FORMALIZED MATHEMATICS
Vol. 20, No. 2, Pages 161–174, 2012
DOI: 10.2478/v10037-012-0019-8

VERSITA
versita.com/fm/

Simple Graphs as Simplicial Complexes: the Mycielskian of a Graph[1]

Piotr Rudnicki
University of Alberta
Edmonton, Canada

Lorna Stewart
University of Alberta
Edmonton, Canada

Summary. Harary [10, p. 7] claims that Veblen [20, p. 2] first suggested to formalize simple graphs using simplicial complexes. We have developed basic terminology for simple graphs as at most 1-dimensional complexes.

We formalize this new setting and then reprove Mycielski's [12] construction resulting in a triangle-free graph with arbitrarily large chromatic number. A different formalization of similar material is in [15].

MML identifier: `SCMYCIEL`, version: `7.12.02 4.181.1147`

The papers [5], [1], [4], [16], [14], [6], [9], [18], [7], [15], [2], [11], [3], [17], [13], [19], and [8] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For all sets x, X holds $\langle x, X\rangle \notin X$.

(2) For all sets x, X holds $\langle x, X\rangle \neq X$.

(3) For all sets x, X holds $\langle x, X\rangle \neq x$.

(4) For all sets x_1, y_1, x_2, y_2, X such that x_1, $x_2 \in X$ and $\{x_1, \langle y_1, X\rangle\} = \{x_2, \langle y_2, X\rangle\}$ holds $x_1 = x_2$ and $y_1 = y_2$.

(5) For all sets X, v such that $3 \subseteq \overline{\overline{X}}$ there exist sets v_1, v_2 such that v_1, $v_2 \in X$ and $v_1 \neq v$ and $v_2 \neq v$ and $v_1 \neq v_2$.

(6) For every set x holds $S_{\{x\}} = \{\{x\}\}$.

[1]This work has been partially supported by the NSERC grant OGP 9207.

ISSN 1426–2630(p), 1898-9934(e)

Let us observe that there exists a finite sequence which is finite-yielding.

The following proposition is true

(7) Let X be a non empty finite set and P be a partition of X. If $\overline{\overline{P}} < \overline{\overline{X}}$, then there exist sets p, x, y such that $p \in P$ and x, $y \in p$ and $x \neq y$.

Let us note that $\bigcup\{\emptyset\}$ is empty.

Next we state three propositions:

(8) For every set x holds $\bigcup\{\emptyset, \{x\}\} = \{x\}$.

(9) For every set X and for every subset s of X such that s is 1-element there exists a set x such that $x \in X$ and $s = \{x\}$.

(10) For every set X holds $\overline{\overline{\{\{X, \langle x,\, X\rangle\}; x \text{ ranges over elements of } X\colon x \in X\}}} = \overline{\overline{X}}$.

Let G be a set. The functor $\mathrm{PairsOf}\, G$ yielding a subset of G is defined as follows:

(Def. 1) For every set e holds $e \in \mathrm{PairsOf}\, G$ iff $e \in G$ and $\overline{\overline{e}} = 2$.

The following propositions are true:

(11) For every set X and for every set e such that $e \in \mathrm{PairsOf}\, X$ there exist sets x, y such that $x \neq y$ and x, $y \in \bigcup X$ and $e = \{x, y\}$.

(12) For all sets X, x, y such that $x \neq y$ and $\{x, y\} \in X$ holds $\{x, y\} \in \mathrm{PairsOf}\, X$.

(13) For all sets X, x, y such that $\{x, y\} \in \mathrm{PairsOf}\, X$ holds $x \neq y$ and x, $y \in \bigcup X$.

(14) For all sets G, H such that $G \subseteq H$ holds $\mathrm{PairsOf}\, G \subseteq \mathrm{PairsOf}\, H$.

(15) For every finite set X holds $\overline{\overline{\{\{x, \langle y, \bigcup X\rangle\}; x \text{ ranges over elements of } \bigcup X, y \text{ ranges over elements of } \bigcup X : \{x, y\} \in \mathrm{PairsOf}\, X\}}} = 2 \cdot \overline{\overline{\mathrm{PairsOf}\, X}}$.

(16) For every finite set X holds $\overline{\overline{\{\langle x,\, y\rangle; x \text{ ranges over elements of } \bigcup X, y \text{ ranges over elements of } \bigcup X : \{x, y\} \in \mathrm{PairsOf}\, X\}}} = 2 \cdot \overline{\overline{\mathrm{PairsOf}\, X}}$.

Let X be a finite set. Note that $\mathrm{PairsOf}\, X$ is finite.

Let X be a set. We say that X is void if and only if:

(Def. 2) $X = \{\emptyset\}$.

One can verify that there exists a set which is void.

Let us observe that every set which is void is also finite.

Let G be a void set. Observe that $\bigcup G$ is empty.

Next we state two propositions:

(17) For every set X such that X is void holds $\mathrm{PairsOf}\, X = \emptyset$.

(18) For every set X such that $\bigcup X = \emptyset$ holds $X = \emptyset$ or $X = \{\emptyset\}$.

Let X be a set. We say that X is pair free if and only if:

(Def. 3) PairsOf X is empty.

We now state the proposition

(19) For all sets X, x such that $\overline{\overline{\bigcup X}} = 1$ holds X is pair free.

Let us observe that there exists a set which is finite-membered and non empty.

Let X be a finite-membered set and let Y be a set. Observe that $X \cap Y$ is finite-membered and $X \setminus Y$ is finite-membered.

2. Simple Graphs as Simplicial Complexes

Let n be a natural number and let X be a set. We say that X is at most n-dimensional if and only if:

(Def. 4) For every set x such that $x \in X$ holds $\overline{\overline{x}} \subseteq n + 1$.

Let n be a natural number. Observe that every set which is at most n-dimensional is also finite-membered.

Let n be a natural number. Observe that there exists a set which is at most n-dimensional, subset-closed, and non empty.

Next we state two propositions:

(20) For every subset-closed non empty set G holds $\emptyset \in G$.

(21) Let n be a natural number, X be an at most n-dimensional set, and x be an element of X. If $x \in X$, then $\overline{\overline{x}} \leq n + 1$.

Let n be a natural number and let X, Y be at most n-dimensional sets. Note that $X \cup Y$ is at most n-dimensional.

Let n be a natural number, let X be an at most n-dimensional set, and let Y be a set. Note that $X \cap Y$ is at most n-dimensional and $X \setminus Y$ is at most n-dimensional.

Let n be a natural number and let X be an at most n-dimensional set. Observe that every at most n-dimensional set is at most n-dimensional.

Let s be a set. We say that s is simple graph-like if and only if:

(Def. 5) s is at most 1-dimensional, subset-closed, and non empty.

Let us note that every set which is simple graph-like is also at most 1-dimensional, subset-closed, and non empty and every set which is at most 1-dimensional, subset-closed, and non empty is also simple graph-like.

The following proposition is true

(22) $\{\emptyset\}$ is simple graph-like.

One can verify that $\{\emptyset\}$ is simple graph-like.

One can verify that there exists a set which is simple graph-like.

A simple graph is a simple graph-like set.

One can verify that there exists a simple graph which is void and there exists a simple graph which is finite.

Let G be a set. We introduce Vertices G as a synonym of $\bigcup G$. We introduce Edges G as a synonym of PairsOf G.

Let X be a set. We introduce X is edgeless as a synonym of X is pair free.

We now state three propositions:

(23) For every simple graph G such that Vertices G is finite holds G is finite.

(24) For every simple graph G and for every set x holds $x \in$ Vertices G iff $\{x\} \in G$.

(25) For every set x holds $\{\emptyset, \{x\}\}$ is a simple graph.

Let X be a finite finite-membered set. The functor order X yielding a natural number is defined by:

(Def. 6) order $X = \overline{\overline{\bigcup X}}$.

Let X be a finite set. The functor size X yielding a natural number is defined by:

(Def. 7) size $X = \overline{\overline{\text{PairsOf } X}}$.

Next we state the proposition

(26) For every finite simple graph G holds order $G \leq \overline{\overline{G}}$.

Let G be a simple graph. A vertex of G is an element of Vertices G. An edge of G is an element of Edges G.

The following propositions are true:

(27) For every simple graph G holds $G = \{\emptyset\} \cup S_{(\text{Vertices } G)} \cup \text{Edges } G$.

(28) For every simple graph G such that Vertices $G = \emptyset$ holds G is void.

(29) Let G be a simple graph and x be a set. If $x \in G$ and $x \neq \emptyset$, then there exists a set y such that $x = \{y\}$ and $y \in$ Vertices G or $x \in$ Edges G.

(30) For every simple graph G and for every set x such that Vertices $G = \{x\}$ holds $G = \{\emptyset, \{x\}\}$.

(31) For every set X there exists a simple graph G such that G is edgeless and Vertices $G = X$.

Let G be a simple graph and let v be an element of Vertices G. The functor Adjacent(v) yielding a subset of Vertices G is defined by:

(Def. 8) For every element x of Vertices G holds $x \in$ Adjacent(v) iff $\{v, x\} \in$ Edges G.

Let X be a set. A simple graph is called a simple graph of X if:

(Def. 9) Vertices it $= X$.

Let X be a set. The functor CompleteSGraph X yields a simple graph of X and is defined by:

(Def. 10) CompleteSGraph $X = \{V; V$ ranges over finite subsets of X: $\overline{\overline{V}} \leq 2\}$.

One can prove the following proposition

(32) For every simple graph G such that for all sets x, y such that $x, y \in \mathrm{Vertices}\, G$ holds $\{x, y\} \in G$ holds $G = \mathrm{CompleteSGraph}\, \mathrm{Vertices}\, G$.

Let X be a finite set. One can check that CompleteSGraph X is finite.

The following propositions are true:

(33) For every set X and for every set x such that $x \in X$ holds $\{x\} \in \mathrm{CompleteSGraph}\, X$.

(34) For every set X and for all sets x, y such that $x, y \in X$ holds $\{x, y\} \in \mathrm{CompleteSGraph}\, X$.

(35) $\mathrm{CompleteSGraph}\, \emptyset = \{\emptyset\}$.

(36) For every set x holds $\mathrm{CompleteSGraph}\{x\} = \{\emptyset, \{x\}\}$.

(37) For all sets x, y holds $\mathrm{CompleteSGraph}\{x, y\} = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.

(38) For all sets X, Y such that $X \subseteq Y$ holds $\mathrm{CompleteSGraph}\, X \subseteq \mathrm{CompleteSGraph}\, Y$.

(39) For every simple graph G and for every set x such that $x \in \mathrm{Vertices}\, G$ holds $\mathrm{CompleteSGraph}\{x\} \subseteq G$.

Let G be a simple graph. One can check that there exists a subset of G which is simple graph-like.

Let G be a simple graph. A subgraph of G is a simple graph-like subset of G.

Let G be a simple graph. The functor Complement G yields a simple graph and is defined as follows:

(Def. 11) $\mathrm{Complement}\, G = \mathrm{CompleteSGraph}\, \mathrm{Vertices}\, G \setminus \mathrm{Edges}\, G$.

Let us observe that the functor Complement G is involutive.

Next we state two propositions:

(40) For every simple graph G holds $\mathrm{Vertices}\, G = \mathrm{Vertices}\, \mathrm{Complement}\, G$.

(41) Let G be a simple graph and x, y be sets. If $x \neq y$ and $x, y \in \mathrm{Vertices}\, G$, then $\{x, y\} \in \mathrm{Edges}\, G$ iff $\{x, y\} \notin \mathrm{Edges}\, \mathrm{Complement}\, G$.

3. Induced Subgraphs

Let G be a simple graph and let L be a set. The subgraph induced by G yielding a subset of G is defined by:

(Def. 12) The subgraph induced by $G = G \cap 2^L$.

Let G be a simple graph and let L be a set. Observe that the subgraph induced by G is simple graph-like.

Next we state two propositions:

(42) For every simple graph G holds $G =$ the subgraph induced by G.

(43) For every simple graph G and for every set L holds the subgraph induced by $G =$ the subgraph induced by G.

Let G be a finite simple graph and let L be a set. Observe that the subgraph induced by G is finite.

Let G be a simple graph and let L be a finite set. One can check that the subgraph induced by G is finite.

One can prove the following three propositions:

(44) For all simple graphs G, H such that $G \subseteq H$ holds $G \subseteq$ the subgraph induced by H.

(45) For every simple graph G and for every set L holds Vertices (the subgraph induced by G) $=$ Vertices $G \cap L$.

(46) For every simple graph G and for every set x such that $x \in$ Vertices G holds the subgraph induced by $G = \{\emptyset, \{x\}\}$.

4. Clique, Clique Number, Clique Cover

Let G be a simple graph. We say that G is a clique if and only if:

(Def. 13) $G =$ CompleteSGraph Vertices G.

The following propositions are true:

(47) Let G be a simple graph. Suppose that for all sets x, y such that $x \neq y$ and x, $y \in$ Vertices G holds $\{x, y\} \in$ Edges G. Then G is a clique.

(48) $\{\emptyset\}$ is a clique.

Observe that there exists a simple graph which is a clique. Let G be a simple graph. Note that there exists a subgraph of G which is a clique.

Let G be a simple graph. A clique of G is a clique subgraph of G.

Next we state the proposition

(49) For every set X holds CompleteSGraph X is a clique.

Let X be a set. One can check that CompleteSGraph X is a clique.

Next we state two propositions:

(50) For every simple graph G and for every set x such that $x \in$ Vertices G holds $\{\emptyset, \{x\}\}$ is a clique of G.

(51) Let G be a simple graph and x, y be sets. If x, $y \in$ Vertices G and $\{x, y\} \in G$, then $\{\emptyset, \{x\}, \{y\}, \{x, y\}\}$ is a clique of G.

Let G be a simple graph. Observe that there exists a clique of G which is finite.

We now state two propositions:

(52) For every simple graph G and for every set x such that $x \in \bigcup G$ there exists a finite clique C of G such that Vertices $C = \{x\}$.

(53) For every a clique simple graph C and for all sets u, v such that u, $v \in$ Vertices C holds $\{u, v\} \in C$.

Let G be a simple graph. We say that G has finite clique number if and only if:

(Def. 14) There exists a finite clique C of G such that for every finite clique D of G holds $\operatorname{order} D \leq \operatorname{order} C$.

Let us note that there exists a simple graph which has finite clique number.

Let us observe that every simple graph which is finite also has finite clique number.

Let G be a simple graph with finite clique number. The functor $\omega(G)$ yielding a natural number is defined as follows:

(Def. 15) There exists a finite clique C of G such that $\operatorname{order} C = \omega(G)$ and for every finite clique T of G holds $\operatorname{order} T \leq \omega(G)$.

We now state several propositions:

(54) For every simple graph G with finite clique number such that $\omega(G) = 0$ holds $\operatorname{Vertices} G = \emptyset$.

(55) For every void simple graph G holds $\omega(G) = 0$.

(56) Let G be a simple graph and x, y be sets. If $\{x, y\} \in G$, then the subgraph induced by G is a clique of G.

(57) For every simple graph G with finite clique number such that $\operatorname{Edges} G \neq \emptyset$ holds $\omega(G) \geq 2$.

(58) For all simple graphs G, H with finite clique number such that $G \subseteq H$ holds $\omega(G) \leq \omega(H)$.

(59) For every finite set X holds $\omega(\operatorname{CompleteSGraph} X) = \overline{\overline{X}}$.

Let G be a simple graph and let P be a partition of $\operatorname{Vertices} G$. We say that P is clique-wise if and only if:

(Def. 16) For every set x such that $x \in P$ holds the subgraph induced by G is a clique of G.

Let G be a simple graph. Observe that there exists a partition of $\operatorname{Vertices} G$ which is clique-wise.

Let G be a simple graph. A clique-partition of G is a clique-wise partition of $\operatorname{Vertices} G$.

Let G be a void simple graph. Note that every partition of $\operatorname{Vertices} G$ which is empty is also clique-wise.

Let G be a simple graph. We say that G has finite clique cover if and only if:

(Def. 17) There exists a clique-partition of G which is finite.

One can verify that every simple graph which is finite also has finite clique cover.

Let G be a simple graph with finite clique cover. Note that there exists a clique-partition of G which is finite.

Let G be a simple graph with finite clique cover and let S be a subset of Vertices G. One can verify that the subgraph induced by G has finite clique cover.

Let G be a simple graph with finite clique cover. The functor $\kappa(G)$ yielding a natural number is defined by:

(Def. 18) There exists a finite clique-partition C of G such that $\overline{\overline{C}} = \kappa(G)$ and for every finite clique-partition C of G holds $\kappa(G) \leq \overline{\overline{C}}$.

5. Stable Set, Coloring

Let G be a simple graph and let S be a subset of Vertices G. We say that S is stable if and only if:

(Def. 19) For all sets x, y such that $x \neq y$ and x, $y \in S$ holds $\{x, y\} \notin G$.

We now state two propositions:

(60) For every simple graph G holds $\emptyset_{\mathrm{Vertices}\, G}$ is stable.

(61) For every simple graph G and for every subset S of Vertices G and for every set v such that $S = \{v\}$ holds S is stable.

Let G be a simple graph. Observe that every subset of Vertices G which is trivial is also stable.

Let G be a simple graph. Note that there exists a subset of Vertices G which is stable.

Let G be a simple graph. A stable set of G is a stable subset of Vertices G.

The following two propositions are true:

(62) For every simple graph G and for all sets x, y such that x, $y \in$ Vertices G and $\{x, y\} \notin G$ holds $\{x, y\}$ is a stable set of G.

(63) For every simple graph G with finite clique number such that $\omega(G) = 1$ holds Vertices G is a stable set of G.

Let G be a simple graph. Note that there exists a stable set of G which is finite.

One can prove the following proposition

(64) For every simple graph G and for every stable set A of G holds every subset of A is a stable set of G.

Let G be a simple graph and let P be a partition of Vertices G. We say that P is stable-wise if and only if:

(Def. 20) For every set x such that $x \in P$ holds x is a stable set of G.

The following proposition is true

(65) For every simple graph G holds SmallestPartition(Vertices G) is stable-wise.

Let G be a simple graph. Note that there exists a partition of Vertices G which is stable-wise. A coloring of G is a stable-wise partition of Vertices G. We say that G is finitely colorable if and only if:

(Def. 21) There exists a coloring of G which is finite.

One can verify that there exists a simple graph which is finitely colorable.

Let us note that every simple graph which is finite is also finitely colorable.

Let G be a finitely colorable simple graph. Note that there exists a coloring of G which is finite.

We now state two propositions:

(66) Let G be a simple graph, S be a clique of G, and L be a set. If $L \subseteq$ Vertices S, then the subgraph induced by G is a clique of G.

(67) Let G be a simple graph, C be a coloring of G, and S be a subset of Vertices G. Then $C{\restriction}S$ is a coloring of the subgraph induced by G.

Let G be a finitely colorable simple graph and let S be a set. One can check that the subgraph induced by G is finitely colorable. The functor $\chi(G)$ yielding a natural number is defined as follows:

(Def. 22) There exists a finite coloring C of G such that $\overline{\overline{C}} = \chi(G)$ and for every finite coloring C of G holds $\chi(G) \leq \overline{\overline{C}}$.

One can prove the following three propositions:

(68) For all finitely colorable simple graphs G, H such that $G \subseteq H$ holds $\chi(G) \leq \chi(H)$.

(69) For every finite set X holds $\chi(\mathrm{CompleteSGraph}\,X) = \overline{\overline{X}}$.

(70) Let G be a finitely colorable simple graph, C be a finite coloring of G, and c be a set. Suppose $c \in C$ and $\overline{\overline{C}} = \chi(G)$. Then there exists an element v of Vertices G such that $v \in c$ and for every element d of C such that $d \neq c$ there exists an element w of Vertices G such that $w \in \mathrm{Adjacent}(v)$ and $w \in d$.

Let G be a simple graph. We say that G has finite stability number if and only if:

(Def. 23) There exists a finite stable set A of G such that for every finite stable set B of G holds $\overline{\overline{B}} \leq \overline{\overline{A}}$.

One can check that every simple graph which is finite also has finite stability number.

Let G be a simple graph with finite stability number. Observe that every stable set of G is finite.

Let us note that there exists a simple graph which is non void and has finite stability number.

Let G be a simple graph with finite stability number. The functor $\alpha(G)$ yielding a natural number is defined as follows:

(Def. 24) There exists a finite stable set A of G such that $\overline{\overline{A}} = \alpha(G)$ and for every finite stable set T of G holds $\overline{\overline{T}} \leq \alpha(G)$.

Let G be a non void simple graph with finite stability number. One can check that $\alpha(G)$ is positive.

Next we state the proposition

(71) For every simple graph G with finite stability number such that $\alpha(G) = 1$ holds G is a clique.

Let us observe that every simple graph which has finite clique number and finite stability number is also finite.

We now state four propositions:

(72) For every simple graph G and for every clique C of G holds Vertices C is a stable set of Complement G.

(73) For every simple graph G and for every clique C of Complement G holds Vertices C is a stable set of G.

(74) For every simple graph G and for every stable set C of G holds the subgraph induced by Complement G is a clique of Complement G.

(75) For every simple graph G and for every stable set C of Complement G holds the subgraph induced by G is a clique of G.

Let G be a simple graph with finite clique number. One can check that Complement G has finite stability number.

Let G be a simple graph with finite stability number. Note that Complement G has finite clique number.

We now state several propositions:

(76) For every simple graph G with finite clique number holds $\omega(G) = \alpha(\text{Complement}\,G)$.

(77) For every simple graph G with finite stability number holds $\alpha(G) = \omega(\text{Complement}\,G)$.

(78) For every simple graph G holds every clique-partition of Complement G is a coloring of G.

(79) For every simple graph G holds every clique-partition of G is a coloring of Complement G.

(80) For every simple graph G holds every coloring of G is a clique-partition of Complement G.

(81) For every simple graph G holds every coloring of Complement G is a clique-partition of G.

Let G be a finitely colorable simple graph. One can check that Complement G has finite clique cover.

Let G be a simple graph with finite clique cover.

One can check that Complement G is finitely colorable.

One can prove the following propositions:

(82) For every finitely colorable simple graph G holds $\chi(G) = \kappa(\mathrm{Complement}\, G)$.

(83) For every simple graph G with finite clique cover holds $\kappa(G) = \chi(\mathrm{Complement}\, G)$.

6. Mycielskian of a Graph

Let G be a simple graph. The functor Mycielskian G yielding a simple graph is defined by the condition (Def. 25).

(Def. 25) $\mathrm{Mycielskian}\, G = \{\emptyset\} \cup \{\{x\} : x$ ranges over elements of $\bigcup G \cup \bigcup G \times \{\bigcup G\} \cup \{\bigcup G\}\} \cup \mathrm{Edges}\, G \cup \{\{x, \langle y, \bigcup G\rangle\}; x$ ranges over elements of $\bigcup G, y$ ranges over elements of $\bigcup G : \{x, y\} \in \mathrm{Edges}\, G\} \cup \{\{\bigcup G, \langle x, \bigcup G\rangle\}; x$ ranges over elements of $\bigcup G : x \in \mathrm{Vertices}\, G\}$.

We now state several propositions:

(84) For every simple graph G holds $G \subseteq \mathrm{Mycielskian}\, G$.

(85) Let G be a simple graph and v be a set. Then $v \in \mathrm{Vertices}\,\mathrm{Mycielskian}\, G$ if and only if one of the following conditions is satisfied:
- (i) $v \in \bigcup G$, or
- (ii) there exists a set x such that $x \in \bigcup G$ and $v = \langle x, \bigcup G\rangle$, or
- (iii) $v = \bigcup G$.

(86) For every simple graph G holds $\mathrm{Vertices}\,\mathrm{Mycielskian}\, G = \bigcup G \cup \bigcup G \times \{\bigcup G\} \cup \{\bigcup G\}$.

(87) For every simple graph G holds $\bigcup G \in \bigcup \mathrm{Mycielskian}\, G$.

(88) For every void simple graph G holds $\mathrm{Mycielskian}\, G = \{\emptyset, \{\bigcup G\}\}$.

Let G be a finite simple graph. Note that Mycielskian G is finite.

The following propositions are true:

(89) For every finite simple graph G holds $\mathrm{order}\,\mathrm{Mycielskian}\, G = 2 \cdot \mathrm{order}\, G + 1$.

(90) Let G be a simple graph and e be a set. Then $e \in \mathrm{Edges}\,\mathrm{Mycielskian}\, G$ if and only if one of the following conditions is satisfied:
- (i) $e \in \mathrm{Edges}\, G$, or
- (ii) there exist elements x, y of $\bigcup G$ such that $e = \{x, \langle y, \bigcup G\rangle\}$ and $\{x, y\} \in \mathrm{Edges}\, G$, or
- (iii) there exists an element y of $\bigcup G$ such that $e = \{\bigcup G, \langle y, \bigcup G\rangle\}$ and $y \in \bigcup G$.

(91) Let G be a simple graph. Then $\mathrm{Edges}\,\mathrm{Mycielskian}\, G = \mathrm{Edges}\, G \cup \{\{x, \langle y, \bigcup G\rangle\}; x$ ranges over elements of $\bigcup G, y$ ranges over elements of $\bigcup G : \{x, y\} \in \mathrm{Edges}\, G\} \cup \{\{\bigcup G, \langle y, \bigcup G\rangle\}; y$ ranges over elements of $\bigcup G : y \in \bigcup G\}$.

(92) For every finite simple graph G holds $\operatorname{size}\operatorname{Mycielskian} G = 3 \cdot \operatorname{size} G + \operatorname{order} G$.

(93) Let G be a simple graph and s, t be sets. Suppose $\{s,t\} \in \operatorname{Edges}\operatorname{Mycielskian} G$. Then

(i) $\{s,t\} \in \operatorname{Edges} G$, or

(ii) $s \in \bigcup G$ or $s = \bigcup G$ but there exists a set y such that $y \in \bigcup G$ and $t = \langle y, \bigcup G\rangle$, or

(iii) $t \in \bigcup G$ or $t = \bigcup G$ but there exists a set y such that $y \in \bigcup G$ and $s = \langle y, \bigcup G\rangle$.

(94) For every simple graph G and for every set u such that $\{\bigcup G, u\} \in \operatorname{Edges}\operatorname{Mycielskian} G$ there exists a set x such that $x \in \bigcup G$ and $u = \langle x, \bigcup G\rangle$.

(95) For every simple graph G and for every set u such that $u \in \operatorname{Vertices} G$ holds $\{\langle u, \bigcup G\rangle\} \in \operatorname{Mycielskian} G$.

(96) For every simple graph G and for every set u such that $u \in \operatorname{Vertices} G$ holds $\{\langle u, \bigcup G\rangle, \bigcup G\} \in \operatorname{Mycielskian} G$.

(97) For every simple graph G and for all sets x, y holds $\{\langle x, \bigcup G\rangle, \langle y, \bigcup G\rangle\} \notin \operatorname{Edges}\operatorname{Mycielskian} G$.

(98) For every simple graph G and for all sets x, y such that $x \neq y$ holds $\{\langle x, \bigcup G\rangle, \langle y, \bigcup G\rangle\} \notin \operatorname{Mycielskian} G$.

(99) For every simple graph G and for all sets x, y such that $\{\langle x, \bigcup G\rangle, y\} \in \operatorname{Edges}\operatorname{Mycielskian} G$ holds $x \neq y$ but $x \in \bigcup G$ but $y \in \bigcup G$ or $y = \bigcup G$.

(100) For every simple graph G and for all sets x, y such that $\{\langle x, \bigcup G\rangle, y\} \in \operatorname{Mycielskian} G$ holds $x \neq y$.

(101) For every simple graph G and for all sets x, y such that $y \in \bigcup G$ and $\{\langle x, \bigcup G\rangle, y\} \in \operatorname{Mycielskian} G$ holds $\{x, y\} \in G$.

(102) For every simple graph G and for all sets x, y such that $\{x, y\} \in \operatorname{Edges} G$ holds $\{\langle x, \bigcup G\rangle, y\} \in \operatorname{Mycielskian} G$.

(103) For every simple graph G and for all sets x, y such that $x, y \in \operatorname{Vertices} G$ and $\{x, y\} \in \operatorname{Mycielskian} G$ holds $\{x, y\} \in G$.

(104) For every simple graph G holds $G =$ the subgraph induced by $\operatorname{Mycielskian} G$.

(105) Let G be a simple graph and C be a finite clique of $\operatorname{Mycielskian} G$. If $3 \leq \operatorname{order} C$, then for every vertex v of C holds $v \neq \bigcup G$.

(106) For every simple graph G with finite clique number such that $\omega(G) = 0$ and for every finite clique D of $\operatorname{Mycielskian} G$ holds $\operatorname{order} D \leq 1$.

(107) For every simple graph G and for every set x such that $\operatorname{Vertices} G = \{x\}$ holds $\operatorname{Mycielskian} G = \{\emptyset, \{x\}, \{\langle x, \bigcup G\rangle\}, \{\bigcup G\}, \{\langle x, \bigcup G\rangle, \bigcup G\}\}$.

(108) For every simple graph G with finite clique number such that $\omega(G) = 1$

and for every finite clique D of Mycielskian G holds order $D \leq 2$.

(109) For every simple graph G with finite clique number such that $2 \leq \omega(G)$ and for every finite clique D of Mycielskian G holds order $D \leq \omega(G)$.

Let G be a simple graph with finite clique number. Note that Mycielskian G has finite clique number.

We now state two propositions:

(110) For every simple graph G with finite clique number such that $2 \leq \omega(G)$ holds $\omega(\text{Mycielskian}\, G) = \omega(G)$.

(111) For every finitely colorable simple graph G there exists a coloring E of Mycielskian G such that $\overline{\overline{E}} = 1 + \chi(G)$.

Let G be a finitely colorable simple graph. Observe that Mycielskian G is finitely colorable.

We now state the proposition

(112) For every finitely colorable simple graph G holds $\chi(\text{Mycielskian}\, G) = 1 + \chi(G)$.

Let G be a simple graph. The Mycielskian sequence of G yields a many sorted set indexed by $\mathbb{N}$ and is defined by the condition (Def. 26).

(Def. 26) There exists a function m_1 such that

(i) the Mycielskian sequence of $G = m_1$,

(ii) $m_1(0) = G$, and

(iii) for every natural number k and for every simple graph G such that $G = m_1(k)$ holds $m_1(k+1) = \text{Mycielskian}\, G$.

We now state two propositions:

(113) For every simple graph G holds (the Mycielskian sequence of $G)(0) = G$.

(114) Let G be a simple graph and n be a natural number. Then (the Mycielskian sequence of $G)(n)$ is a simple graph.

Let G be a simple graph and let n be a natural number. Observe that (the Mycielskian sequence of $G)(n)$ is simple graph-like.

The following proposition is true

(115) Let G, H be simple graphs and n be a natural number. Then (the Mycielskian sequence of $G)(n+1) = \text{Mycielskian}$ (the Mycielskian sequence of $G)(n)$.

Let G be a simple graph with finite clique number and let n be a natural number. One can check that (the Mycielskian sequence of $G)(n)$ has finite clique number.

Let G be a finitely colorable simple graph and let n be a natural number. One can check that (the Mycielskian sequence of $G)(n)$ is finitely colorable.

Let G be a finite simple graph and let n be a natural number. Observe that (the Mycielskian sequence of $G)(n)$ is finite.

One can prove the following propositions:

(116) Let G be a finite simple graph and n be a natural number. Then order (the Mycielskian sequence of $G)(n) = (2^n \cdot \operatorname{order} G + 2^n) - 1$.

(117) Let G be a finite simple graph and n be a natural number. Then size (the Mycielskian sequence of $G)(n) = 3^n \cdot \operatorname{size} G + (3^n - 2^n) \cdot \operatorname{order} G + ((n + 1) \operatorname{block} 3)$.

(118) Let n be a natural number. Then ω((the Mycielskian sequence of CompleteSGraph 2)$(n)) = 2$ and χ((the Mycielskian sequence of CompleteSGraph 2)$(n)) = n + 2$.

(119) For every natural number n there exists a finite simple graph G such that $\omega(G) = 2$ and $\chi(G) > n$.

(120) For every natural number n there exists a finite simple graph G such that $\alpha(G) = 2$ and $\kappa(G) > n$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek. Tarski's classes and ranks. *Formalized Mathematics*, 1(**3**):563–567, 1990.
[5] Grzegorz Bancerek. Mizar analysis of algorithms: Preliminaries. *Formalized Mathematics*, 15(**3**):87–110, 2007, doi:10.2478/v10037-007-0011-x.
[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Frank Harary. *Graph theory*. Addison-Wesley, 1969.
[11] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.
[12] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.
[13] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[14] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[15] Piotr Rudnicki and Lorna Stewart. The Mycielskian of a graph. *Formalized Mathematics*, 19(**1**):27–34, 2011, doi: 10.2478/v10037-011-0005-6.
[16] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[17] Andrzej Trybulec. Many sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[18] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.
[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[20] Oswald Veblen. *Analysis Situs*, volume V. AMS Colloquium Publications, 1931.

Received February 7, 2012