrm{Ball}(p, r_2)$ and for every function f from $T_7 \restriction (A \setminus U)$ into T_9 such that f is continuous there exists a function h from $T_7 \restriction A$ into T_9 such that h is continuous and $h \restriction (A \setminus U) = f$. Reconsider $S_4 = \mathrm{Sphere}(p, r_2)$ as a non empty subset of T_7. Consider a being an object such that $a \in S_4$. Reconsider $C_2 = \overline{\mathrm{Ball}}(p, r_2)$ as a non empty subset of T_7. Reconsider $s_2 = S_4$ as a non empty subset of $T_7 \restriction C_2$. Reconsider $A_1 = A \setminus U$ as a non empty subset of T_7. Set $T_1 = T_7 \restriction A_1$. Set $t = \mathrm{transl}(-p, T_7)$. Set $T = t \restriction T_1$. $\mathrm{rng}\, T \subseteq c$ by [7, (47)], [42, (21)]. Reconsider $T_1 1 = T$ as a continuous function from T_1 into T_8. For every point p of T_7 such that $p \in c$ holds $b(p) = \frac{1}{|p|} \cdot p$ and $|\frac{1}{|p|} \cdot p| = 1$ by [22, (84)], [7, (49)], [26, (72)], [12, (56)]. $\mathrm{rng}\, b \subseteq S$ by [42, (13)]. Reconsider $B = b$ as a function from T_8 into T_9. Set $m = r_2 \bullet T_7$. Set $M = m \restriction T_9$. Reconsider $M = m \restriction T_9$ as a continuous function from T_9 into T_7. Reconsider $c_2 = C_2$ as a subset of $T_7 \restriction A$. Consider h being a function from $T_7 \restriction A$ into T_9 such

that h is continuous and $h{\restriction}(A \setminus U) = B \cdot T_1 1$. Reconsider $G_2 = G \cdot (M \cdot h)$ as a continuous function from $T_7{\restriction}A$ into T_7. $\operatorname{rng} G_2 \subseteq S_4$ by [7, (12), (11), (47)], [42, (28), (15)]. Reconsider $g_2 = G_2$ as a function from $T_7{\restriction}A$ into $T_7{\restriction}S_4$. Reconsider $g_1 = g_2{\restriction}((T_7{\restriction}A){\restriction}c_2)$ as a continuous function from $T_7{\restriction}C_2$ into $(T_7{\restriction}C_2){\restriction}s_2$. For every point w of $T_7{\restriction}C_2$ such that $w \in S_4$ holds $g_1(w) = w$ by [7, (11), (12)], [44, (61)], [7, (47)]. □

(9) Suppose $p \in \operatorname{Fr} A$ and A is closed. Suppose $r > 0$. Then there exists an open subset U of $\mathcal{E}^n_{\mathrm{T}}{\restriction}A$ such that

  (i) $p \in U$, and
  (ii) $U \subseteq \operatorname{Ball}(p, r)$, and
  (iii) for every function f from $\mathcal{E}^n_{\mathrm{T}}{\restriction}(A \setminus U)$ into TopUnitCircle n such that f is continuous there exists a function h from $\mathcal{E}^n_{\mathrm{T}}{\restriction}A$ into TopUnitCircle n such that h is continuous and $h{\restriction}(A \setminus U) = f$.

Proof: $n > 0$ by [14, (77), (22)], [12, (33)]. Set $r_3 = \frac{r}{3}$. Set $r_2 = 2 \cdot r_3$. Set $B = \operatorname{Ball}(p, r_3)$. Consider x being an object such that $x \in A^{\mathrm{c}}$ and $x \in B$. Set $u = \operatorname{Ball}(x, r_2)$. $u \subseteq \operatorname{Ball}(p, r)$. □

4. Brouwer Invariance of Domain Theorem – Special Case

Let us consider a function h from $\mathcal{E}^n_{\mathrm{T}}{\restriction}A$ into $\mathcal{E}^n_{\mathrm{T}}{\restriction}B$. Now we state the propositions:

(10) If A is closed and $p \in \operatorname{Fr} A$, then if h is a homeomorphism, then $h(p) \in \operatorname{Fr} B$. The theorem is a consequence of (9) and (8).

(11) If B is closed and $p \in \operatorname{Int} A$, then if h is a homeomorphism, then $h(p) \in \operatorname{Int} B$. The theorem is a consequence of (8) and (9).

(12) Suppose A is closed and B is closed. Then if h is a homeomorphism, then $h^{\circ}(\operatorname{Int} A) = \operatorname{Int} B$ and $h^{\circ}(\operatorname{Fr} A) = \operatorname{Fr} B$. Proof: $h^{\circ}(\operatorname{Int} A) = \operatorname{Int} B$ by (11), (10), [46, (39)]. □

5. Topological Invariance of Dimension – An Introduction to Manifolds

Now we state the proposition:

(13) Suppose $r > 0$. Let us consider a subset U of $\operatorname{Tdisk}(p, r)$. Suppose U is open and non empty. Let us consider a subset A of $\mathcal{E}^n_{\mathrm{T}}$. If $A = U$, then $\operatorname{Int} A$ is not empty.

Let us consider a non empty topological space T, subsets A, B of T, r, s, a point p_1 of $\mathcal{E}^n_{\mathrm{T}}$, and a point p_2 of $\mathcal{E}^m_{\mathrm{T}}$.

Let us assume that $r > 0$ and $s > 0$. Now we state the propositions:

(14) Suppose $T\restriction A$ and $\mathrm{Tdisk}(p_1, r)$ are homeomorphic and $T\restriction B$ and $\mathrm{Tdisk}(p_2, s)$ are homeomorphic and $\mathrm{Int}\,A$ meets $\mathrm{Int}\,B$. Then $n = m$. The theorem is a consequence of (13) and (6).

(15) Suppose $T\restriction A$ and $\mathcal{E}^n_{\mathrm{T}}\restriction \mathrm{Ball}(p_1, r)$ are homeomorphic and $T\restriction B$ and $\mathrm{Tdisk}(p_2, s)$ are homeomorphic and $\mathrm{Int}\,A$ meets $\mathrm{Int}\,B$. Then $n = m$. The theorem is a consequence of (13) and (6).

Now we state the propositions:

(16) (i) $(\mathrm{transl}(p, \mathcal{E}^n_{\mathrm{T}}))^\circ(\mathrm{Ball}(q, r)) = \mathrm{Ball}(q + p, r)$, and

(ii) $(\mathrm{transl}(p, \mathcal{E}^n_{\mathrm{T}}))^\circ(\overline{\mathrm{Ball}}(q, r)) = \overline{\mathrm{Ball}}(q + p, r)$, and

(iii) $(\mathrm{transl}(p, \mathcal{E}^n_{\mathrm{T}}))^\circ(\mathrm{Sphere}(q, r)) = \mathrm{Sphere}((q + p), r)$.

Proof: Set $T_5 = \mathcal{E}^n_{\mathrm{T}}$. Set $T = \mathrm{transl}(p, T_5)$. $T^\circ(\mathrm{Ball}(q, r)) = \mathrm{Ball}(q + p, r)$ by [28, (7)], [42, (27)]. $T^\circ(\overline{\mathrm{Ball}}(q, r)) = \overline{\mathrm{Ball}}(q+p, r)$ by [28, (8)], [42, (27)]. $T^\circ(\mathrm{Sphere}(q, r)) \subseteq \mathrm{Sphere}((q + p), r)$ by [28, (9)]. $\square$

(17) Suppose $s > 0$. Then

(i) $(s \bullet \mathcal{E}^n_{\mathrm{T}})^\circ(\mathrm{Ball}(p, r)) = \mathrm{Ball}(s \cdot p, r \cdot s)$, and

(ii) $(s \bullet \mathcal{E}^n_{\mathrm{T}})^\circ(\overline{\mathrm{Ball}}(p, r)) = \overline{\mathrm{Ball}}(s \cdot p, r \cdot s)$, and

(iii) $(s \bullet \mathcal{E}^n_{\mathrm{T}})^\circ(\mathrm{Sphere}(p, r)) = \mathrm{Sphere}((s \cdot p), (r \cdot s))$.

Proof: Set $T_5 = \mathcal{E}^n_{\mathrm{T}}$. Set $M = s \bullet T_5$. $M^\circ(\mathrm{Ball}(p, r)) = \mathrm{Ball}(s \cdot p, r \cdot s)$ by [42, (34)], [14, (11)], [28, (7)]. $M^\circ(\overline{\mathrm{Ball}}(p, r)) = \overline{\mathrm{Ball}}(s \cdot p, r \cdot s)$ by [42, (34)], [14, (11)], [28, (8)]. $M^\circ(\mathrm{Sphere}(p, r)) \subseteq \mathrm{Sphere}((s \cdot p), (r \cdot s))$ by [42, (34)], [14, (11)], [28, (9)]. $\square$

(18) Let us consider a rotation homogeneous additive function f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose f is onto. Then

(i) $f^\circ(\mathrm{Ball}(p, r)) = \mathrm{Ball}(f(p), r)$, and

(ii) $f^\circ(\overline{\mathrm{Ball}}(p, r)) = \overline{\mathrm{Ball}}(f(p), r)$, and

(iii) $f^\circ(\mathrm{Sphere}(p, r)) = \mathrm{Sphere}((f(p)), r)$.

Proof: $f^\circ(\mathrm{Ball}(p, r)) = \mathrm{Ball}(f(p), r)$ by [28, (7)]. $f^\circ(\overline{\mathrm{Ball}}(p, r)) = \overline{\mathrm{Ball}}(f(p), r)$ by [28, (8)]. $f^\circ(\mathrm{Sphere}(p, r)) \subseteq \mathrm{Sphere}((f(p)), r)$ by [28, (9)]. Consider x being an object such that $x \in \mathrm{dom}\, f$ and $f(x) = y$. $\square$

(19) Let us consider points p, q of $\mathcal{E}^{n+1}_{\mathrm{T}}$, r, and s. Suppose

(i) $s \leqslant r \leqslant |p - q|$, and

(ii) $s < |p - q| < s + r$.

Then there exists a function h from $\mathcal{E}^{n+1}_{\mathrm{T}}\restriction(\mathrm{Sphere}(p, r) \cap \overline{\mathrm{Ball}}(q, s))$ into $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ such that

(iii) h is a homeomorphism, and

(iv) $h^\circ(\mathrm{Sphere}(p, r) \cap \mathrm{Sphere}(q, s)) = \mathrm{Sphere}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$.

Proof: Set $n_1 = n + 1$. Set $T_6 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Set $y = \frac{1}{r} \cdot (q - p)$. Set $Y = \langle\underbrace{0, \ldots, 0}_{n_1}\rangle +\cdot (n_1, |y|)$. There exists a homogeneous additive rotation function R from T_6 into T_6 such that R is a homeomorphism and $R(y) = Y$ by [34, (40), (41)]. Consider R being a homogeneous additive rotation function from T_6 into T_6 such that R is a homeomorphism and $R(y) = Y$. $s > 0$. □

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