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Commutativeness of Fundamental Groups of Topological Groups

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Summary. In this article we prove that fundamental groups based at the unit point of topological groups are commutative [11].

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The notation and terminology used in this paper have been introduced in the following articles: [3], [19], [9], [10], [16], [20], [4], [5], [22], [23], [21], [1], [6], [17], [18], [2], [25], [26], [24], [15], [12], [13], [8], [14], and [7].

Let A be a non empty set, x be an element, and a be an element of A. Let us observe that $(A \longmapsto x)(a)$ reduces to x.

Let A, B be non empty topological spaces, C be a set, and f be a function from $A \times B$ into C. Let b be an element of B. Let us note that the functor $f(a, b)$ yields an element of C. Let G be a multiplicative magma and g be an element of G. We say that g is unital if and only if

(Def. 1) $g = \mathbf{1}_G$.

One can check that $\mathbf{1}_G$ is unital.

Let G be a unital multiplicative magma. Let us note that there exists an element of G which is unital.

Let g be an element of G and h be a unital element of G. One can check that $g \cdot h$ reduces to g. One can check that $h \cdot g$ reduces to g.

Let G be a group. One can verify that $(\mathbf{1}_G)^{-1}$ reduces to $\mathbf{1}_G$.

The scheme *TopFuncEx* deals with non empty topological spaces $\mathcal{S}$, $\mathcal{T}$ and a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and states that

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(Sch. 1) There exists a function f from $\mathcal{S} \times \mathcal{T}$ into $\mathcal{X}$ such that for every point s of $\mathcal{S}$ for every point t of $\mathcal{T}$, $f(s,t) = \mathcal{F}(s,t)$.

The scheme *TopFuncEq* deals with non empty topological spaces $\mathcal{S}$, $\mathcal{T}$ and a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and states that

(Sch. 2) For every functions f, g from $\mathcal{S} \times \mathcal{T}$ into $\mathcal{X}$ such that for every point s of $\mathcal{S}$ and for every point t of $\mathcal{T}$, $f(s,t) = \mathcal{F}(s,t)$ and for every point s of $\mathcal{S}$ and for every point t of $\mathcal{T}$, $g(s,t) = \mathcal{F}(s,t)$ holds $f = g$.

Let X be a non empty set, T be a non empty multiplicative magma, and f, g be functions from X into T. The functor $f \cdot g$ yielding a function from X into T is defined by

(Def. 2) Let us consider an element x of X. Then $it(x) = f(x) \cdot g(x)$.

Now we state the proposition:

(1) Let us consider a non empty set X, an associative non empty multiplicative magma T, and functions f, g, h from X into T. Then $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

Let X be a non empty set, T be a commutative non empty multiplicative magma, and f, g be functions from X into T. Observe that the functor $f \cdot g$ is commutative.

Let T be a non empty topological group structure, t be a point of T, and f, g be loops of t. The functor $f \bullet g$ yielding a function from $\mathbb{I}$ into T is defined by the term

(Def. 3) $f \cdot g$.

In this paper T denotes a continuous unital topological space-like non empty topological group structure, x, y denote points of $\mathbb{I}$, s, t denote unital points of T, f, g denote loops of t, and c denotes a constant loop of t.

Let us consider T, t, f, and g. One can check that the functor $f \bullet g$ yields a loop of t. Let T be an inverse-continuous semi topological group. Observe that $\cdot_T^{-1}$ is continuous.

Let T be a semi topological group, t be a point of T, and f be a loop of t. The functor f^{-1} yielding a function from $\mathbb{I}$ into T is defined by the term

(Def. 4) $\cdot_T^{-1} \cdot f$.

Let us consider a semi topological group T, a point t of T, and a loop f of t. Now we state the propositions:

(2) $(f^{-1})(x) = f(x)^{-1}$.

(3) $(f^{-1})(x) \cdot f(x) = \mathbf{1}_T$.

(4) $f(x) \cdot (f^{-1})(x) = \mathbf{1}_T$.

Let T be an inverse-continuous semi topological group, t be a unital point of T, and f be a loop of t. One can check that the functor f^{-1} yields a loop of

t. Let s, t be points of $\mathbb{I}$. One can check that the functor $s \cdot t$ yields a point of $\mathbb{I}$. The functor $\otimes_{\mathbb{R}^1}$ yielding a function from $\mathbb{R}^{\mathbf{1}} \times \mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by

(Def. 5) Let us consider points x, y of $\mathbb{R}^{\mathbf{1}}$. Then $it(x, y) = x \cdot y$.

Observe that $\otimes_{\mathbb{R}^1}$ is continuous.

Now we state the proposition:

(5) $(\mathbb{R}^{\mathbf{1}} \times \mathbb{R}^{\mathbf{1}}) \restriction (R^1[0, 1] \times R^1[0, 1]) = \mathbb{I} \times \mathbb{I}$.

The functor $\otimes_{\mathbb{I}}$ yielding a function from $\mathbb{I} \times \mathbb{I}$ into $\mathbb{I}$ is defined by the term

(Def. 6) $\otimes_{\mathbb{R}^1} \restriction R^1[0, 1]$.

Now we state the proposition:

(6) $(\otimes_{\mathbb{I}})(x, y) = x \cdot y$.

One can verify that $\otimes_{\mathbb{I}}$ is continuous.

Now we state the proposition:

(7) Let us consider points a, b of $\mathbb{I}$ and a neighbourhood N of $a \cdot b$. Then there exists a neighbourhood N_1 of a and there exists a neighbourhood N_2 of b such that for every points x, y of $\mathbb{I}$ such that $x \in N_1$ and $y \in N_2$ holds $x \cdot y \in N$. The theorem is a consequence of (6).

Let T be a non empty multiplicative magma and F, G be functions from $\mathbb{I} \times \mathbb{I}$ into T. The functor $F * G$ yielding a function from $\mathbb{I} \times \mathbb{I}$ into T is defined by

(Def. 7) Let us consider points a, b of $\mathbb{I}$. Then $it(a, b) = F(a, b) \cdot G(a, b)$.

Now we state the proposition:

(8) Let us consider functions F, G from $\mathbb{I} \times \mathbb{I}$ into T and subsets M, N of $\mathbb{I} \times \mathbb{I}$. Then $(F * G)^{\circ}(M \cap N) \subseteq F^{\circ}M \cdot G^{\circ}N$.

Let us consider T. Let F, G be continuous functions from $\mathbb{I} \times \mathbb{I}$ into T. Observe that $F * G$ is continuous.

Now we state the propositions:

(9) Let us consider loops f_1, f_2, g_1, g_2 of t. Suppose

(i) f_1, f_2 are homotopic, and

(ii) g_1, g_2 are homotopic.

Then $f_1 \bullet g_1$, $f_2 \bullet g_2$ are homotopic.

(10) Let us consider loops f_1, f_2, g_1, g_2 of t, a homotopy F between f_1 and f_2, and a homotopy G between g_1 and g_2. Suppose

(i) f_1, f_2 are homotopic, and

(ii) g_1, g_2 are homotopic.

Then $F * G$ is a homotopy between $f_1 \bullet g_1$ and $f_2 \bullet g_2$. The theorem is a consequence of (9).

(11) $f + g = (f + c) \bullet (c + g)$.

(12) $f \bullet g$, $(f + c) \bullet (c + g)$ are homotopic. The theorem is a consequence of (9).

Let T be a semi topological group, t be a point of T, and f, g be loops of t. The functor HopfHomotopy(f, g) yielding a function from $\mathbb{I} \times \mathbb{I}$ into T is defined by

(Def. 8) Let us consider points a, b of $\mathbb{I}$. Then $it(a, b) = (((f^{-1})(a \cdot b) \cdot f(a)) \cdot g(a)) \cdot f(a \cdot b)$.

Note that HopfHomotopy(f, g) is continuous.

In the sequel T denotes a topological group, t denotes a unital point of T, and f, g denote loops of t.

Now we state the proposition:

(13) $f \bullet g$, $g \bullet f$ are homotopic.

Let us consider T, t, f, and g. Let us note that the functor HopfHomotopy(f, g) yields a homotopy between $f \bullet g$ and $g \bullet f$.

Now we are at the position where we can present the Main Theorem of the paper: $\pi_1(T, t)$ is commutative.

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