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Introduction to Rational Functions

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Summary. In this article we formalize rational functions as pairs of polynomials and define some basic notions including the degree and evaluation of rational functions [8]. The main goal of the article is to provide properties of rational functions necessary to prove a theorem on the stability of networks.

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The notation and terminology used in this paper are introduced in the following articles: [14], [3], [4], [5], [18], [20], [16], [17], [1], [15], [2], [6], [12], [10], [11], [22], [19], [21], [9], [13], [23], and [7].

1. Preliminaries

One can prove the following three propositions:

(1) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, a be an element of L, and p, q be finite sequences of elements of L. Suppose $\operatorname{len} p = \operatorname{len} q$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} p$ holds $q_i = a \cdot p_i$. Then $\sum q = a \cdot \sum p$.

(2) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, f be a finite sequence of elements of L, and i, j be elements of $\mathbb{N}$. If $i \in \operatorname{dom} f$ and $j = i - 1$, then $\operatorname{Ins}(f_{\restriction i}, j, f_i) = f$.

(3) Let L be an add-associative right zeroed right complementable associative unital right distributive commutative non empty double loop structure, f be a finite sequence of elements of L, and i be an element of $\mathbb{N}$. If $i \in \operatorname{dom} f$, then $\prod f = f_i \cdot \prod(f_{\restriction i})$.

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Let L be an add-associative right zeroed right complementable well unital associative left distributive commutative almost left invertible integral domain-like non trivial double loop structure and let x, y be non zero elements of L. Note that $\frac{x}{y}$ is non zero.

Let us note that every add-associative right zeroed right complementable right distributive non empty double loop structure which is integral domain-like is also almost left cancelable and every add-associative right zeroed right complementable left distributive non empty double loop structure which is integral domain-like is also almost right cancelable.

Let x, y be integers. Note that $\max(x, y)$ is integer and $\min(x, y)$ is integer.

One can prove the following proposition

(4) For all integers x, y, z holds $\max(x + y, x + z) = x + \max(y, z)$.

2. More on Polynomials

Let L be a non empty zero structure and let p be a polynomial of L. We say that p is zero if and only if:

(Def. 1) $p = \mathbf{0}.\, L$.

We say that p is constant if and only if:

(Def. 2) $\deg p \leq 0$.

Let L be a non trivial zero structure. One can verify that there exists a polynomial of L which is non zero.

Let L be a non empty zero structure. One can verify that $\mathbf{0}.\, L$ is zero and constant.

Let L be a non degenerated multiplicative loop with zero structure. Note that $\mathbf{1}.\, L$ is non zero.

Let L be a non empty multiplicative loop with zero structure. Note that $\mathbf{1}.\, L$ is constant.

Let L be a non empty zero structure. One can verify that every polynomial of L which is zero is also constant. Note that every polynomial of L which is non constant is also non zero.

Let L be a non trivial zero structure. One can verify that there exists a polynomial of L which is non constant.

Let L be a well unital non degenerated non empty double loop structure, let z be an element of L, and let k be an element of $\mathbb{N}$. Observe that $\mathrm{rpoly}(k, z)$ is non zero.

Let L be an add-associative right zeroed right complementable distributive non degenerated double loop structure. One can check that Polynom-Ring L is non degenerated.

Let L be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure. Observe that Polynom-Ring L is integral domain-like.

Next we state two propositions:

(5) Let L be an add-associative right zeroed right complementable right distributive associative non empty double loop structure, p, q be polynomials of L, and a be an element of L. Then $(a \cdot p) * q = a \cdot (p * q)$.

(6) Let L be an add-associative right zeroed right complementable right distributive commutative associative non empty double loop structure, p, q be polynomials of L, and a be an element of L. Then $p*(a \cdot q) = a \cdot (p*q)$.

Let L be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure, let p be a non zero polynomial of L, and let a be a non zero element of L. Note that $a \cdot p$ is non zero.

Let L be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let p_1, p_2 be non zero polynomials of L. Observe that $p_1 * p_2$ is non zero.

One can prove the following proposition

(7) Let L be an add-associative right zeroed right complementable distributive Abelian integral domain-like non trivial double loop structure, p_1, p_2 be polynomials of L, and p_3 be a non zero polynomial of L. If $p_1 * p_3 = p_2 * p_3$, then $p_1 = p_2$.

Let L be a non trivial zero structure and let p be a non zero polynomial of L. One can check that degree(p) is natural.

Next we state several propositions:

(8) Let L be an add-associative right zeroed right complementable unital right distributive non empty double loop structure and p be a polynomial of L. If $\deg p = 0$, then for every element x of L holds $\mathrm{eval}(p, x) \neq 0_L$.

(9) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non degenerated double loop structure, p be a polynomial of L, and x be an element of L. Then $\mathrm{eval}(p, x) = 0_L$ if and only if $\mathrm{rpoly}(1, x) \mid p$.

(10) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible integral domain-like non degenerated double loop structure, p, q be polynomials of L, and x be an element of L. If $\mathrm{rpoly}(1, x) \mid p * q$, then $\mathrm{rpoly}(1, x) \mid p$ or $\mathrm{rpoly}(1, x) \mid q$.

(11) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non degenerated double loop structure and f be a finite sequence of elements

of Polynom-Ring L. Suppose that for every natural number i such that $i \in \mathrm{dom}\, f$ there exists an element z of L such that $f(i) = \mathrm{rpoly}(1, z)$. Let p be a polynomial of L. If $p = \prod f$, then $p \neq \mathbf{0}.\, L$.

(12) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible integral domain-like non degenerated double loop structure and f be a finite sequence of elements of Polynom-Ring L. Suppose that for every natural number i such that $i \in \mathrm{dom}\, f$ there exists an element z of L such that $f(i) = \mathrm{rpoly}(1, z)$. Let p be a polynomial of L. Suppose $p = \prod f$. Let x be an element of L. Then $\mathrm{eval}(p, x) = 0_L$ if and only if there exists a natural number i such that $i \in \mathrm{dom}\, f$ and $f(i) = \mathrm{rpoly}(1, x)$.

3. Common Roots of Polynomials

Let L be a unital non empty double loop structure, let p_1, p_2 be polynomials of L, and let x be an element of L. We say that x is a common root of p_1 and p_2 if and only if:

(Def. 3) x is a root of p_1 and x is a root of p_2.

Let L be a unital non empty double loop structure and let p_1, p_2 be polynomials of L. We say that p_1 and p_2 have a common root if and only if:

(Def. 4) There exists an element of L which is a common root of p_1 and p_2.

Let L be a unital non empty double loop structure and let p_1, p_2 be polynomials of L. We introduce p_1 and p_2 have common roots as a synonym of p_1 and p_2 have a common root. We introduce p_1 and p_2 have no common roots as an antonym of p_1 and p_2 have a common root.

Next we state several propositions:

(13) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p be a polynomial of L, and x be an element of L. If x is a root of p, then x is a root of $-p$.

(14) Let L be an Abelian add-associative right zeroed right complementable unital left distributive non empty double loop structure, p_1, p_2 be polynomials of L, and x be an element of L. If x is a common root of p_1 and p_2, then x is a root of $p_1 + p_2$.

(15) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p_1, p_2 be polynomials of L, and x be an element of L. If x is a common root of p_1 and p_2, then x is a root of $-(p_1 + p_2)$.

(16) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p, q be polynomials

of L, and x be an element of L. If x is a common root of p and q, then x is a root of $p+q$.

(17) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non trivial double loop structure and p_1, p_2 be polynomials of L. If $p_1 \mid p_2$ and p_1 has roots, then p_1 and p_2 have common roots.

Let L be a unital non empty double loop structure and let p, q be polynomials of L. The common roots of p and q yields a subset of L and is defined by:

(Def. 5) The common roots of p and $q = \{x \in L$: x is a common root of p and $q\}$.

4. Normalized Polynomials

Let L be a non empty zero structure and let p be a polynomial of L. The leading coefficient of p yields an element of L and is defined by:

(Def. 6) The leading coefficient of $p = p(\operatorname{len} p -' 1)$.

We introduce $\mathrm{LC}\, p$ as a synonym of the leading coefficient of p.

Let L be a non trivial double loop structure and let p be a non zero polynomial of L. One can check that $\mathrm{LC}\, p$ is non zero.

One can prove the following proposition

(18) Let L be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non empty double loop structure, p be a polynomial of L, and a be an element of L. Then $\mathrm{LC}(a \cdot p) = a \cdot \mathrm{LC}\, p$.

Let L be a non empty double loop structure and let p be a polynomial of L. We say that p is normalized if and only if:

(Def. 7) $\mathrm{LC}\, p = 1_L$.

Let L be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure and let p be a non zero polynomial of L. One can check that $\frac{1_L}{\mathrm{LC}\, p} \cdot p$ is normalized.

Let L be a field and let p be a non zero polynomial of L. One can verify that $\mathrm{NormPolynomial}\, p$ is normalized.

5. Rational Functions

Let L be a non trivial multiplicative loop with zero structure. Rational function of L is defined by:

(Def. 8) There exists a polynomial p_1 of L and there exists a non zero polynomial p_2 of L such that it $= \langle p_1, p_2 \rangle$.

Let L be a non trivial multiplicative loop with zero structure, let p_1 be a polynomial of L, and let p_2 be a non zero polynomial of L. Then $\langle p_1, p_2 \rangle$ is a rational function of L.

Let L be a non trivial multiplicative loop with zero structure and let z be a rational function of L. Then $z_{\mathbf{1}}$ is a polynomial of L. Then $z_{\mathbf{2}}$ is a non zero polynomial of L.

Let L be a non trivial multiplicative loop with zero structure and let z be a rational function of L. We say that z is zero if and only if:

(Def. 9) $z_{\mathbf{1}} = \mathbf{0}.\,L$.

Let L be a non trivial multiplicative loop with zero structure. One can check that there exists a rational function of L which is non zero.

Next we state the proposition

(19) Let L be a non trivial multiplicative loop with zero structure and z be a rational function of L. Then $z = \langle z_{\mathbf{1}}, z_{\mathbf{2}} \rangle$.

Let L be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let z be a rational function of L. We say that z is irreducible if and only if:

(Def. 10) $z_{\mathbf{1}}$ and $z_{\mathbf{2}}$ have no common roots.

Let L be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let z be a rational function of L. We introduce z is reducible as an antonym of z is irreducible.

Let L be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let z be a rational function of L. We say that z is normalized if and only if:

(Def. 11) z is irreducible and $z_{\mathbf{2}}$ is normalized.

Let L be an add-associative right zeroed right complementable distributive unital non trivial double loop structure. Observe that every rational function of L which is normalized is also irreducible.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L. Note that $\mathrm{LC}(z_{\mathbf{2}})$ is non zero.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L. The norm rational function of z yields a rational function of L and is defined by:

(Def. 12) The norm rational function of $z = \langle \frac{1_L}{\mathrm{LC}(z_{\mathbf{2}})} \cdot z_{\mathbf{1}}, \frac{1_L}{\mathrm{LC}(z_{\mathbf{2}})} \cdot z_{\mathbf{2}} \rangle$.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L. We introduce NormRatF z as a synonym of the norm rational function of z.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a non zero rational function of L. Observe that the norm rational function of z is non zero.

Let L be a non degenerated multiplicative loop with zero structure. The functor $0.L$ yields a rational function of L and is defined by:

(Def. 13) $0.L = \langle \mathbf{0}.L, \mathbf{1}.L \rangle$.

The functor $1.L$ yields a rational function of L and is defined as follows:

(Def. 14) $1.L = \langle \mathbf{1}.L, \mathbf{1}.L \rangle$.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can check that $0.L$ is normalized.

Let L be a non degenerated multiplicative loop with zero structure. Note that $1.L$ is non zero.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can verify that $1.L$ is irreducible.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. Observe that there exists a rational function of L which is irreducible and non zero.

Let L be an add-associative right zeroed right complementable distributive Abelian associative well unital non degenerated double loop structure and let x be an element of L. One can check that $\langle \mathrm{rpoly}(1, x), \mathrm{rpoly}(1, x) \rangle$ is reducible and non zero as a rational function of L.

Let L be an add-associative right zeroed right complementable distributive Abelian associative well unital non degenerated double loop structure. Observe that there exists a rational function of L which is reducible and non zero.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can verify that there exists a rational function of L which is normalized.

Let L be a non degenerated multiplicative loop with zero structure. One can verify that $0.L$ is zero.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can check that $1.L$ is normalized.

Let L be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let p, q be rational functions of L. The functor $p + q$ yields a rational function of L and is defined by:

(Def. 15) $p + q = \langle \mathbf{p_1} * \mathbf{q_2} + \mathbf{p_2} * \mathbf{q_1}, \mathbf{p_2} * \mathbf{q_2} \rangle$.

Let L be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let p, q be rational functions of L. The functor $p * q$ yielding a rational function of L is defined by:

(Def. 16) $p * q = \langle \mathbf{p_1} * \mathbf{q_1}, \mathbf{p_2} * \mathbf{q_2} \rangle$.

One can prove the following proposition

(20) Let L be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure, p be a rational function of L, and a be a non zero element of L. Then $\langle a \cdot \mathbf{p_1}, a \cdot \mathbf{p_2} \rangle$ is irreducible if and only if p is irreducible.

6. Normalized Rational Functions

We now state the proposition

(21) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative integral domain-like non trivial double loop structure and z be a rational function of L. Then there exists a rational function z_1 of L and there exists a non zero polynomial z_2 of L such that

(i) $z = \langle z_2 * (z_1)_{\mathbf{1}}, z_2 * (z_1)_{\mathbf{2}} \rangle$,

(ii) z_1 is irreducible, and

(iii) there exists a finite sequence f of elements of Polynom-Ring L such that $z_2 = \prod f$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} f$ there exists an element x of L such that x is a common root of $\mathbf{z_1}$ and $\mathbf{z_2}$ and $f(i) = \operatorname{rpoly}(1, x)$.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L. The functor $\operatorname{NF} z$ yielding a rational function of L is defined by:

(Def. 17)(i) There exists a rational function z_1 of L and there exists a non zero polynomial z_2 of L such that $z = \langle z_2 * (z_1)_{\mathbf{1}}, z_2 * (z_1)_{\mathbf{2}} \rangle$ and z_1 is irreducible and $\operatorname{NF} z =$ the norm rational function of z_1 and there exists a finite sequence f of elements of Polynom-Ring L such that $z_2 = \prod f$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} f$ there exists an element x of L such that x is a common root of $\mathbf{z_1}$ and $\mathbf{z_2}$ and $f(i) = \operatorname{rpoly}(1, x)$ if z is non zero,

(ii) $\operatorname{NF} z = 0. L$, otherwise.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L. Observe that NF z is normalized and irreducible.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a non zero rational function of L. One can verify that NF z is non zero.

One can prove the following propositions:

(22) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure, z be a non zero rational function of L, z_1 be a rational function of L, and z_2 be a non zero polynomial of L. Suppose that

(i) $z = \langle z_2 * (z_1)_{\mathbf{1}}, z_2 * (z_1)_{\mathbf{2}} \rangle$,

(ii) z_1 is irreducible, and

(iii) there exists a finite sequence f of elements of Polynom-Ring L such that $z_2 = \prod f$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} f$ there exists an element x of L such that x is a common root of $z_{\mathbf{1}}$ and $z_{\mathbf{2}}$ and $f(i) = \operatorname{rpoly}(1, x)$.

Then NF z = the norm rational function of z_1.

(23) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure. Then NF $0. L = 0. L$.

(24) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure. Then NF $1. L = 1. L$.

(25) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and z be an irreducible non zero rational function of L. Then NF z = the norm rational function of z.

(26) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure, z be a rational function of L, and x be an element of L. Then NF $\langle \operatorname{rpoly}(1, x) * z_{\mathbf{1}}, \operatorname{rpoly}(1, x) * z_{\mathbf{2}} \rangle$ = NF z.

(27) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and z be a rational function of L. Then NF NF z = NF z.

(28) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible in-

tegral domain-like non degenerated double loop structure and z be a non zero rational function of L. Then z is irreducible if and only if there exists an element a of L such that $a \neq 0_L$ and $\langle a \cdot z_{\mathbf{1}},\, a \cdot z_{\mathbf{2}} \rangle = \mathrm{NF}\, z$.

7. Degree of Rational Functions

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L. The functor degree(z) yielding an integer is defined as follows:

(Def. 18) $\mathrm{degree}(z) = \max(\mathrm{degree}((\mathrm{NF}\, z)_{\mathbf{1}}), \mathrm{degree}((\mathrm{NF}\, z)_{\mathbf{2}}))$.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L. We introduce $\deg z$ as a synonym of degree(z).

Next we state two propositions:

(29) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and z be a rational function of L. Then $\mathrm{degree}(z) \leq \max(\mathrm{degree}(z_{\mathbf{1}}), \mathrm{degree}(z_{\mathbf{2}}))$.

(30) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and z be a non zero rational function of L. Then z is irreducible if and only if $\mathrm{degree}(z) = \max(\mathrm{degree}(z_{\mathbf{1}}), \mathrm{degree}(z_{\mathbf{2}}))$.

8. Evaluation of Rational Functions

Let L be a field, let z be a rational function of L, and let x be an element of L. The functor eval(z, x) yielding an element of L is defined by:

(Def. 19) $\mathrm{eval}(z, x) = \frac{\mathrm{eval}(z_{\mathbf{1}}, x)}{\mathrm{eval}(z_{\mathbf{2}}, x)}$.

The following propositions are true:

(31) For every field L and for every element x of L holds $\mathrm{eval}(0.\, L, x) = 0_L$.

(32) For every field L and for every element x of L holds $\mathrm{eval}(1.\, L, x) = 1_L$.

(33) Let L be a field, p, q be rational functions of L, and x be an element of L. If $\mathrm{eval}(p_{\mathbf{2}}, x) \neq 0_L$ and $\mathrm{eval}(q_{\mathbf{2}}, x) \neq 0_L$, then $\mathrm{eval}(p + q, x) = \mathrm{eval}(p, x) + \mathrm{eval}(q, x)$.

(34) Let L be a field, p, q be rational functions of L, and x be an element of L. If $\mathrm{eval}(p_{\mathbf{2}}, x) \neq 0_L$ and $\mathrm{eval}(q_{\mathbf{2}}, x) \neq 0_L$, then $\mathrm{eval}(p * q, x) = \mathrm{eval}(p, x) \cdot \mathrm{eval}(q, x)$.

(35) Let L be a field, p be a rational function of L, and x be an element of L. If $\mathrm{eval}(p_{\mathbf{2}}, x) \neq 0_L$, then eval(the norm rational function of p, x) $= \mathrm{eval}(p, x)$.

(36) Let L be a field, p be a rational function of L, and x be an element of L. If $\mathrm{eval}(p_{\mathbf{2}}, x) \neq 0_L$, then x is a common root of $p_{\mathbf{1}}$ and $p_{\mathbf{2}}$ or $\mathrm{eval}(\mathrm{NF}\, p, x) = \mathrm{eval}(p, x)$.

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