

Coproducts in Categories without Uniqueness of cod and dom

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Summary. The paper introduces coproducts in categories without uniqueness of cod and dom . It is proven that set-theoretical disjoint union is the coproduct in the category Ens [9].

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The notation and terminology used in this paper have been introduced in the following articles: [10], [7], [6], [1], [11], [2], [3], [8], [4], [12], [14], [13], and [5].

From now on I denotes a set and E denotes a non empty set.

Let I be a non empty set, A be a many sorted set indexed by I , and i be an element of I . Let us observe that $\text{coprod}(i, A)$ is relation-like and function-like.

Let C be a non empty category structure, o be an object of C , I be a set, and f be an objects family of I and C . A morphisms family of f and o is a many sorted set indexed by I and is defined by

(Def. 1) Let us consider an element i . Suppose $i \in I$. Then there exists an object o_1 of C such that

- (i) $o_1 = f(i)$, and
- (ii) $it(i)$ is a morphism from o_1 to o .

Let I be a non empty set. Let us note that a morphisms family of f and o can equivalently be formulated as follows:

(Def. 2) Let us consider an element i of I . Then $it(i)$ is a morphism from $f(i)$ to o .

Let M be a morphisms family of f and o and i be an element of I . Note that the functor $M(i)$ yields a morphism from $f(i)$ to o . Let C be a functional non empty category structure. Let I be a set. Let us note that every morphisms family of f and o is function yielding.

Now we state the proposition:

- (1) Let us consider a non empty category structure C , an object o of C , and an objects family f of \emptyset and C . Then \emptyset is a morphisms family of f and o .

Let C be a non empty category structure, I be a set, A be an objects family of I and C , B be an object of C , and P be a morphisms family of A and B . We say that P is feasible if and only if

- (Def. 3) Let us consider a set i . Suppose $i \in I$. Then there exists an object o of C such that

- (i) $o = A(i)$, and
- (ii) $P(i) \in \langle o, B \rangle$.

Let I be a non empty set. Let us observe that P is feasible if and only if the condition (Def. 4) is satisfied.

- (Def. 4) Let us consider an element i of I . Then $P(i) \in \langle A(i), B \rangle$.

Let C be a category and I be a set. We say that P is coprojection morphisms if and only if

- (Def. 5) Let us consider an object X of C and a morphisms family F of A and X . Suppose F is feasible. Then there exists a morphism f from B to X such that

- (i) $f \in \langle B, X \rangle$, and
- (ii) for every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to B such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$, and
- (iii) for every morphism f_1 from B to X such that for every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to B such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f_1 \cdot P_i$ holds $f = f_1$.

Let I be a non empty set. Let us note that P is coprojection morphisms if and only if the condition (Def. 6) is satisfied.

- (Def. 6) Let us consider an object X of C and a morphisms family F of A and X . Suppose F is feasible. Then there exists a morphism f from B to X such that

- (i) $f \in \langle B, X \rangle$, and
- (ii) for every element i of I , $F(i) = f \cdot P(i)$, and
- (iii) for every morphism f_1 from B to X such that for every element i of I , $F(i) = f_1 \cdot P(i)$ holds $f = f_1$.

Let A be an objects family of \emptyset and C . Note that every morphisms family of A and B is feasible.

Now we state the propositions:

- (2) Let us consider a category C , an objects family A of \emptyset and C , and an object B of C . Suppose B is initial. Then there exists a morphisms family P of A and B such that P is empty and coprojection morphisms. The theorem is a consequence of (1).
- (3) Let us consider an objects family A of I and $\text{Ens}_{\{\emptyset\}}$ and an object o of $\text{Ens}_{\{\emptyset\}}$. Then $I \mapsto \emptyset$ is a morphisms family of A and o .
- (4) Let us consider an objects family A of I and $\text{Ens}_{\{\emptyset\}}$, an object o of $\text{Ens}_{\{\emptyset\}}$, and a morphisms family P of A and o . If $P = I \mapsto \emptyset$, then P is feasible and coprojection morphisms. PROOF: P is feasible by [11, (7)]. Reconsider $f = \emptyset$ as a morphism from o to Y . For every set i such that $i \in I$ there exists an object s_i of C and there exists a morphism P_i from s_i to o such that $s_i = A(i)$ and $P_i = P(i)$ and $F(i) = f \cdot P_i$ by [11, (7)]. \square

Let C be a category. We say that C has coproducts if and only if

- (Def. 7) Let us consider a set I and an objects family A of I and C . Then there exists an object B of C and there exists a morphisms family P of A and B such that P is feasible and coprojection morphisms.

Note that $\text{Ens}_{\{\emptyset\}}$ has coproducts and there exists a category which is strict and has products and coproducts.

Let C be a category, I be a set, A be an objects family of I and C , and B be an object of C . We say that B is A -category coproduct-like if and only if

- (Def. 8) There exists a morphisms family P of A and B such that P is feasible and coprojection morphisms.

Let C be a category with coproducts. Let us observe that there exists an object of C which is A -category coproduct-like.

Let C be a category and A be an objects family of \emptyset and C . Note that every object of C which is A -category coproduct-like is also initial.

Now we state the propositions:

- (5) Let us consider a category C , an objects family A of \emptyset and C , and an object B of C . If B is initial, then B is A -category coproduct-like. The theorem is a consequence of (2).
- (6) Let us consider a category C , an objects family A of I and C , and objects C_1, C_2 of C . Suppose
 - (i) C_1 is A -category coproduct-like, and
 - (ii) C_2 is A -category coproduct-like.

Then C_1, C_2 are iso.

From now on A denotes an objects family of I and Ens_E .

Let us consider I , E , and A . Assume $\bigcup \text{coprod}(A) \in E$. The functor $\coprod A$ yielding an object of Ens_E is defined by the term

(Def. 9) $\bigcup \text{coprod}(A)$.

The functor $\text{Coprod}(A)$ yielding a many sorted set indexed by I is defined by

(Def. 10) Let us consider an element i . Suppose $i \in I$. Then there exists a function F from $A(i)$ into $\bigcup \text{coprod}(A)$ such that

(i) $it(i) = F$, and

(ii) for every element x such that $x \in A(i)$ holds $F(x) = \langle x, i \rangle$.

Observe that $\text{Coprod}(A)$ is function yielding.

Assume $\bigcup \text{coprod}(A) \in E$. The functor $\coprod_{i \in I} A(i)$ yielding a morphisms family

of A and $\coprod A$ is defined by the term

(Def. 11) $\text{Coprod}(A)$.

Now we state the propositions:

(7) If $\bigcup \text{coprod}(A) = \emptyset$, then $\text{Coprod}(A)$ is empty yielding.

(8) If $\bigcup \text{coprod}(A) = \emptyset$, then A is empty yielding.

(9) If $\bigcup \text{coprod}(A) \in E$ and $\bigcup \text{coprod}(A) = \emptyset$, then $\coprod_{i \in I} A(i) = I \mapsto \emptyset$. The theorem is a consequence of (7).

(10) If $\bigcup \text{coprod}(A) \in E$, then $\coprod_{i \in I} A(i)$ is feasible and coprojection morphisms. The theorem is a consequence of (7) and (8).

(11) If $\bigcup \text{coprod}(A) \in E$, then $\coprod A$ is A -category coproduct-like. The theorem is a consequence of (10).

(12) If for every I and A , $\bigcup \text{coprod}(A) \in E$, then Ens_E has coproducts. The theorem is a consequence of (10).

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