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Some Properties of the Sorgenfrey Line and the Sorgenfrey Plane

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Summary. We first provide a modified version of the proof in [3] that the Sorgenfrey line is T_1. Here, we prove that it is in fact T_2, a stronger result. Next, we prove that all subspaces of $\mathbb{R}^1$ (that is the real line with the usual topology) are Lindelöf. We utilize this result in the proof that the Sorgenfrey line is Lindelöf, which is based on the proof found in [8]. Next, we construct the Sorgenfrey plane, as the product topology of the Sorgenfrey line and itself. We prove that the Sorgenfrey plane is not Lindelöf, and therefore the product space of two Lindelöf spaces need not be Lindelöf. Further, we note that the Sorgenfrey line is regular, following from [3]:59. Next, we observe that the Sorgenfrey line is normal since it is both regular and Lindelöf. Finally, we prove that the Sorgenfrey plane is not normal, and hence the product of two normal spaces need not be normal. The proof that the Sorgenfrey plane is not normal and many of the lemmas leading up to this result are modelled after the proof in [3], that the Niemytzki plane is not normal. Information was also gathered from [15].

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The notation and terminology used in this paper have been introduced in the following articles: [16], [1], [13], [12], [11], [14], [19], [18], [9], [2], [10], [3], [7], [20], and [6].

In this paper T denotes a topological space, x, y, a, b, U, U_1, r_1 denote sets, p, q denote rational numbers, F, G denote families of subsets of T, and U_2, I denote families of subsets of Sorgenfrey line.

Observe that Sorgenfrey line is T_2.

Now we state the proposition:

(1) Let us consider real numbers x, a, b. Suppose $x \in]a, b[$. Then there exist rational numbers p, r such that

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(i) $x \in]p, r[$, and

(ii) $]p, r[\subseteq]a, b[$.

PROOF: Consider p being a rational number such that $p > a$ and $x > p$. Consider r being a rational number such that $x < r < b$. $]p, r[\subseteq]a, b[$. □

Let us observe that every subspace of $\mathbb{R}^{\mathbf{1}}$ is Lindelöf and Sorgenfrey line is Lindelöf.

The Sorgenfrey plane yielding a non empty strict topological space is defined by the term

(Def. 1) Sorgenfrey line × Sorgenfrey line.

The functor real-anti-diagonal yielding a subset of $\mathbb{R} \times \mathbb{R}$ is defined by the term

(Def. 2) $\{\langle x, y\rangle$, where x, y are real numbers : $y = -x\}$.

Now we state the propositions:

(2) $\mathbb{Q} \times \mathbb{Q}$ is a dense subset of the Sorgenfrey plane. PROOF: $\mathbb{Q} \times \mathbb{Q} \subseteq \Omega_\alpha$, where α is the Sorgenfrey plane by [17, (12)]. Reconsider $C = \mathbb{Q} \times \mathbb{Q}$ as a subset of the Sorgenfrey plane. For every subset A of the Sorgenfrey plane such that $A \neq \emptyset$ and A is open holds A meets C by [16, (5)], [6, (90)], [4, (31)]. □

(3) $\overline{\overline{\text{real-anti-diagonal}}} = \mathfrak{c}$. PROOF: $\mathbb{R} \approx$ real-anti-diagonal by [5, (4)]. □

(4) real-anti-diagonal is a closed subset of the Sorgenfrey plane. PROOF: Set $L =$ real-anti-diagonal. Set $S =$ the Sorgenfrey plane. $L \subseteq \Omega_S$. Reconsider $L =$ real-anti-diagonal as a subset of the Sorgenfrey plane. Define $\mathcal{P}[\text{element}, \text{element}] \equiv$ there exist real numbers x, y such that $\$_1 = \langle x, y\rangle$ and $\$_2 = x + y$. For every element z such that $z \in$ the carrier of S there exists an element u such that $u \in$ the carrier of $\mathbb{R}^{\mathbf{1}}$ and $\mathcal{P}[z, u]$ by [7, (17)]. Consider f being a function from S into $\mathbb{R}^{\mathbf{1}}$ such that for every element z such that $z \in$ the carrier of S holds $\mathcal{P}[z, f(z)]$ from [5, Sch. 1]. For every elements x, y of $\mathbb{R}$ such that $\langle x, y\rangle \in$ the carrier of S holds $f(\langle x, y\rangle) = x + y$. For every point p of S and for every positive real number r, there exists an open subset W of S such that $p \in W$ and $f^\circ W \subseteq]f(p) - r, f(p) + r[$ by [2, (11)], [16, (6)]. Reconsider $z_1 = 0$ as an element of $\mathbb{R}$. Reconsider $k = \{z_1\}$ as a subset of $\mathbb{R}^{\mathbf{1}}$. $L = f^{-1}(k)$ by [5, (38)]. □

(5) Let us consider a subset A of the Sorgenfrey plane. Suppose $A =$ real-anti-diagonal. Then Der A is empty.

(6) Every subset of real-anti-diagonal is a closed subset of the Sorgenfrey plane. The theorem is a consequence of (4) and (5).

Note that the Sorgenfrey plane is non Lindelöf and Sorgenfrey line is regular and Sorgenfrey line is normal and the Sorgenfrey plane is non normal.

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