

# Constructing Binary Huffman Tree<sup>1</sup>

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**Summary.** Huffman coding is one of a most famous entropy encoding methods for lossless data compression [16]. JPEG and ZIP formats employ variants of Huffman encoding as lossless compression algorithms. Huffman coding is a bijective map from source letters into leaves of the Huffman tree constructed by the algorithm. In this article we formalize an algorithm constructing a binary code tree, Huffman tree.

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The notation and terminology used in this paper have been introduced in the following articles: [9], [1], [20], [8], [14], [11], [12], [23], [22], [2], [3], [18], [19], [17], [25], [26], [24], [4], [5], [6], [7], and [13].

## 1. CONSTRUCTING BINARY DECODED TREES

Let  $D$  be a non empty set and  $x$  be an element of  $D$ . Observe that the root tree of  $x$  is binary as a decorated tree.

The functor  $\mathbb{R}_{\mathbb{N}}$  yielding a set is defined by the term

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(Def. 1)  $\mathbb{N} \times \mathbb{R}$ .

Note that  $\mathbb{R}_{\mathbb{N}}$  is non empty.

Let  $D$  be a non empty set. The binary finite trees of  $D$  yielding a set of trees decorated with elements of  $D$  is defined by

(Def. 2) Let us consider a tree  $T$  decorated with elements of  $D$ . Then  $\text{dom } T$  is finite and  $T$  is binary if and only if  $T \in \text{it}$ .

The Boolean binary finite trees of  $D$  yielding a non empty subset of  $2^{\text{the binary finite trees of } D}$  is defined by the term

(Def. 3)  $\{x, \text{ where } x \text{ is an element of } 2^\alpha : x \text{ is finite and } x \neq \emptyset\}$ , where  $\alpha$  is the binary finite trees of  $D$ .

In this paper  $\mathbb{S}$  denotes a non empty finite set,  $p$  denotes a probability on the trivial  $\sigma$ -field of  $\mathbb{S}$ ,  $T_1$  denotes a finite sequence of elements of the Boolean binary finite trees of  $\mathbb{R}_{\mathbb{N}}$ , and  $q$  denotes a finite sequence of elements of  $\mathbb{N}$ .

Let us consider  $\mathbb{S}$  and  $p$ . The functor  $\text{InitTrees } p$  yielding a non empty finite subset of the binary finite trees of  $\mathbb{R}_{\mathbb{N}}$  is defined by the term

(Def. 4)  $\{T, \text{ where } T \text{ is an element of } \text{FinTrees}(\mathbb{R}_{\mathbb{N}}) : T \text{ is a finite binary tree decorated with elements of } \mathbb{R}_{\mathbb{N}} \text{ and there exists an element } x \text{ of } \mathbb{S} \text{ such that } T = \text{the root tree of } \langle (\text{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$ .

Let  $p$  be a tree decorated with elements of  $\mathbb{R}_{\mathbb{N}}$ . The value of root from right of  $p$  yielding a real number is defined by the term

(Def. 5)  $p(\emptyset)_2$ .

The value of root from left of  $p$  yielding a natural number is defined by the term

(Def. 6)  $p(\emptyset)_1$ .

Let  $T$  be a finite binary tree decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  and  $p$  be an element of  $\text{dom } T$ . The value of tree of  $p$  yielding a real number is defined by the term

(Def. 7)  $T(p)_2$ .

Let  $p, q$  be finite binary trees decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  and  $k$  be a natural number. The functor  $\text{MakeTree}(p, q, k)$  yielding a finite binary tree decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  is defined by the term

(Def. 8)  $\langle k, (\text{the value of root from right of } p) + (\text{the value of root from right of } q) \rangle\text{-tree}(p, q)$ .

Let  $X$  be a non empty finite subset of the binary finite trees of  $\mathbb{R}_{\mathbb{N}}$ . The maximal value of  $X$  yielding a natural number is defined by

(Def. 9) There exists a non empty finite subset  $L$  of  $\mathbb{N}$  such that

- (i)  $L = \{\text{the value of root from left of } p, \text{ where } p \text{ is an element of the binary finite trees of } \mathbb{R}_{\mathbb{N}} : p \in X\}$ , and
- (ii)  $\text{it} = \max L$ .

Now we state the propositions:

- (1) Let us consider a non empty finite subset  $X$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$  and a finite binary tree  $w$  decorated with elements of  $\mathbb{R}_\mathbb{N}$ . Suppose  $X = \{w\}$ . Then the maximal value of  $X$  = the value of root from left of  $w$ . PROOF: Consider  $L$  being a non empty finite subset of  $\mathbb{N}$  such that  $L = \{\text{the value of root from left of } p, \text{ where } p \text{ is an element of the binary finite trees of } \mathbb{R}_\mathbb{N} : p \in X\}$  and the maximal value of  $X = \max L$ . For every element  $n$  such that  $n \in L$  holds  $n = \text{the value of root from left of } w$ . For every element  $n$  such that  $n = \text{the value of root from left of } w$  holds  $n \in L$ .  $\square$
- (2) Let us consider non empty finite subsets  $X, Y, Z$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$ . Suppose  $Z = X \cup Y$ . Then the maximal value of  $Z = \max(\text{the maximal value of } X, \text{the maximal value of } Y)$ .
- (3) Let us consider non empty finite subsets  $X, Z$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$  and a set  $Y$ . Suppose  $Z = X \setminus Y$ . Then the maximal value of  $Z \leq \text{the maximal value of } X$ . The theorem is a consequence of (2).
- (4) Let us consider a non empty finite subset  $X$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$  and an element  $p$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$ . Suppose  $p \in X$ . Then the value of root from left of  $p \leq \text{the maximal value of } X$ .

Let  $X$  be a non empty finite subset of the binary finite trees of  $\mathbb{R}_\mathbb{N}$ . A minimal value tree of  $X$  is a finite binary tree decorated with elements of  $\mathbb{R}_\mathbb{N}$  and is defined by

- (Def. 10) (i)  $it \in X$ , and
- (ii) for every finite binary tree  $q$  decorated with elements of  $\mathbb{R}_\mathbb{N}$  such that  $q \in X$  holds the value of root from right of  $it \leq \text{the value of root from right of } q$ .

Now we state the propositions:

- (5)  $\overline{\text{InitTrees } p} = \overline{\mathbb{S}}$ . PROOF: Reconsider  $f_1 = (\text{CFS}(\mathbb{S}))^{-1}$  as a function from  $\mathbb{S}$  into  $\text{Seg } \overline{\mathbb{S}}$ . Define  $\mathcal{P}[\text{element}, \text{element}] \equiv \$_2 = \text{the root tree of } \langle f_1(\$_1), p(\{\$_1\}) \rangle$ . For every element  $x$  such that  $x \in \mathbb{S}$  there exists an element  $y$  such that  $y \in \text{InitTrees } p$  and  $\mathcal{P}[x, y]$  by [12, (5)], [13, (87)], [7, (3)]. Consider  $f$  being a function from  $\mathbb{S}$  into  $\text{InitTrees } p$  such that for every element  $x$  such that  $x \in \mathbb{S}$  holds  $\mathcal{P}[x, f(x)]$  from [12, Sch. 1].  $\square$
- (6) Let us consider a non empty finite subset  $X$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$  and finite binary trees  $s, t$  decorated with elements of  $\mathbb{R}_\mathbb{N}$ . Then  $\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)) \notin X$ .

Let  $X$  be a set. The set of leaves of  $X$  yielding a subset of  $2^{\mathbb{R}_\mathbb{N}}$  is defined by the term

- (Def. 11)  $\{\text{Leaves}(p), \text{ where } p \text{ is an element of the binary finite trees of } \mathbb{R}_\mathbb{N} : p \in X\}$ .

Now we state the propositions:

- (7) Let us consider a finite binary tree  $X$  decorated with elements of  $\mathbb{R}_N$ . Then the set of leaves of  $\{X\} = \{\text{Leaves}(X)\}$ . PROOF: For every element  $x$ ,  $x \in$  the set of leaves of  $\{X\}$  iff  $x \in \{\text{Leaves}(X)\}$ .  $\square$
- (8) Let us consider sets  $X, Y$ . Then the set of leaves of  $X \cup Y =$  (the set of leaves of  $X$ )  $\cup$  (the set of leaves of  $Y$ ). PROOF: For every element  $x$ ,  $x \in$  the set of leaves of  $X \cup Y$  iff  $x \in$  (the set of leaves of  $X$ )  $\cup$  (the set of leaves of  $Y$ ).  $\square$
- (9) Let us consider trees  $s, t$ . Then  $\emptyset \notin \text{Leaves}(\widehat{t, s})$ . PROOF: For every element  $p$ ,  $p \in \widehat{t, s}$  iff  $p \in$  the elementary tree of 0 by [4, (19), (29)], [10, (130)].  $\square$
- (10) Let us consider trees  $s, t$ . Then  $\text{Leaves}(\widehat{t, s}) = \{\langle 0 \rangle \wedge p, \text{ where } p \text{ is an element of } t : p \in \text{Leaves}(t)\} \cup \{\langle 1 \rangle \wedge p, \text{ where } p \text{ is an element of } s : p \in \text{Leaves}(s)\}$ . The theorem is a consequence of (9). PROOF: Set  $L = \{\langle 0 \rangle \wedge p, \text{ where } p \text{ is an element of } t : p \in \text{Leaves}(t)\}$ . Set  $R = \{\langle 1 \rangle \wedge p, \text{ where } p \text{ is an element of } s : p \in \text{Leaves}(s)\}$ . Set  $H = \text{Leaves}(\widehat{t, s})$ . For every element  $x$ ,  $x \in H$  iff  $x \in L \cup R$  by [2, (23)], [9, (6)].  $\square$

Let us consider decorated trees  $s, t$ , an element  $x$ , and a finite sequence  $q$  of elements of  $\mathbb{N}$ . Now we state the propositions:

- (11) If  $q \in \text{dom } t$ , then  $(x\text{-tree}(t, s))(\langle 0 \rangle \wedge q) = t(q)$ .
- (12) If  $q \in \text{dom } s$ , then  $(x\text{-tree}(t, s))(\langle 1 \rangle \wedge q) = s(q)$ .

Now we state the propositions:

- (13) Let us consider decorated trees  $s, t$  and an element  $x$ . Then  $\text{Leaves}(x\text{-tree}(t, s)) = \text{Leaves}(t) \cup \text{Leaves}(s)$ . The theorem is a consequence of (10), (11), and (12). PROOF: Set  $L = \{\langle 0 \rangle \wedge p, \text{ where } p \text{ is an element of } \text{dom } t : p \in \text{Leaves}(\text{dom } t)\}$ . Set  $R = \{\langle 1 \rangle \wedge p, \text{ where } p \text{ is an element of } \text{dom } s : p \in \text{Leaves}(\text{dom } s)\}$ . For every element  $z$ ,  $z \in (x\text{-tree}(t, s))^\circ L$  iff  $z \in t^\circ(\text{Leaves}(\text{dom } t))$ . For every element  $z$ ,  $z \in (x\text{-tree}(t, s))^\circ R$  iff  $z \in s^\circ(\text{Leaves}(\text{dom } s))$ .  $\square$
- (14) Let us consider a natural number  $k$  and finite binary trees  $s, t$  decorated with elements of  $\mathbb{R}_N$ . Then  $\bigcup \text{the set of leaves of } \{s, t\} = \bigcup \text{the set of leaves of } \{\text{MakeTree}(t, s, k)\}$ . The theorem is a consequence of (8), (7), and (13).
- (15)  $\text{Leaves}(\text{the elementary tree of } 0) = \text{the elementary tree of } 0$ . PROOF: For every element  $x$ ,  $x \in \text{Leaves}(\text{the elementary tree of } 0)$  iff  $x \in$  the elementary tree of 0 by [4, (29), (54)].  $\square$
- (16) Let us consider an element  $x$ , a non empty set  $D$ , and a finite binary tree  $T$  decorated with elements of  $D$ . Suppose  $T =$  the root tree of  $x$ . Then  $\text{Leaves}(T) = \{x\}$ . The theorem is a consequence of (15).

## 2. BINARY HUFFMAN TREE

Let us consider  $\mathbb{S}$ ,  $p$ ,  $T_1$ , and  $q$ . We say that  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree if and only if

- (Def. 12) (i)  $T_1(1) = \text{InitTrees } p$ , and
- (ii)  $\text{len } T_1 = \overline{\overline{\mathbb{S}}}$ , and
- (iii) for every natural number  $i$  such that  $1 \leq i < \text{len } T_1$  there exist non empty finite subsets  $X$ ,  $Y$  of the binary finite trees of  $\mathbb{R}_{\mathbb{N}}$  and there exists a minimal value tree  $s$  of  $X$  and there exists a minimal value tree  $t$  of  $Y$  and there exists a finite binary tree  $v$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  such that  $T_1(i) = X$  and  $Y = X \setminus \{s\}$  and  $v \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$  and  $T_1(i + 1) = (X \setminus \{t, s\}) \cup \{v\}$ , and
- (iv) there exists a finite binary tree  $T$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  such that  $\{T\} = T_1(\text{len } T_1)$ , and
- (v)  $\text{dom } q = \text{Seg } \overline{\overline{\mathbb{S}}}$ , and
- (vi) for every natural number  $k$  such that  $k \in \text{Seg } \overline{\overline{\mathbb{S}}}$  holds  $q(k) = \overline{\overline{T_1(k)}}$  and  $q(k) \neq 0$ , and
- (vii) for every natural number  $k$  such that  $k < \overline{\overline{\mathbb{S}}}$  holds  $q(k + 1) = q(1) - k$ , and
- (viii) for every natural number  $k$  such that  $1 \leq k < \overline{\overline{\mathbb{S}}}$  holds  $2 \leq q(k)$ .

Now we state the proposition:

- (17) There exists  $T_1$  and there exists  $q$  such that  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree. The theorem is a consequence of (5) and (6). PROOF: Define  $\mathcal{A}[\text{natural number, set, set}] \equiv$  if there exist elements  $u$ ,  $v$  such that  $u \neq v$  and  $u, v \in \mathbb{S}_2$ , then there exist non empty finite subsets  $X$ ,  $Y$  of the binary finite trees of  $\mathbb{R}_{\mathbb{N}}$  and there exists a minimal value tree  $s$  of  $X$  and there exists a minimal value tree  $t$  of  $Y$  and there exists a finite binary tree  $w$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  such that  $\mathbb{S}_2 = X$  and  $Y = X \setminus \{s\}$  and  $w \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$  and  $\mathbb{S}_3 = (X \setminus \{t, s\}) \cup \{w\}$ . For every natural number  $n$  such that  $1 \leq n < \overline{\overline{\mathbb{S}}}$  for every element  $x$  of the Boolean binary finite trees of  $\mathbb{R}_{\mathbb{N}}$ , there exists an element  $y$  of the Boolean binary finite trees of  $\mathbb{R}_{\mathbb{N}}$  such that  $\mathcal{A}[n, x, y]$ . Reconsider  $I = \text{InitTrees } p$  as an element of the Boolean binary finite trees of  $\mathbb{R}_{\mathbb{N}}$ . Consider  $T_1$  being a finite sequence of elements of the Boolean binary finite trees of  $\mathbb{R}_{\mathbb{N}}$  such that  $\text{len } T_1 = \overline{\overline{\mathbb{S}}}$  and  $T_1(1) = I$  or  $\overline{\overline{\mathbb{S}}} = 0$  and for every natural number  $n$  such that  $1 \leq n < \overline{\overline{\mathbb{S}}}$  holds  $\mathcal{A}[n, T_1(n), T_1(n + 1)]$  from [15, Sch. 4]. Define  $\mathcal{B}[\text{element, element}] \equiv$  there exists a finite set  $X$  such that

$T_1(\$_1) = X$  and  $\$_2 = \overline{X}$  and  $\$_2 \neq 0$ . For every natural number  $k$  such that  $k \in \text{Seg } \overline{\mathbb{S}}$  there exists an element  $x$  of  $\mathbb{N}$  such that  $\mathcal{B}[k, x]$  by [11, (3)]. Consider  $q$  being a finite sequence of elements of  $\mathbb{N}$  such that  $\text{dom } q = \text{Seg } \overline{\mathbb{S}}$  and for every natural number  $k$  such that  $k \in \text{Seg } \overline{\mathbb{S}}$  holds  $\mathcal{B}[k, q(k)]$  from [8, Sch. 5]. For every natural number  $k$  such that  $k \in \text{Seg } \overline{\mathbb{S}}$  holds  $q(k) = \overline{T_1(k)}$  and  $q(k) \neq 0$ . For every natural number  $k$  such that  $1 \leq k < \overline{\mathbb{S}}$  holds if  $2 \leq q(k)$ , then  $q(k+1) = q(k) - 1$  by [8, (1)], [2, (11), (13)]. Define  $\mathcal{C}[\text{natural number}] \equiv$  if  $\$_1 < \overline{\mathbb{S}}$ , then  $q(\$_1 + 1) = q(1) - \$_1$ . For every natural number  $n$  such that  $\mathcal{C}[n]$  holds  $\mathcal{C}[n+1]$  by [2, (11)], [8, (1)], [2, (14), (13)]. For every natural number  $n$ ,  $\mathcal{C}[n]$  from [2, Sch. 2]. For every natural number  $n$  such that  $1 \leq n < \overline{\mathbb{S}}$  holds  $2 \leq q(n)$  by [2, (21), (13)]. For every natural number  $k$  such that  $1 \leq k < \text{len } T_1$  there exist non empty finite subsets  $X, Y$  of the binary finite trees of  $\mathbb{R}_{\mathbb{N}}$  and there exists a minimal value tree  $s$  of  $X$  and there exists a minimal value tree  $t$  of  $Y$  and there exists a finite binary tree  $w$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  such that  $T_1(k) = X$  and  $Y = X \setminus \{s\}$  and  $w \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$  and  $T_1(k+1) = (X \setminus \{t, s\}) \cup \{w\}$  by [8, (1)]. Consider  $T_2$  being a finite set such that  $T_1(\overline{\mathbb{S}}) = T_2$  and  $q(\overline{\mathbb{S}}) = \overline{T_2}$  and  $q(\overline{\mathbb{S}}) \neq 0$ . Consider  $u$  being an element such that  $T_2 = \{u\}$ .  $\square$

Let us consider  $\mathbb{S}$  and  $p$ . A binary Huffman tree of  $p$  is a finite binary tree decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  and is defined by

- (Def. 13) There exists a finite sequence  $T_1$  of elements of the Boolean binary finite trees of  $\mathbb{R}_{\mathbb{N}}$  and there exists a finite sequence  $q$  of elements of  $\mathbb{N}$  such that  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree and  $\{it\} = T_1(\text{len } T_1)$ .

In this paper  $T$  denotes a binary Huffman tree of  $p$ .

Now we state the propositions:

- (18)  $\bigcup$  the set of leaves of  $\text{InitTrees } p = \{z, \text{ where } z \text{ is an element of } \mathbb{N} \times \mathbb{R} : \text{there exists an element } x \text{ of } \mathbb{S} \text{ such that } z = \langle (\text{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$ . The theorem is a consequence of (16). PROOF: Set  $L = \bigcup$  the set of leaves of  $\text{InitTrees } p$ . Set  $R = \{z, \text{ where } z \text{ is an element of } \mathbb{N} \times \mathbb{R} : \text{there exists an element } x \text{ of } \mathbb{S} \text{ such that } z = \langle (\text{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$ . For every element  $x$ ,  $x \in L$  iff  $x \in R$  by [13, (87)], [7, (3)].  $\square$
- (19) Suppose  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree. Let us consider a natural number  $i$ . Suppose  $1 \leq i \leq \text{len } T_1$ . Then  $\bigcup$  the set of leaves of  $T_1(i) = \{z, \text{ where } z \text{ is an element of } \mathbb{N} \times \mathbb{R} : \text{there exists an element } x \text{ of } \mathbb{S} \text{ such that } z = \langle (\text{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$ . The theorem is a consequence of (18), (8), and (14). PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv$  if  $\$_1 < \text{len } T_1$ , then  $\bigcup$  the set of leaves of  $T_1(\$_1 + 1) = \{z, \text{ where } z \text{ is an element of } \mathbb{N} \times \mathbb{R} : \text{there exists an element } x \text{ of } \mathbb{S} \text{ such that } z = \langle (\text{CFS}(\mathbb{S}))^{-1}(x),$

$p(\{x\})\}$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k+1]$  by [2, (11)], [13, (78), (32)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [2, Sch. 2].  
 $\square$

- (20)  $\text{Leaves}(T) = \{z, \text{ where } z \text{ is an element of } \mathbb{N} \times \mathbb{R} : \text{there exists an element } x \text{ of } \mathbb{S} \text{ such that } z = \langle (\text{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$ . The theorem is a consequence of (19) and (7).
- (21) Suppose  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree. Let us consider a natural number  $i$ , a finite binary tree  $T$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$ , and elements  $t$ ,  $s$ ,  $r$  of  $\text{dom } T$ . Suppose
- (i)  $T \in T_1(i)$ , and
  - (ii)  $t \in \text{dom } T \setminus \text{Leaves}(\text{dom } T)$ , and
  - (iii)  $s = t \frown \langle 0 \rangle$ , and
  - (iv)  $r = t \frown \langle 1 \rangle$ .

Then the value of tree of  $t = (\text{the value of tree of } s) + (\text{the value of tree of } r)$ . The theorem is a consequence of (15), (11), and (12). PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv \text{if } 1 \leq \$_1 \leq \text{len } T_1$ , then for every finite binary tree  $T$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  and for every elements  $a$ ,  $b$ ,  $c$  of  $\text{dom } T$  such that  $T \in T_1(\$_1)$  and  $a \in \text{dom } T \setminus \text{Leaves}(\text{dom } T)$  and  $b = a \frown \langle 0 \rangle$  and  $c = a \frown \langle 1 \rangle$  holds the value of tree of  $a = (\text{the value of tree of } b) + (\text{the value of tree of } c)$ . For every natural number  $i$  such that  $\mathcal{P}[i]$  holds  $\mathcal{P}[i+1]$  by [2, (16), (14)], [8, (44)]. For every natural number  $i$ ,  $\mathcal{P}[i]$  from [2, Sch. 2].  
 $\square$

- (22) Let us consider elements  $t$ ,  $s$ ,  $r$  of  $\text{dom } T$ . Suppose
- (i)  $t \in \text{dom } T \setminus \text{Leaves}(\text{dom } T)$ , and
  - (ii)  $s = t \frown \langle 0 \rangle$ , and
  - (iii)  $r = t \frown \langle 1 \rangle$ .

Then the value of tree of  $t = (\text{the value of tree of } s) + (\text{the value of tree of } r)$ . The theorem is a consequence of (21).

- (23) Let us consider a non empty finite subset  $X$  of the binary finite trees of  $\mathbb{R}_{\mathbb{N}}$ . Suppose a finite binary tree  $T$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$ . Suppose  $T \in X$ . Let us consider an element  $p$  of  $\text{dom } T$  and an element  $r$  of  $\mathbb{N}$ . Suppose  $r = T(p)_1$ . Then  $r \leq \text{the maximal value of } X$ . Let us consider finite binary trees  $s$ ,  $t$ ,  $w$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$ . Suppose
- (i)  $s, t \in X$ , and
  - (ii)  $w = \text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1))$ .

Let us consider an element  $p$  of  $\text{dom } w$  and an element  $r$  of  $\mathbb{N}$ . Suppose  $r = w(p)_1$ . Then  $r \leq (\text{the maximal value of } X) + 1$ . The theorem is a consequence of (11) and (12). PROOF: For every element  $a$  such that

$a \in \text{dom } d$  holds  $a = \emptyset$  or there exists an element  $f$  of  $\text{dom } t$  such that  $a = \langle 0 \rangle \frown f$  or there exists an element  $f$  of  $\text{dom } s$  such that  $a = \langle 1 \rangle \frown f$  by [2, (23)].  $\square$

- (24) Suppose  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree. Let us consider a natural number  $i$ . Suppose  $1 \leq i < \text{len } T_1$ . Let us consider non empty finite subsets  $X, Y$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$ . Suppose

- (i)  $X = T_1(i)$ , and
- (ii)  $Y = T_1(i + 1)$ .

Then the maximal value of  $Y = (\text{the maximal value of } X) + 1$ . PROOF: Consider  $X, Y$  being non empty finite subsets of the binary finite trees of  $\mathbb{R}_\mathbb{N}$ ,  $s$  being a minimal value tree of  $X$ ,  $t$  being a minimal value tree of  $Y$ ,  $v$  being a finite binary tree decorated with elements of  $\mathbb{R}_\mathbb{N}$  such that  $T_1(i) = X$  and  $Y = X \setminus \{s\}$  and  $v \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$  and  $T_1(i + 1) = (X \setminus \{t, s\}) \cup \{v\}$ . Consider  $L_1$  being a non empty finite subset of  $\mathbb{N}$  such that  $L_1 = \{\text{the value of root from left of } p, \text{ where } p \text{ is an element of the binary finite trees of } \mathbb{R}_\mathbb{N} : p \in X0\}$  and the maximal value of  $X0 = \max L_1$ . Consider  $L_4$  being a non empty finite subset of  $\mathbb{N}$  such that  $L_4 = \{\text{the value of root from left of } p, \text{ where } p \text{ is an element of the binary finite trees of } \mathbb{R}_\mathbb{N} : p \in Y0\}$  and the maximal value of  $Y0 = \max L_4$ . Reconsider  $p_1 = v$  as an element of the binary finite trees of  $\mathbb{R}_\mathbb{N}$ . For every extended real  $x$  such that  $x \in L_4$  holds  $x \leq$  the value of root from left of  $p_1$  by [2, (16)].  $\square$

Let us consider a natural number  $i$ , a non empty finite subset  $X$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$ , a finite binary tree  $T$  decorated with elements of  $\mathbb{R}_\mathbb{N}$ , an element  $p$  of  $\text{dom } T$ , and an element  $r$  of  $\mathbb{N}$ . Now we state the propositions:

- (25) Suppose  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree. Then if  $X = T_1(i)$ , then if  $T \in X$ , then if  $r = T(p)_1$ , then  $r \leq$  the maximal value of  $X$ .
- (26) Suppose  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree. Then if  $X = T_1(i)$ , then if  $T \in X$ , then if  $r = T(p)_1$ , then  $r \leq$  the maximal value of  $X$ .

Now we state the proposition:

- (27) Suppose  $T_1$ ,  $q$ , and  $p$  are constructing binary Huffman tree. Let us consider a natural number  $i$ , finite binary trees  $s, t$  decorated with elements of  $\mathbb{R}_\mathbb{N}$ , and a non empty finite subset  $X$  of the binary finite trees of  $\mathbb{R}_\mathbb{N}$ . Suppose
- (i)  $X = T_1(i)$ , and
  - (ii)  $s, t \in X$ .

Let us consider a finite binary tree  $z$  decorated with elements of  $\mathbb{R}_N$ . Suppose  $z \in X$ . Then  $\langle (\text{the maximal value of } X) + 1, (\text{the value of root from right of } t) + (\text{the value of root from right of } s) \rangle \notin \text{rng } z$ . The theorem is a consequence of (26).

Let  $x$  be an element. Note that the root tree of  $x$  is one-to-one.

Now we state the propositions:

- (28) Let us consider a non empty finite subset  $X$  of the binary finite trees of  $\mathbb{R}_N$  and finite binary trees  $s, t, w$  decorated with elements of  $\mathbb{R}_N$ . Suppose
- (i) for every finite binary tree  $T$  decorated with elements of  $\mathbb{R}_N$  such that  $T \in X$  for every element  $p$  of  $\text{dom } T$  for every element  $r$  of  $\mathbb{N}$  such that  $r = T(p)_1$  holds  $r \leq \text{the maximal value of } X$ , and
  - (ii) for every finite binary trees  $p, q$  decorated with elements of  $\mathbb{R}_N$  such that  $p, q \in X$  and  $p \neq q$  holds  $\text{rng } p \cap \text{rng } q = \emptyset$ , and
  - (iii)  $s, t \in X$ , and
  - (iv)  $s \neq t$ , and
  - (v)  $w \in X \setminus \{s, t\}$ .

Then  $\text{rng } \text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)) \cap \text{rng } w = \emptyset$ . The theorem is a consequence of (11) and (12). PROOF: Set  $d = \text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1))$ . For every element  $a$  such that  $a \in \text{dom } d$  holds  $a = \emptyset$  or there exists an element  $f$  of  $\text{dom } t$  such that  $a = \langle 0 \rangle \cap f$  or there exists an element  $f$  of  $\text{dom } s$  such that  $a = \langle 1 \rangle \cap f$  by [2, (23)]. Consider  $n_2$  being an element such that  $n_2 \in \text{rng } d \cap \text{rng } w$ . Consider  $a_1$  being an element such that  $a_1 \in \text{dom } d$  and  $n_2 = d(a_1)$ . Consider  $b_1$  being an element such that  $b_1 \in \text{dom } w$  and  $n_2 = w(b_1)$ .  $w \in X$  and  $w \neq s$  and  $w \neq t$ .  $\square$

- (29) Suppose  $T_1, q$ , and  $p$  are constructing binary Huffman tree. Let us consider a natural number  $i$  and finite binary trees  $T, S$  decorated with elements of  $\mathbb{R}_N$ . Suppose
- (i)  $T, S \in T_1(i)$ , and
  - (ii)  $T \neq S$ .

Then  $\text{rng } T \cap \text{rng } S = \emptyset$ . The theorem is a consequence of (26) and (28). PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv \text{if } 1 \leq \$1 \leq \text{len } T_1$ , then for every finite binary trees  $T, S$  decorated with elements of  $\mathbb{R}_N$  such that  $T, S \in T_1(\$1)$  and  $T \neq S$  holds  $\text{rng } T \cap \text{rng } S = \emptyset$ . For every natural number  $i$  such that  $\mathcal{P}[i]$  holds  $\mathcal{P}[i+1]$  by [21, (8)], [2, (16), (14)]. For every natural number  $i$ ,  $\mathcal{P}[i]$  from [2, Sch. 2].  $\square$

- (30) Let us consider a non empty finite subset  $X$  of the binary finite trees of  $\mathbb{R}_N$  and finite binary trees  $s, t$  decorated with elements of  $\mathbb{R}_N$ . Suppose
- (i)  $s$  is one-to-one, and

- (ii)  $t$  is one-to-one, and
- (iii)  $t, s \in X$ , and
- (iv)  $\text{rng } s \cap \text{rng } t = \emptyset$ , and
- (v) for every finite binary tree  $z$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  such that  $z \in X$  holds  $\langle (\text{the maximal value of } X) + 1, (\text{the value of root from right of } t) + (\text{the value of root from right of } s) \rangle \notin \text{rng } z$ .

Then  $\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1))$  is one-to-one. The theorem is a consequence of (11) and (12). PROOF: Set  $d = \text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1))$ . For every element  $a$  such that  $a \in \text{dom } d$  holds  $a = \emptyset$  or there exists an element  $f$  of  $\text{dom } t$  such that  $a = \langle 0 \rangle \frown f$  or there exists an element  $f$  of  $\text{dom } s$  such that  $a = \langle 1 \rangle \frown f$  by [2, (23)]. For every element  $x$  such that  $x \in \text{dom } d$  and  $x \neq \emptyset$  holds  $d(x) \neq d(\emptyset)$  by [11, (3)]. For every elements  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom } d$  and  $d(x_1) = d(x_2)$  holds it is not true that there exists an element  $f$  of  $\text{dom } s$  such that  $x_1 = \langle 1 \rangle \frown f$  and there exists an element  $f$  of  $\text{dom } t$  such that  $x_2 = \langle 0 \rangle \frown f$  by [11, (3)]. For every elements  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom } d$  and  $d(x_1) = d(x_2)$  holds  $x_1 = x_2$ .  $\square$

- (31) Suppose  $T_1, q$ , and  $p$  are constructing binary Huffman tree. Let us consider a natural number  $i$  and a finite binary tree  $T$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$ . If  $T \in T_1(i)$ , then  $T$  is one-to-one. The theorem is a consequence of (27), (29), and (30). PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv$  if  $1 \leq \$_1 \leq \text{len } T_1$ , then for every finite binary tree  $T$  decorated with elements of  $\mathbb{R}_{\mathbb{N}}$  such that  $T \in T_1(\$_1)$  holds  $T$  is one-to-one. For every natural number  $i$  such that  $\mathcal{P}[i]$  holds  $\mathcal{P}[i+1]$  by [2, (16), (14)]. For every natural number  $i$ ,  $\mathcal{P}[i]$  from [2, Sch. 2].  $\square$

Let us consider  $p$ .

NOW WE ARE AT THE POSITION WHERE WE CAN PRESENT THE MAIN THEOREM OF THE PAPER: Every binary Huffman tree of  $p$  is one-to-one.

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