

Relational Formal Characterization of Rough Sets

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Summary. The notion of a rough set, developed by Pawlak [10], is an important tool to describe situation of incomplete or partially unknown information. In this article, which is essentially the continuation of [6], we try to give the characterization of approximation operators in terms of ordinary properties of underlying relations (some of them, as serial and mediate relations, were not available in the Mizar Mathematical Library). Here we drop the classical equivalence- and tolerance-based models of rough sets [12] trying to formalize some parts of [19] following also [18] in some sense (Propositions 1–8, Corr. 1 and 2; the complete description is available in the Mizar script). Our main problem was that informally, there is a direct correspondence between relations and underlying properties, in our approach however [7], which uses relational structures rather than relations, we had to switch between classical (based on pure set theory) and abstract (using the notion of a structure) parts of the Mizar Mathematical Library. Our next step will be translation of these properties into the pure language of Mizar attributes.

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The notation and terminology used in this paper have been introduced in the following articles: [13], [11], [5], [1], [2], [14], [3], [9], [16], [6], [15], [17], [8], and [4].

1. PRELIMINARIES

One can verify that there exists a relational structure which is non empty and void.

Now we state the propositions:

- (1) Let us consider a total non empty relational structure R and an element x of R . Then $x \in \text{field the internal relation of } R$.
- (2) Let us consider a non empty 1-sorted structure R and a subset X of R . Then $\{x \text{ where } x \text{ is an element of } R : \emptyset \subseteq X\} = \Omega_R$. PROOF: $y \in \{x \text{ where } x \text{ is an element of } R : \emptyset \subseteq X\}$. \square
- (3) Let us consider a 1-sorted structure R and a subset X of R . Then $\{x \text{ where } x \text{ is an element of } R : \emptyset \text{ meets } X\} = \emptyset_R$.

2. MISSING ORDINARY PROPERTIES OF BINARY RELATIONS

Let R be a binary relation and X be a set. We say that R is serial in X if and only if

- (Def. 1) Let us consider an element x . Suppose $x \in X$. Then there exists an element y such that
- (i) $y \in X$, and
 - (ii) $\langle x, y \rangle \in R$.

We say that R is serial if and only if

- (Def. 2) R is serial in field R .

Let R be a relational structure. We say that R is serial if and only if

- (Def. 3) the internal relation of R is serial in the carrier of R .

One can check that every relational structure which is reflexive is also serial.

Let R be a non empty relational structure. One can verify that R is serial if and only if the condition (Def. 4) is satisfied.

- (Def. 4) Let us consider an element x of R . Then there exists an element y of R such that $x \leq y$.

Let us observe that every relational structure which is total is also serial and every relational structure which is serial is also total.

Let R be a non empty serial relational structure and x be an element of R .

Let us note that $[x]_{\text{the internal relation of } R}$ is non empty.

Now we state the proposition:

- (4) Let us consider a non empty reflexive relational structure R and an element x of R . Then $x \in [x]_\alpha$, where α is the internal relation of R . The theorem is a consequence of (1).

Let R be a non empty reflexive relational structure and x be an element of R . Note that $[x]_{\text{the internal relation of } R}$ is non empty.

Let R be a binary relation and X be a set. We say that R is mediate in X if and only if

- (Def. 5) Let us consider elements x, y . Suppose $x, y \in X$. If $\langle x, y \rangle \in R$, then there exists an element z such that $z \in X$ and $\langle x, z \rangle, \langle z, y \rangle \in R$.

We say that R is mediate if and only if

(Def. 6) R is mediate in field R .

Let R be a relational structure. We say that R is mediate if and only if

(Def. 7) the internal relation of R is mediate in the carrier of R .

Let us note that every relational structure which is reflexive is also mediate.

3. APPROXIMATIONS REVISITED

Now we state the proposition:

- (5) Let us consider a non empty relational structure R and elements a, b of R . Suppose $a \in \text{UAp}(\{b\})$. Then $\langle a, b \rangle \in$ the internal relation of R .

Let R be a non empty relational structure and X be a subset of R . The functor $\text{Uap } X$ yielding a subset of R is defined by the term

(Def. 8) $(\text{LAp}(X^c))^c$.

The functor $\text{Lap } X$ yielding a subset of R is defined by the term

(Def. 9) $(\text{UAp}(X^c))^c$.

Now we state the propositions:

- (6) Let us consider a non empty relational structure R , a subset X of R , and an element x . If $x \in \text{LAp}(X)$, then $[x]_\alpha \subseteq X$, where α is the internal relation of R .
- (7) Let us consider a non empty relational structure R , a subset X of R , and a set x . If $x \in \text{UAp}(X)$, then $[x]_\alpha$ meets X , where α is the internal relation of R .

Let us consider a non empty relational structure R and a subset X of R . Now we state the propositions:

(8) $\text{Uap } X = \text{UAp}(X)$.

(9) $\text{Lap } X = \text{LAp}(X)$.

Let us consider a non empty void relational structure R and a subset X of R . Now we state the propositions:

(10) $\text{LAp}(X) = \Omega_R$.

(11) $\text{UAp}(X) = \emptyset_R$.

4. GENERAL PROPERTIES OF APPROXIMATIONS

Let R be a non empty relational structure. Observe that $\text{LAp}(\Omega_R)$ reduces to Ω_R .

Let R be a non empty serial relational structure. One can check that $\text{UAp}(\Omega_R)$ reduces to Ω_R .

One can check that $\text{LAp}(\emptyset_R)$ reduces to \emptyset_R .

Let R be a non empty relational structure. Note that $\text{UAp}(\emptyset_R)$ reduces to \emptyset_R .

Let us consider a non empty relational structure R and subsets X, Y of R . Now we state the propositions:

- (12) $\text{LAp}(X \cap Y) = \text{LAp}(X) \cap \text{LAp}(Y)$.
- (13) $\text{UAp}(X \cup Y) = \text{UAp}(X) \cup \text{UAp}(Y)$.
- (14) If $X \subseteq Y$, then $\text{LAp}(X) \subseteq \text{LAp}(Y)$.
- (15) If $X \subseteq Y$, then $\text{UAp}(X) \subseteq \text{UAp}(Y)$.

Now we state the propositions:

- (16) Let us consider a non empty relational structure R and a subset X of R . Then $\text{LAp}(X^c) = (\text{UAp}(X))^c$.
- (17) Let us consider a non empty serial relational structure R and a subset X of R . Then $\text{LAp}(X) \subseteq \text{UAp}(X)$.

5. AUXILIARY OPERATIONS ON APPROXIMATION OPERATORS

Let R be a non empty relational structure. The functors $\text{LAp}(R)$ and $\text{UAp}(R)$ yielding functions from $2^{\text{the carrier of } R}$ into $2^{\text{the carrier of } R}$ are defined by the conditions, respectively.

(Def. 10) Let us consider a subset X of R . Then $(\text{LAp}(R))(X) = \text{LAp}(X)$.

(Def. 11) Let us consider a subset X of R . Then $(\text{UAp}(R))(X) = \text{UAp}(X)$.

Let A be a non empty set and U be a function from 2^A into 2^A . We say that U preserves empty set if and only if

(Def. 12) $U(\emptyset) = \emptyset$.

We say that U preserves universe if and only if

(Def. 13) $U(A) = A$.

Observe that id_{2^A} preserves empty set and universe as a function from 2^A into 2^A .

One can verify that there exists a function from 2^A into 2^A which preserves empty set and universe.

Let X be a set and f be a function from 2^X into 2^X . The functor $\text{Flip } f$ yielding a function from 2^X into 2^X is defined by

(Def. 14) Let us consider a subset x of X . Then $it(x) = f(x^c)^c$.

Let us consider a set X and a function f from 2^X into 2^X . Now we state the propositions:

- (18) If $f(\emptyset) = \emptyset$, then $(\text{Flip } f)(X) = X$.
- (19) If $f(X) = X$, then $(\text{Flip } f)(\emptyset) = \emptyset$.
- (20) If $f = \text{id}_{2^X}$, then $\text{Flip } f = f$.

Let us consider a set X , a function f from 2^X into 2^X , and subsets A, B of X . Now we state the propositions:

- (21) If for every subsets A, B of X , $f(A \cup B) = f(A) \cup f(B)$, then $(\text{Flip } f)(A \cap B) = (\text{Flip } f)(A) \cap (\text{Flip } f)(B)$.
- (22) If for every subsets A, B of X , $f(A \cap B) = f(A) \cap f(B)$, then $(\text{Flip } f)(A \cup B) = (\text{Flip } f)(A) \cup (\text{Flip } f)(B)$.

Now we state the proposition:

- (23) Let us consider a set X and a function f from 2^X into 2^X . Then $\text{Flip } \text{Flip } f = f$. PROOF: Set $g = \text{Flip } \text{Flip } f$. For every subset x of X , $g(x) = f(x)$. \square

Let A be a non empty set and f be a function from 2^A into 2^A . Observe that $\text{Flip } f$ preserves empty set.

Let f be a function from 2^A into 2^A . One can verify that $\text{Flip } f$ preserves universe.

Now we state the proposition:

- (24) Let us consider a non empty set A and functions L, U from 2^A into 2^A . Suppose

- (i) $U = \text{Flip } L$, and
- (ii) for every subset X of A , $L(L(X)) \subseteq L(X)$.

Let us consider a subset X of A . Then $U(X) \subseteq U(U(X))$.

6. TOWARDS TOPOLOGICAL MODELS OF ROUGH SETS

Let T be a topological space. The functors $\text{ClMap } T$ and $\text{IntMap } T$ yielding functions from $2^{\text{the carrier of } T}$ into $2^{\text{the carrier of } T}$ are defined by the conditions, respectively.

(Def. 15) Let us consider a subset X of T . Then $(\text{ClMap } T)(X) = \overline{X}$.

(Def. 16) Let us consider a subset X of T . Then $(\text{IntMap } T)(X) = \text{Int } X$.

Let f be a function from $2^{\text{the carrier of } T}$ into $2^{\text{the carrier of } T}$. We say that f is closed-valued if and only if

(Def. 17) Let us consider a subset X of T . Then $f(X)$ is closed.

We say that f is open-valued if and only if

(Def. 18) Let us consider a subset X of T . Then $f(X)$ is open.

Note that $\text{ClMap } T$ is closed-valued and $\text{IntMap } T$ is open-valued.

Let us observe that there exists a function from $2^{\text{the carrier of } T}$ into $2^{\text{the carrier of } T}$ which is closed-valued and there exists a function from $2^{\text{the carrier of } T}$ into $2^{\text{the carrier of } T}$ which is open-valued.

Let us consider a topological space T . Now we state the propositions:

- (25) $\text{Flip } \text{ClMap } T = \text{IntMap } T$.

$$(26) \quad \text{Flip IntMap } T = \text{ClMap } T.$$

Let T be a non empty topological space. One can verify that $\text{ClMap } T$ preserves empty set and universe and $\text{IntMap } T$ preserves empty set and universe.

7. FORMALIZATION OF ZHU'S PAPER [19]

Let us consider a non empty relational structure R . Now we state the propositions:

$$(27) \quad \text{Flip UAp}(R) = \text{LAp}(R).$$

$$(28) \quad \text{Flip LAp}(R) = \text{UAp}(R).$$

Now we state the proposition:

$$(29) \quad \text{Let us consider a non empty finite set } A \text{ and a function } U \text{ from } 2^A \text{ into } 2^A. \text{ Suppose}$$

$$(i) \quad U(\emptyset) = \emptyset, \text{ and}$$

$$(ii) \quad \text{for every subsets } X, Y \text{ of } A, U(X \cup Y) = U(X) \cup U(Y).$$

Then there exists a non empty finite relational structure R such that

$$(iii) \quad \text{the carrier of } R = A, \text{ and}$$

$$(iv) \quad U = \text{UAp}(R).$$

The theorem is a consequence of (13). PROOF: Define $\mathcal{P}[\text{set}, \text{set}] \equiv \$_1 \in L(\{\$2\})$. Consider R being a binary relation on A such that for every elements x, y of A , $\langle x, y \rangle \in R$ iff $\mathcal{P}[x, y]$. Reconsider $RR = \langle A, R \rangle$ as a non empty finite relational structure. For every element y of RR and for every subset Y of RR such that $Y = \{y\}$ holds $\text{UAp}(Y) = L(Y)$. For every element x such that $x \in \text{dom UAp}(RR)$ holds $(\text{UAp}(RR))(x) = L(x)$. \square

Let us consider a non empty finite set A and a function L from 2^A into 2^A . Now we state the propositions:

$$(30) \quad \text{Suppose } L(A) = A \text{ and for every subsets } X, Y \text{ of } A, L(X \cap Y) = L(X) \cap L(Y). \text{ Then there exists a non empty finite relational structure } R \text{ such that}$$

$$(i) \quad \text{the carrier of } R = A, \text{ and}$$

$$(ii) \quad L = \text{LAp}(R).$$

$$(31) \quad \text{Suppose } L(A) = A \text{ and } L(\emptyset) = \emptyset \text{ and for every subsets } X, Y \text{ of } A, L(X \cap Y) = L(X) \cap L(Y). \text{ Then there exists a non empty serial relational structure } R \text{ such that}$$

$$(i) \quad \text{the carrier of } R = A, \text{ and}$$

$$(ii) \quad L = \text{LAp}(R).$$

Now we state the propositions:

(32) Let us consider a non empty finite set A and a function U from 2^A into 2^A . Suppose

- (i) $U(A) = A$, and
- (ii) $U(\emptyset) = \emptyset$, and
- (iii) for every subsets X, Y of A , $U(X \cup Y) = U(X) \cup U(Y)$.

Then there exists a non empty finite serial relational structure R such that

- (iv) the carrier of $R = A$, and
- (v) $U = \text{UAp}(R)$.

The theorem is a consequence of (29). PROOF: Consider R being a non empty finite relational structure such that the carrier of $R = A$ and $U = \text{UAp}(R)$. For every element x such that $x \in$ the carrier of R there exists an element y such that $y \in$ the carrier of R and $\langle x, y \rangle \in$ the internal relation of R . \square

(33) Let us consider a non empty finite set A and a function L from 2^A into 2^A . Suppose

- (i) $L(A) = A$, and
- (ii) for every subset X of A , $L(X) \subseteq L(X^c)^c$, and
- (iii) for every subsets X, Y of A , $L(X \cap Y) = L(X) \cap L(Y)$.

Then there exists a non empty finite serial relational structure R such that

- (iv) the carrier of $R = A$, and
- (v) $L = \text{LAp}(R)$.

The theorem is a consequence of (30). PROOF: Consider R being a non empty finite relational structure such that the carrier of $R = A$ and $L = \text{LAp}(R)$. For every element x such that $x \in$ the carrier of R there exists an element y such that $y \in$ the carrier of R and $\langle x, y \rangle \in$ the internal relation of R . \square

(34) Let us consider a non empty finite set A and a function U from 2^A into 2^A . Suppose

- (i) $U(\emptyset) = \emptyset$, and
- (ii) for every subset X of A , $U(X^c)^c \subseteq U(X)$, and
- (iii) for every subsets X, Y of A , $U(X \cup Y) = U(X) \cup U(Y)$.

Then there exists a non empty serial relational structure R such that

- (iv) the carrier of $R = A$, and
- (v) $U = \text{UAp}(R)$.

The theorem is a consequence of (29), (19), and (27). PROOF: Consider R being a non empty finite relational structure such that the carrier of $R = A$ and $U = \text{UAp}(R)$. For every element x such that $x \in$ the carrier of R there exists an element y such that $y \in$ the carrier of R and $\langle x, y \rangle \in$ the internal relation of R . \square

Let us consider a non empty reflexive relational structure R and a subset X of R . Now we state the propositions:

$$(35) \quad \text{LAp}(X) \subseteq X.$$

$$(36) \quad X \subseteq \text{UAp}(X).$$

Now we state the propositions:

$$(37) \quad \text{Let us consider a non empty finite set } A \text{ and a function } U \text{ from } 2^A \text{ into } 2^A. \text{ Suppose}$$

$$(i) \quad U(\emptyset) = \emptyset, \text{ and}$$

$$(ii) \quad \text{for every subset } X \text{ of } A, X \subseteq U(X), \text{ and}$$

$$(iii) \quad \text{for every subsets } X, Y \text{ of } A, U(X \cup Y) = U(X) \cup U(Y).$$

Then there exists a non empty finite reflexive relational structure R such that

$$(iv) \quad \text{the carrier of } R = A, \text{ and}$$

$$(v) \quad U = \text{UAp}(R).$$

The theorem is a consequence of (32). PROOF: Consider R being a non empty finite serial relational structure such that the carrier of $R = A$ and $U = \text{UAp}(R)$. For every element x such that $x \in$ the carrier of R holds $\langle x, x \rangle \in$ the internal relation of R . \square

$$(38) \quad \text{Let us consider a non empty finite set } A \text{ and a function } L \text{ from } 2^A \text{ into } 2^A. \text{ Suppose}$$

$$(i) \quad L(A) = A, \text{ and}$$

$$(ii) \quad \text{for every subset } X \text{ of } A, L(X) \subseteq X, \text{ and}$$

$$(iii) \quad \text{for every subsets } X, Y \text{ of } A, L(X \cap Y) = L(X) \cap L(Y).$$

Then there exists a non empty finite reflexive relational structure R such that

$$(iv) \quad \text{the carrier of } R = A, \text{ and}$$

$$(v) \quad L = \text{LAp}(R).$$

The theorem is a consequence of (19), (22), (37), (23), and (27). PROOF: Set $U = \text{Flip } L$. For every subset X of A , $X \subseteq U(X)$. Consider R being a non empty finite reflexive relational structure such that the carrier of $R = A$ and $U = \text{UAp}(R)$. \square

Let us consider a non empty mediate relational structure R and a subset X of R . Now we state the propositions:

$$(39) \quad \text{UAp}(X) \subseteq \text{UAp}(\text{UAp}(X)).$$

$$(40) \quad \text{LAp}(\text{LAp}(X)) \subseteq \text{LAp}(X).$$

Now we state the proposition:

(41) Let us consider a non empty finite set A and a function U from 2^A into 2^A . Suppose

$$(i) \quad U(\emptyset) = \emptyset, \text{ and}$$

$$(ii) \quad \text{for every subset } X \text{ of } A, U(X) \subseteq U(U(X)), \text{ and}$$

$$(iii) \quad \text{for every subsets } X, Y \text{ of } A, U(X \cup Y) = U(X) \cup U(Y).$$

Then there exists a non empty mediate finite relational structure R such that

$$(iv) \quad \text{the carrier of } R = A, \text{ and}$$

$$(v) \quad U = \text{UAp}(R).$$

The theorem is a consequence of (29) and (5). PROOF: Consider R being a non empty finite relational structure such that the carrier of $R = A$ and $U = \text{UAp}(R)$. For every elements x, y such that $x, y \in$ the carrier of R holds if $\langle x, y \rangle \in$ the internal relation of R , then there exists an element z such that $z \in$ the carrier of R and $\langle x, z \rangle, \langle z, y \rangle \in$ the internal relation of R . \square

Let us consider a non empty finite set A and a function L from 2^A into 2^A . Now we state the propositions:

(42) Suppose $L(A) = A$ and for every subset X of A , $L(L(X)) \subseteq L(X)$ and for every subsets X, Y of A , $L(X \cap Y) = L(X) \cap L(Y)$. Then there exists a non empty mediate finite relational structure R such that

$$(i) \quad \text{the carrier of } R = A, \text{ and}$$

$$(ii) \quad L = \text{LAp}(R).$$

(43) Suppose $L(A) = A$ and for every subsets X, Y of A , $L(X \cap Y) = L(X) \cap L(Y)$. Then for every subset X of A , $L(X) \subseteq L(X^c)^c$ if and only if $L(\emptyset) = \emptyset$.

Now we state the proposition:

(44) Let us consider a non empty finite set A and a function U from 2^A into 2^A . Suppose

$$(i) \quad U(\emptyset) = \emptyset, \text{ and}$$

$$(ii) \quad \text{for every subsets } X, Y \text{ of } A, U(X \cup Y) = U(X) \cup U(Y).$$

Then for every subset X of A , $U(X^c)^c \subseteq U(X)$ if and only if $U(A) = A$.

The theorem is a consequence of (34), (32), (27), and (17).

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