

Products in Categories without Uniqueness of \mathbf{cod} and \mathbf{dom} ¹

Artur Kornilowicz
 Institute of Informatics
 University of Białystok
 Sosnowa 64, 15-887 Białystok
 Poland

Summary. The paper introduces Cartesian products in categories without uniqueness of \mathbf{cod} and \mathbf{dom} . It is proven that set-theoretical product is the product in the category \mathbf{Ens} [7].

MML identifier: **ALTCAT_5**, version: 8.0.01 5.3.1162

The papers [10], [6], [1], [8], [2], [3], [4], [9], [12], [11], and [5] provide the terminology and notation for this paper.

In this paper I denotes a set and E denotes a non empty set.

Let us mention that every binary relation which is empty is also \emptyset -defined.

Let C be a graph. We say that C is functional if and only if:

(Def. 1) For all objects a, b of C holds $\langle a, b \rangle$ is functional.

Let us consider E . One can verify that \mathbf{Ens}_E is functional.

Let us observe that there exists a category which is functional and strict.

Let C be a functional category structure. One can verify that the graph of C is functional.

Let us observe that there exists a graph which is functional and strict.

Let us note that there exists a category which is functional and strict.

Let C be a functional graph and let a, b be objects of C . Observe that $\langle a, b \rangle$ is functional.

¹This work has been supported by the Polish Ministry of Science and Higher Education project “Managing a Large Repository of Computer-verified Mathematical Knowledge” (N N519 385136).

Let C be a non empty category structure and let I be a set. An objects family of I and C is a function from I into C .

Let C be a non empty category structure, let o be an object of C , let I be a set, and let f be an objects family of I and C . A many sorted set indexed by I is said to be a morphisms family of o and f if:

(Def. 2) For every set i such that $i \in I$ there exists an object o_1 of C such that $o_1 = f(i)$ and $it(i)$ is a morphism from o to o_1 .

Let C be a non empty category structure, let o be an object of C , let I be a non empty set, and let f be an objects family of I and C . Let us note that the morphisms family of o and f can be characterized by the following (equivalent) condition:

(Def. 3) For every element i of I holds $it(i)$ is a morphism from o to $f(i)$.

Let C be a non empty category structure, let o be an object of C , let I be a non empty set, let f be an objects family of I and C , let M be a morphisms family of o and f , and let i be an element of I . Then $M(i)$ is a morphism from o to $f(i)$.

Let C be a functional non empty category structure, let o be an object of C , let I be a set, and let f be an objects family of I and C . Observe that every morphisms family of o and f is function yielding.

Next we state the proposition

(1) Let C be a non empty category structure, o be an object of C , and f be an objects family of \emptyset and C . Then \emptyset is a morphisms family of o and f .

Let C be a non empty category structure, let I be a set, let A be an objects family of I and C , let B be an object of C , and let P be a morphisms family of B and A . We say that P is feasible if and only if:

(Def. 4) For every set i such that $i \in I$ there exists an object o of C such that $o = A(i)$ and $P(i) \in \langle B, o \rangle$.

Let C be a non empty category structure, let I be a non empty set, let A be an objects family of I and C , let B be an object of C , and let P be a morphisms family of B and A . Let us observe that P is feasible if and only if:

(Def. 5) For every element i of I holds $P(i) \in \langle B, A(i) \rangle$.

Let C be a category, let I be a set, let A be an objects family of I and C , let B be an object of C , and let P be a morphisms family of B and A . We say that P is projection morphisms family if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let X be an object of C and F be a morphisms family of X and A . Suppose F is feasible. Then there exists a morphism f from X to B such that

(i) $f \in \langle X, B \rangle$,

- (ii) for every set i such that $i \in I$ there exists an object s_1 of C and there exists a morphism P_1 from B to s_1 such that $s_1 = A(i)$ and $P_1 = P(i)$ and $F(i) = P_1 \cdot f$, and
- (iii) for every morphism f_1 from X to B such that for every set i such that $i \in I$ there exists an object s_1 of C and there exists a morphism P_1 from B to s_1 such that $s_1 = A(i)$ and $P_1 = P(i)$ and $F(i) = P_1 \cdot f_1$ holds $f = f_1$.

Let C be a category, let I be a non empty set, let A be an objects family of I and C , let B be an object of C , and let P be a morphisms family of B and A . Let us observe that P is projection morphisms family if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let X be an object of C and F be a morphisms family of X and A . Suppose F is feasible. Then there exists a morphism f from X to B such that

- (i) $f \in \langle X, B \rangle$,
- (ii) for every element i of I holds $F(i) = P(i) \cdot f$, and
- (iii) for every morphism f_1 from X to B such that for every element i of I holds $F(i) = P(i) \cdot f_1$ holds $f = f_1$.

Let C be a category, let A be an objects family of \emptyset and C , and let B be an object of C . Note that every morphisms family of B and A is feasible.

One can prove the following propositions:

- (2) Let C be a category, A be an objects family of \emptyset and C , and B be an object of C . If B is terminal, then there exists a morphisms family of B and A which is empty and projection morphisms family.
- (3) For every objects family A of I and Ens_1 and for every object o of Ens_1 holds $I \mapsto \emptyset$ is a morphisms family of o and A .
- (4) Let A be an objects family of I and Ens_1 , o be an object of Ens_1 , and P be a morphisms family of o and A . If $P = I \mapsto \emptyset$, then P is feasible and projection morphisms family.

Let C be a category. We say that C has products if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let I be a set and A be an objects family of I and C . Then there exists an object B of C such that there exists a morphisms family of B and A which is feasible and projection morphisms family.

Let us note that Ens_1 has products.

One can check that there exists a category which has products.

Let C be a category, let I be a set, let A be an objects family of I and C , and let B be an object of C . We say that B is A -cat product-like if and only if:

(Def. 9) There exists a morphisms family of B and A which is feasible and projection morphisms family.

Let C be a category with products, let I be a set, and let A be an objects family of I and C . One can check that there exists an object of C which is A -cat product-like.

Let C be a category and let A be an objects family of \emptyset and C . Note that every object of C which is A -cat product-like is also terminal.

We now state two propositions:

- (5) Let C be a category, A be an objects family of \emptyset and C , and B be an object of C . If B is terminal, then B is A -cat product-like.
- (6) Let C be a category, A be an objects family of I and C , and C_1, C_2 be objects of C . Suppose C_1 is A -cat product-like and C_2 is A -cat product-like. Then C_1, C_2 are iso.

In the sequel A is an objects family of I and Ens_E .

Let us consider I, E, A . Let us assume that $\coprod A \in E$. The functor $\text{EnsCatProductObj } A$ yielding an object of Ens_E is defined by:

(Def. 10) $\text{EnsCatProductObj } A = \coprod A$.

Let us consider I, E, A . Let us assume that $\coprod A \in E$. The functor $\text{EnsCatProduct } A$ yields a morphisms family of $\text{EnsCatProductObj } A$ and A and is defined by:

(Def. 11) For every set i such that $i \in I$ holds $(\text{EnsCatProduct } A)(i) = \text{proj}(A, i)$.

We now state four propositions:

- (7) If $\coprod A \in E$ and $\coprod A = \emptyset$, then $\text{EnsCatProduct } A = I \longrightarrow \emptyset$.
- (8) If $\coprod A \in E$, then $\text{EnsCatProduct } A$ is feasible and projection morphisms family.
- (9) If $\coprod A \in E$, then $\text{EnsCatProductObj } A$ is A -cat product-like.
- (10) If for all I, A holds $\coprod A \in E$, then Ens_E has products.

REFERENCES

- [1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [6] Beata Madras. Basic properties of objects and morphisms. *Formalized Mathematics*, 6(3):329–334, 1997.
- [7] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów*, volume 45 of *Biblioteka Matematyczna*. PWN, Warszawa, 1978.
- [8] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [9] Andrzej Trybulec. Many sorted sets. *Formalized Mathematics*, 4(1):15–22, 1993.
- [10] Andrzej Trybulec. Categories without uniqueness of **cod** and **dom**. *Formalized Mathematics*, 5(2):259–267, 1996.
- [11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.

- [12] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.

Received August 19, 2012
