

Isometric Differentiable Functions on Real Normed Space¹

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Summary. In this article, we formalize isometric differentiable functions on real normed space [17], and their properties.

MSC: 58C20 46G05 03B35

Keywords: isometric differentiable function

MML identifier: NDIFF_7, version: 8.1.02 5.22.1194

The notation and terminology used in this paper have been introduced in the following articles: [3], [2], [8], [4], [5], [18], [10], [11], [19], [14], [16], [1], [6], [9], [15], [23], [24], [21], [22], [13], [25], and [7].

1. PRELIMINARIES

From now on S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T , Z denotes a subset of S , and i, n denote natural numbers.

Now we state the propositions:

- (1) Let us consider a set X and functions I, f . Then $(f \upharpoonright X) \cdot I = (f \cdot I) \upharpoonright I^{-1}(X)$.
- (2) Let us consider real normed spaces S, T , a linear operator L from S into T , and points x, y of S . Then $L(x) - L(y) = L(x - y)$.

¹This work was supported by JSPS KAKENHI 23500029 and 22300285.

- (3) Let us consider real normed spaces X, Y, W , a function I from X into Y , and partial functions f_1, f_2 from Y to W . Then

- (i) $(f_1 + f_2) \cdot I = f_1 \cdot I + f_2 \cdot I$, and
- (ii) $(f_1 - f_2) \cdot I = f_1 \cdot I - f_2 \cdot I$.

PROOF: Set $D_1 =$ the carrier of X . For every element s of D_1 , $s \in \text{dom}((f_1 + f_2) \cdot I)$ iff $s \in \text{dom}(f_1 \cdot I + f_2 \cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}((f_1 + f_2) \cdot I)$ holds $((f_1 + f_2) \cdot I)(z) = (f_1 \cdot I + f_2 \cdot I)(z)$ by [4, (11), (12)]. For every element s of D_1 , $s \in \text{dom}((f_1 - f_2) \cdot I)$ iff $s \in \text{dom}(f_1 \cdot I - f_2 \cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}((f_1 - f_2) \cdot I)$ holds $((f_1 - f_2) \cdot I)(z) = (f_1 \cdot I - f_2 \cdot I)(z)$ by [4, (11), (12)]. \square

- (4) Let us consider real normed spaces X, Y, W , a function I from X into Y , a partial function f from Y to W , and a real number r . Then $r \cdot (f \cdot I) = (r \cdot f) \cdot I$. PROOF: Set $D_1 =$ the carrier of X . For every element s of D_1 , $s \in \text{dom}((r \cdot f) \cdot I)$ iff $s \in \text{dom}(f \cdot I)$ by [4, (11)]. For every element s of D_1 , $s \in \text{dom}(r \cdot (f \cdot I))$ iff $I(s) \in \text{dom}(r \cdot f)$ by [4, (11)]. For every element z of D_1 such that $z \in \text{dom}(r \cdot (f \cdot I))$ holds $(r \cdot (f \cdot I))(z) = ((r \cdot f) \cdot I)(z)$ by [4, (12)]. \square

- (5) Let us consider a partial function f from T to W , a function g from S into T , and a point x of S . Suppose

- (i) $x \in \text{dom } g$, and
- (ii) $g_x \in \text{dom } f$, and
- (iii) g is continuous in x , and
- (iv) f is continuous in g_x .

Then $f \cdot g$ is continuous in x . PROOF: Set $h = f \cdot g$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in \text{dom } h$ and $\|x_1 - x\| < s$ holds $\|h_{x_1} - h_x\| < r$ by [14, (7)], [12, (3), (4)]. \square

Let X, Y be real normed spaces and x be an element of $X \times Y$. The functor $\text{reproj1}(x)$ yielding a function from X into $X \times Y$ is defined by

(Def. 1) Let us consider an element r of X . Then $it(r) = \langle r, x_2 \rangle$.

The functor $\text{reproj2}(x)$ yielding a function from Y into $X \times Y$ is defined by

(Def. 2) Let us consider an element r of Y . Then $it(r) = \langle x_1, r \rangle$.

2. ISOMETRIES

Now we state the propositions:

- (6) Let us consider a linear operator I from S into T and a point x of S . If I is isometric, then I is continuous in x .

- (7) Let us consider real normed spaces S , T and a linear operator f from S into T . Then f is isometric if and only if for every element x of S , $\|f(x)\| = \|x\|$. The theorem is a consequence of (2).
- (8) Let us consider a linear operator I from S into T and a subset Z of S . If I is isometric, then I is continuous on Z . The theorem is a consequence of (6).
- (9) Let us consider a linear operator I from S into T . Suppose I is one-to-one, onto, and isometric. Then there exists a linear operator J from T into S such that
 - (i) $J = I^{-1}$, and
 - (ii) J is one-to-one, onto, and isometric.

The theorem is a consequence of (7). PROOF: Reconsider $J = I^{-1}$ as a function from T into S . For every points v, w of T , $J(v+w) = J(v) + J(w)$ by [5, (113)], [4, (34)]. For every point v of T and for every real number r , $J(r \cdot v) = r \cdot J(v)$ by [5, (113)], [4, (34)]. For every point v of T , $\|J(v)\| = \|v\|$ by [5, (113)], [4, (34)]. \square

Let us consider a linear operator I from S into T and a sequence s_1 of S . Now we state the propositions:

- (10) If I is isometric and s_1 is convergent, then $I \cdot s_1$ is convergent and $\lim(I \cdot s_1) = I(\lim s_1)$.
- (11) If I is one-to-one, onto, and isometric, then s_1 is convergent iff $I \cdot s_1$ is convergent.

Let us consider a linear operator I from S into T and a subset Z of S . Now we state the propositions:

- (12) If I is one-to-one, onto, and isometric, then Z is closed iff $I^\circ Z$ is closed.
- (13) If I is one-to-one, onto, and isometric, then Z is open iff $I^\circ Z$ is open.
- (14) If I is one-to-one, onto, and isometric, then Z is compact iff $I^\circ Z$ is compact.

Now we state the propositions:

- (15) Let us consider a partial function f from T to W and a linear operator I from S into T . Suppose I is one-to-one, onto, and isometric. Let us consider a point x of S . Suppose $I(x) \in \text{dom } f$. Then $f \cdot I$ is continuous in x if and only if f is continuous in $I(x)$. The theorem is a consequence of (9), (6), and (5).
- (16) Let us consider a partial function f from T to W , a linear operator I from S into T , and a set X . Suppose
 - (i) $X \subseteq \text{the carrier of } T$, and
 - (ii) I is one-to-one, onto, and isometric.

Then f is continuous on X if and only if $f \cdot I$ is continuous on $I^{-1}(X)$. The theorem is a consequence of (15) and (1). PROOF: For every point y of T such that $y \in X$ holds $f|_X$ is continuous in y by [5, (113)], [23, (57)]. \square

Let X, Y be real normed spaces. The functor $\text{IsoCPNrSP}(X, Y)$ yielding a linear operator from $X \times Y$ into $\prod\langle X, Y \rangle$ is defined by

(Def. 3) Let us consider a point x of X and a point y of Y . Then $it(x, y) = \langle x, y \rangle$.

Now we state the proposition:

(17) Let us consider real normed spaces X, Y . Then $0_{\prod\langle X, Y \rangle} = (\text{IsoCPNrSP}(X, Y))(0_{X \times Y})$.

Let X, Y be real normed spaces. Observe that $\text{IsoCPNrSP}(X, Y)$ is one-to-one onto and isometric.

Let us note that there exists a linear operator from $X \times Y$ into $\prod\langle X, Y \rangle$ which is one-to-one, onto, and isometric.

Let f be a one-to-one onto isometric linear operator from $X \times Y$ into $\prod\langle X, Y \rangle$. Let us note that the functor f^{-1} yields a linear operator from $\prod\langle X, Y \rangle$ into $X \times Y$. One can verify that f^{-1} is one-to-one onto and isometric as a linear operator from $\prod\langle X, Y \rangle$ into $X \times Y$.

Observe that there exists a linear operator from $\prod\langle X, Y \rangle$ into $X \times Y$ which is one-to-one, onto, and isometric.

Now we state the propositions:

(18) Let us consider real normed spaces X, Y , a point x of X , and a point y of Y . Then $(\text{IsoCPNrSP}(X, Y))^{-1}(\langle x, y \rangle) = \langle x, y \rangle$. PROOF: Set $I = \text{IsoCPNrSP}(X, Y)$. Set $J = I^{-1}$. For every point x of X and for every point y of Y , $J(\langle x, y \rangle) = \langle x, y \rangle$ by [4, (34)]. \square

(19) Let us consider real normed spaces X, Y . Then $(\text{IsoCPNrSP}(X, Y))^{-1}(0_{\prod\langle X, Y \rangle}) = 0_{X \times Y}$. The theorem is a consequence of (17).

(20) Let us consider real normed spaces X, Y and a subset Z of $X \times Y$. Then $\text{IsoCPNrSP}(X, Y)$ is continuous on Z .

(21) Let us consider real normed spaces X, Y and a subset Z of $\prod\langle X, Y \rangle$. Then $(\text{IsoCPNrSP}(X, Y))^{-1}$ is continuous on Z .

(22) Let us consider real normed spaces S, T, W , a point f of the real norm space of bounded linear operators from S into W , a point g of the real norm space of bounded linear operators from T into W , and a linear operator I from S into T . Suppose

- (i) I is one-to-one, onto, and isometric, and
- (ii) $f = g \cdot I$.

Then $\|f\| = \|g\|$. The theorem is a consequence of (9) and (7). PROOF: Consider J being a linear operator from T into S such that $J = I^{-1}$ and

J is one-to-one, onto, and isometric. Reconsider $g_0 = g$ as a Lipschitzian linear operator from T into W . Reconsider $g_3 = g \cdot I$ as a Lipschitzian linear operator from S into W . For every element x , $x \in \{\|g_0(t)\|$, where t is a vector of $T : \|t\| \leq 1\}$ iff $x \in \{\|g_3(w)\|$, where w is a vector of $S : \|w\| \leq 1\}$ by [4, (13), (35)]. \square

Let us consider S and T . One can verify that every linear operator from S into T which is isometric is also Lipschitzian.

3. ISOMETRIC DIFFERENTIABLE FUNCTIONS ON REAL NORMED SPACE

Let us consider a real norm space sequence G , a real normed space F , a set i , partial functions f, g from $\prod G$ to F , and a subset X of $\prod G$. Now we state the propositions:

- (23) Suppose X is open and $i \in \text{dom } G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i . Then
- (i) $f + g$ is partially differentiable on X w.r.t. i , and
 - (ii) $(f + g)|^i X = (f|^i X) + (g|^i X)$.
- (24) Suppose X is open and $i \in \text{dom } G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i . Then
- (i) $f - g$ is partially differentiable on X w.r.t. i , and
 - (ii) $(f - g)|^i X = (f|^i X) - (g|^i X)$.

Now we state the propositions:

- (25) Let us consider a real norm space sequence G , a real normed space F , a set i , a partial function f from $\prod G$ to F , a real number r , and a subset X of $\prod G$. Suppose
- (i) X is open, and
 - (ii) $i \in \text{dom } G$, and
 - (iii) f is partially differentiable on X w.r.t. i .

Then

- (iv) $r \cdot f$ is partially differentiable on X w.r.t. i , and
- (v) $r \cdot f|^i X = r \cdot (f|^i X)$.

PROOF: Set $h = r \cdot f$. For every point x of $\prod G$ such that $x \in X$ holds h is partially differentiable in x w.r.t. i and $\text{partdiff}(h, x, i) = r \cdot \text{partdiff}(f, x, i)$ by [19, (24), (30)]. Set $f_3 = f|^i X$. For every point x of $\prod G$ such that $x \in X$ holds $(r \cdot f_3)_x = \text{partdiff}(h, x, i)$. \square

- (26) Let us consider real normed spaces S, T , a Lipschitzian linear operator L from S into T , and a point x_0 of S . Then

- (i) L is differentiable in x_0 , and
- (ii) $L'(x_0) = L$.

The theorem is a consequence of (2). PROOF: Reconsider $L_0 = L$ as a point of the real norm space of bounded linear operators from S into T . Reconsider $R = (\text{the carrier of } S) \mapsto 0_T$ as a partial function from S to T . Set $N =$ the neighbourhood of x_0 . For every point x of S such that $x \in N$ holds $L_{0x} - L_{0x_0} = L(x - x_0) + R_{x-x_0}$ by [20, (7)], [21, (4)]. \square

- (27) Let us consider a partial function f from T to W , a Lipschitzian linear operator I from S into T , and a point I_0 of the real norm space of bounded linear operators from S into T . Suppose $I_0 = I$. Let us consider a point x of S . Suppose f is differentiable in $I(x)$. Then

- (i) $f \cdot I$ is differentiable in x , and
- (ii) $(f \cdot I)'(x) = f'(I(x)) \cdot I_0$.

The theorem is a consequence of (26).

- (28) Let us consider a partial function f from T to W and a linear operator I from S into T . Suppose

- (i) I is one-to-one and onto, and
- (ii) I is isometric.

Let us consider a point x of S . Then $f \cdot I$ is differentiable in x if and only if f is differentiable in $I(x)$. The theorem is a consequence of (9), (26), and (27).

- (29) Let us consider a partial function f from T to W , a linear operator I from S into T , and a set X . Suppose

- (i) $X \subseteq$ the carrier of T , and
- (ii) I is one-to-one and onto, and
- (iii) I is isometric.

Then f is differentiable on X if and only if $f \cdot I$ is differentiable on $I^{-1}(X)$. The theorem is a consequence of (28) and (1). PROOF: For every point y of T such that $y \in X$ holds $f|_X$ is differentiable in y by [5, (113)]. \square

- (30) Let us consider real normed spaces X, Y , a partial function f from $\prod\langle X, Y \rangle$ to W , and a subset D of $\prod\langle X, Y \rangle$. Suppose f is differentiable on D . Let us consider a point z of $\prod\langle X, Y \rangle$. Suppose $z \in \text{dom } f|_D$. Then $f|_D'(z) = ((f \cdot \text{IsoCPNrSP}(X, Y))'_{|(\text{IsoCPNrSP}(X, Y))^{-1}(D)})_{(\text{IsoCPNrSP}(X, Y))^{-1}(z)} \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$. The theorem is a consequence of (17), (29), and (27). PROOF: Set $I = \text{IsoCPNrSP}(X, Y)$. Set $J = (\text{IsoCPNrSP}(X, Y))^{-1}$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $\prod\langle X, Y \rangle$ such that $z \in \text{dom } f|_D$ holds $f|_D'(z) = (g|_E)'_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [23, (36)]. \square

- (31) Let us consider real normed spaces X, Y , a partial function f from $X \times Y$ to W , and a subset D of $X \times Y$. Suppose f is differentiable on D . Let us consider a point z of $X \times Y$. Suppose $z \in \text{dom } f'_{|D}$. Then $f'_{|D}(z) = ((f \cdot (\text{IsoCPNrSP}(X, Y))^{-1})'_{|((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(D)})_{(\text{IsoCPNrSP}(X, Y))(z)} \cdot ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}$. The theorem is a consequence of (18), (19), (17), (29), and (27). PROOF: Set $I = (\text{IsoCPNrSP}(X, Y))^{-1}$. Set $J = \text{IsoCPNrSP}(X, Y)$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $X \times Y$ such that $z \in \text{dom } f'_{|D}$ holds $f'_{|D}(z) = (g'_{|E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [23, (36)]. \square
- (32) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then
- (i) $\text{reproj1}(z) = (\text{IsoCPNrSP}(X, Y))^{-1} \cdot \text{reproj}(1(\in \text{dom}\langle X, Y \rangle), (\text{IsoCPNrSP}(X, Y))(z))$, and
 - (ii) $\text{reproj2}(z) = (\text{IsoCPNrSP}(X, Y))^{-1} \cdot \text{reproj}(2(\in \text{dom}\langle X, Y \rangle), (\text{IsoCPNrSP}(X, Y))(z))$.

The theorem is a consequence of (18).

Let X, Y be real normed spaces and z be a point of $X \times Y$. Let us note that the functor z_1 yields a point of X . One can verify that the functor z_2 yields a point of Y . Let X, Y, W be real normed spaces. Let f be a partial function from $X \times Y$ to W . We say that f is partially differentiable in z w.r.t. 1 if and only if (Def. 4) $f \cdot \text{reproj1}(z)$ is differentiable in z_1 .

We say that f is partially differentiable in z w.r.t. 2 if and only if

(Def. 5) $f \cdot \text{reproj2}(z)$ is differentiable in z_2 .

Now we state the propositions:

- (33) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then
- (i) $z_1 = \text{the projection onto } 1(\in \text{dom}\langle X, Y \rangle)((\text{IsoCPNrSP}(X, Y))(z))$, and
 - (ii) $z_2 = \text{the projection onto } 2(\in \text{dom}\langle X, Y \rangle)((\text{IsoCPNrSP}(X, Y))(z))$.
- (34) Let us consider real normed spaces X, Y, W , a point z of $X \times Y$, and a partial function f from $X \times Y$ to W . Then
- (i) f is partially differentiable in z w.r.t. 1 iff $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable in $(\text{IsoCPNrSP}(X, Y))(z)$ w.r.t. 1, and
 - (ii) f is partially differentiable in z w.r.t. 2 iff $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable in $(\text{IsoCPNrSP}(X, Y))(z)$ w.r.t. 2.

The theorem is a consequence of (32) and (33).

Let X, Y, W be real normed spaces, z be a point of $X \times Y$, and f be a partial function from $X \times Y$ to W . The functor $\text{partdiff}(f, z)$ w.r.t. 1 yielding a point of the real norm space of bounded linear operators from X into W is defined by the term

(Def. 6) $(f \cdot \text{reproj1}(z))'(z_1)$.

The functor $\text{partdiff}(f, z)$ w.r.t. 2 yielding a point of the real norm space of bounded linear operators from Y into W is defined by the term

(Def. 7) $(f \cdot \text{reproj2}(z))'(z_2)$.

Now we state the proposition:

(35) Let us consider real normed spaces X, Y, W , a point z of $X \times Y$, and a partial function f from $X \times Y$ to W . Then

- (i) $\text{partdiff}(f, z)$ w.r.t. 1 = $\text{partdiff}(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}, (\text{IsoCPNrSP}(X, Y))(z), 1)$, and
- (ii) $\text{partdiff}(f, z)$ w.r.t. 2 = $\text{partdiff}(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}, (\text{IsoCPNrSP}(X, Y))(z), 2)$.

The theorem is a consequence of (32) and (33).

Let us consider real normed spaces X, Y, W , a point z of $X \times Y$, and partial functions f_1, f_2 from $X \times Y$ to W . Now we state the propositions:

(36) Suppose f_1 is partially differentiable in z w.r.t. 1 and f_2 is partially differentiable in z w.r.t. 1. Then

- (i) $f_1 + f_2$ is partially differentiable in z w.r.t. 1, and
- (ii) $\text{partdiff}(f_1 + f_2, z)$ w.r.t. 1 = $\text{partdiff}(f_1, z)$ w.r.t. 1 + $\text{partdiff}(f_2, z)$ w.r.t. 1, and
- (iii) $f_1 - f_2$ is partially differentiable in z w.r.t. 1, and
- (iv) $\text{partdiff}(f_1 - f_2, z)$ w.r.t. 1 = $\text{partdiff}(f_1, z)$ w.r.t. 1 - $\text{partdiff}(f_2, z)$ w.r.t. 1.

(37) Suppose f_1 is partially differentiable in z w.r.t. 2 and f_2 is partially differentiable in z w.r.t. 2. Then

- (i) $f_1 + f_2$ is partially differentiable in z w.r.t. 2, and
- (ii) $\text{partdiff}(f_1 + f_2, z)$ w.r.t. 2 = $\text{partdiff}(f_1, z)$ w.r.t. 2 + $\text{partdiff}(f_2, z)$ w.r.t. 2, and
- (iii) $f_1 - f_2$ is partially differentiable in z w.r.t. 2, and
- (iv) $\text{partdiff}(f_1 - f_2, z)$ w.r.t. 2 = $\text{partdiff}(f_1, z)$ w.r.t. 2 - $\text{partdiff}(f_2, z)$ w.r.t. 2.

Let us consider real normed spaces X, Y, W , a point z of $X \times Y$, a real number r , and a partial function f from $X \times Y$ to W . Now we state the propositions:

(38) Suppose f is partially differentiable in z w.r.t. 1. Then

- (i) $r \cdot f$ is partially differentiable in z w.r.t. 1, and
- (ii) $\text{partdiff}(r \cdot f, z)$ w.r.t. 1 = $r \cdot \text{partdiff}(f, z)$ w.r.t. 1.

(39) Suppose f is partially differentiable in z w.r.t. 2. Then

- (i) $r \cdot f$ is partially differentiable in z w.r.t. 2, and
- (ii) $\text{partdiff}(r \cdot f, z) \text{ w.r.t. } 2 = r \cdot \text{partdiff}(f, z) \text{ w.r.t. } 2$.

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W . We say that f is partially differentiable on Z w.r.t. 1 if and only if

- (Def. 8) (i) $Z \subseteq \text{dom } f$, and
- (ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f|_Z$ is partially differentiable in z w.r.t. 1.

We say that f is partially differentiable on Z w.r.t. 2 if and only if

- (Def. 9) (i) $Z \subseteq \text{dom } f$, and
- (ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f|_Z$ is partially differentiable in z w.r.t. 2.

Now we state the proposition:

- (40) Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W . Then

- (i) f is partially differentiable on Z w.r.t. 1 iff $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable on $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ w.r.t. 1, and
- (ii) f is partially differentiable on Z w.r.t. 2 iff $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable on $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ w.r.t. 2.

The theorem is a consequence of (18), (19), (17), (34), and (1). PROOF: Set $I = (\text{IsoCPNrSP}(X, Y))^{-1}$. Set $g = f \cdot I$. Set $E = I^{-1}(Z)$. f is partially differentiable on Z w.r.t. 1 iff g is partially differentiable on E w.r.t. 1 by [5, (113)], [4, (34)], [5, (38)]. f is partially differentiable on Z w.r.t. 2 iff g is partially differentiable on E w.r.t. 2 by [5, (113)], [4, (34)], [5, (38)]. \square

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W . Assume f is partially differentiable on Z w.r.t. 1. The functor $f|_Z$ yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from X into W is defined by

- (Def. 10) (i) $\text{dom } it = Z$, and
- (ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \text{partdiff}(f, z) \text{ w.r.t. } 1$.

Assume f is partially differentiable on Z w.r.t. 2. The functor $f|_Z$ yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from Y into W is defined by

- (Def. 11) (i) $\text{dom } it = Z$, and
- (ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \text{partdiff}(f, z) \text{ w.r.t. } 2$.

Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W . Now we state the propositions:

- (41) Suppose f is partially differentiable on Z w.r.t. 1. Then $f \upharpoonright^1 Z = (f \cdot (\text{IsoCPNrSP}(X, Y))^{-1} \upharpoonright^1 ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y)$.
- (42) Suppose f is partially differentiable on Z w.r.t. 2. Then $f \upharpoonright^2 Z = (f \cdot (\text{IsoCPNrSP}(X, Y))^{-1} \upharpoonright^2 ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y)$.
- (43) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 1 if and only if $Z \subseteq \text{dom } f$ and for every point x of $X \times Y$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 1.
- (44) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 2 if and only if $Z \subseteq \text{dom } f$ and for every point x of $X \times Y$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 2.

Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, and partial functions f, g from $X \times Y$ to W . Now we state the propositions:

- (45) Suppose Z is open and f is partially differentiable on Z w.r.t. 1 and g is partially differentiable on Z w.r.t. 1. Then
 - (i) $f + g$ is partially differentiable on Z w.r.t. 1, and
 - (ii) $(f + g) \upharpoonright^1 Z = (f \upharpoonright^1 Z) + (g \upharpoonright^1 Z)$.
- (46) Suppose Z is open and f is partially differentiable on Z w.r.t. 1 and g is partially differentiable on Z w.r.t. 1. Then
 - (i) $f - g$ is partially differentiable on Z w.r.t. 1, and
 - (ii) $(f - g) \upharpoonright^1 Z = (f \upharpoonright^1 Z) - (g \upharpoonright^1 Z)$.
- (47) Suppose Z is open and f is partially differentiable on Z w.r.t. 2 and g is partially differentiable on Z w.r.t. 2. Then
 - (i) $f + g$ is partially differentiable on Z w.r.t. 2, and
 - (ii) $(f + g) \upharpoonright^2 Z = (f \upharpoonright^2 Z) + (g \upharpoonright^2 Z)$.
- (48) Suppose Z is open and f is partially differentiable on Z w.r.t. 2 and g is partially differentiable on Z w.r.t. 2. Then
 - (i) $f - g$ is partially differentiable on Z w.r.t. 2, and
 - (ii) $(f - g) \upharpoonright^2 Z = (f \upharpoonright^2 Z) - (g \upharpoonright^2 Z)$.

Let us consider real normed spaces X, Y, W , a subset Z of $X \times Y$, a real number r , and a partial function f from $X \times Y$ to W . Now we state the propositions:

- (49) Suppose Z is open and f is partially differentiable on Z w.r.t. 1. Then
 - (i) $r \cdot f$ is partially differentiable on Z w.r.t. 1, and
 - (ii) $r \cdot f \upharpoonright^1 Z = r \cdot (f \upharpoonright^1 Z)$.
- (50) Suppose Z is open and f is partially differentiable on Z w.r.t. 2. Then

- (i) $r \cdot f$ is partially differentiable on Z w.r.t. 2, and
- (ii) $r \cdot f \restriction^2 Z = r \cdot (f \restriction^2 Z)$.

Let us consider real normed spaces X , Y , W , a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W . Now we state the propositions:

- (51) Suppose f is differentiable on Z . Then $f'_{\restriction Z}$ is continuous on Z if and only if $(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1})'_{\restriction ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)}$ is continuous on $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$.
- (52) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 1 and f is partially differentiable on Z w.r.t. 2 and $f \restriction^1 Z$ is continuous on Z and $f \restriction^2 Z$ is continuous on Z if and only if f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z .

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Received December 31, 2013
