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The Gödel Completeness Theorem for Uncountable Languages[1]

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Summary. This article is the second in a series of two Mizar articles constituting a formal proof of the Gödel Completeness theorem [15] for uncountably large languages. We follow the proof given in [16]. The present article contains the techniques required to expand a theory such that the expanded theory contains witnesses and is negation faithful. Then the completeness theorem follows immediately.

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The notation and terminology used here have been introduced in the following papers: [8], [1], [3], [10], [19], [5], [14], [11], [12], [7], [6], [22], [2], [4], [17], [18], [23], [20], [9], [21], and [13].

[1]This article is part of the first author's Bachelor thesis under the supervision of the second author.

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1. Formula-Constant Extension

For simplicity, we use the following convention: A_1 denotes an alphabet, P_1 denotes a consistent subset of CQC-WFF A_1, P_2 denotes a subset of CQC-WFF A_1, p, q, r, s denote elements of CQC-WFF A_1, A denotes a non empty set, J denotes an interpretation of A_1 and A, v denotes an element of the valuations in A_1 and A, n, k denote elements of $\mathbb{N}$, x denotes a bound variable of A_1, and A_2 denotes an A_1-expanding alphabet.

Let us consider A_1 and let P_1 be a subset of CQC-WFF A_1. We say that P_1 is satisfiable if and only if:

(Def. 1) There exist A, J, v such that $J \models_v P_1$.

In the sequel J_2 is an interpretation of A_2 and A and J_1 is an interpretation of A_1 and A.

One can prove the following proposition

(1) There exists a set s such that for all p, x holds $\langle s, \langle x, p\rangle\rangle \notin \mathrm{Symb}\, A_1$.

Let us consider A_1. A set is called a free symbol of A_1 if:

(Def. 2) For all p, x holds $\langle \mathrm{it}, \langle x, p\rangle\rangle \notin \mathrm{Symb}\, A_1$.

Let us consider A_1. The functor FCEx A_1 yielding an A_1-expanding alphabet is defined as follows:

(Def. 3) $\mathrm{FCEx}\, A_1 = \mathbb{N} \times (\mathrm{Symb}\, A_1 \cup \{\langle \text{ the free symbol of } A_1,\ \langle x, p\rangle\rangle\})$.

Let us consider A_1, p, x. The example of p and x yielding a bound variable of FCEx A_1 is defined as follows:

(Def. 4) The example of p and $x = \langle 4, \langle \text{ the free symbol of } A_1,\ \langle x, p\rangle\rangle\rangle$.

Let us consider A_1, p, x. The example formula of p and x yielding an element of CQC-WFF FCEx A_1 is defined by:

(Def. 5) The example formula of p and $x = \neg\exists_{\mathrm{FCEx}\, A_1\text{-Cast}\, x}(\mathrm{FCEx}\, A_1\text{-Cast}\, p) \vee (\mathrm{FCEx}\, A_1\text{-Cast}\, p)(\mathrm{FCEx}\, A_1\text{-Cast}\, x, \text{the example of } p \text{ and } x)$.

Let us consider A_1. The example formulae of A_1 yields a subset of CQC-WFF FCEx A_1 and is defined as follows:

(Def. 6) The example formulae of $A_1 = \{$the example formula of p and $x\}$.

One can prove the following proposition

(2) Let k be an element of $\mathbb{N}$. Suppose $k > 0$. Then there exists a k-element finite sequence F such that

(i) for every natural number n such that $n \leq k$ and $1 \leq n$ holds $F(n)$ is an alphabet,

(ii) $F(1) = A_1$, and

(iii) for every natural number n such that $n < k$ and $1 \leq n$ there exists an alphabet A_2 such that $F(n) = A_2$ and $F(n+1) = \mathrm{FCEx}\, A_2$.

Let us consider A_1 and let k be a natural number. A $k+1$-element finite sequence is said to be a FCEx-sequence of A_1 and k if it satisfies the conditions (Def. 7).

(Def. 7)(i) For every natural number n such that $n \leq k+1$ and $1 \leq n$ holds it(n) is an alphabet,

(ii) it$(1) = A_1$, and

(iii) for every natural number n such that $n < k+1$ and $1 \leq n$ there exists an alphabet A_2 such that it$(n) = A_2$ and it$(n+1) = \mathrm{FCEx}\, A_2$.

The following propositions are true:

(3) For every natural number k and for every FCEx-sequence S of A_1 and k holds $S(k+1)$ is an alphabet.

(4) For every natural number k and for every FCEx-sequence S of A_1 and k holds $S(k+1)$ is an A_1-expanding alphabet.

Let us consider A_1 and let k be a natural number. The k-th FCEx of A_1 yielding an A_1-expanding alphabet is defined as follows:

(Def. 8) The k-th FCEx of $A_1 =$ the FCEx-sequence of A_1 and $k(k+1)$.

Let us consider A_1, P_1. A function is called an EF-sequence of A_1 and P_1 if it satisfies the conditions (Def. 9).

(Def. 9)(i) $\mathrm{dom}\,\mathrm{it} = \mathbb{N}$,

(ii) it$(0) = P_1$, and

(iii) for every natural number n holds it$(n+1) =$ it$(n) \cup$ the example formulae of the n-th FCEx of A_1.

Next we state two propositions:

(5) For every natural number k holds FCEx (the k-th FCEx of A_1) = the $(k+1)$-th FCEx of A_1.

(6) For all k, n such that $n \leq k$ holds the n-th FCEx of $A_1 \subseteq$ the k-th FCEx of A_1.

Let us consider A_1, P_1 and let k be a natural number. The k-th EF of A_1 and P_1 yields a subset of CQC-WFF (the k-th FCEx of A_1) and is defined as follows:

(Def. 10) The k-th EF of A_1 and $P_1 =$ the EF-sequence of A_1 and $P_1(k)$.

One can prove the following propositions:

(7) For all r, s, x holds $A_2\text{-Cast}(r \vee s) = A_2\text{-Cast}\, r \vee A_2\text{-Cast}\, s$ and $A_2\text{-Cast}\, \exists_x r = \exists_{A_2\text{-Cast}\, x}(A_2\text{-Cast}\, r)$.

(8) For all p, q, A, J, v holds $J \models_v p$ or $J \models_v q$ iff $J \models_v p \vee q$.

(9) $P_1 \cup$ the example formulae of A_1 is a consistent subset of CQC-WFF FCEx A_1.

2. The Completeness Theorem

We now state four propositions:

(10) There exists an A_1-expanding alphabet A_2 and there exists a consistent subset P_2 of CQC-WFF A_2 such that $P_1 \subseteq P_2$ and P_2 has examples.

(11) $P_1 \cup \{p\}$ is consistent or $P_1 \cup \{\neg p\}$ is consistent.

(12) Let P_2 be a consistent subset of CQC-WFF A_1. Then there exists a consistent subset T_1 of CQC-WFF A_1 such that T_1 is negation faithful and $P_2 \subseteq T_1$.

(13) For every consistent subset T_1 of CQC-WFF A_1 such that $P_1 \subseteq T_1$ and P_1 has examples holds T_1 has examples.

Let us consider A_1. One can check that every subset of CQC-WFF A_1 which is consistent is also satisfiable.

We now state the proposition

(14)[2] If $P_2 \models p$, then $P_2 \vdash p$.

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