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Continuity of Barycentric Coordinates in Euclidean Topological Spaces

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Summary. In this paper we present selected properties of barycentric coordinates in the Euclidean topological space. We prove the topological correspondence between a subset of an affine closed space of $\mathcal{E}^n$ and the set of vectors created from barycentric coordinates of points of this subset.

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The terminology and notation used here have been introduced in the following articles: [1], [3], [15], [25], [13], [18], [5], [4], [6], [12], [7], [8], [33], [21], [24], [2], [22], [20], [17], [30], [31], [23], [10], [28], [26], [11], [16], [29], [14], [19], [27], [32], and [9].

1. Preliminaries

For simplicity, we adopt the following rules: x denotes a set, n, m, k denote natural numbers, r denotes a real number, V denotes a real linear space, v, w denote vectors of V, A_1 denotes a finite subset of V, and A_2 denotes a finite affinely independent subset of V.

One can prove the following propositions:

(1) For all real-valued finite sequences f_1, f_2 and for every real number r holds $(\mathrm{Intervals}(f_1, r)) ^\frown \mathrm{Intervals}(f_2, r) = \mathrm{Intervals}(f_1 ^\frown f_2, r)$.

(2) Let f_1, f_2 be finite sequences. Then $x \in \prod(f_1 ^\frown f_2)$ if and only if there exist finite sequences p_1, p_2 such that $x = p_1 ^\frown p_2$ and $p_1 \in \prod f_1$ and $p_2 \in \prod f_2$.

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(3) V is finite dimensional iff Ω_V is finite dimensional.

Let V be a finite dimensional real linear space. One can verify that every affinely independent subset of V is finite.

Let us consider n. One can check that $\mathcal{E}^n_{\mathrm{T}}$ is add-continuous and mult-continuous and $\mathcal{E}^n_{\mathrm{T}}$ is finite dimensional.

In the sequel p_3 denotes a point of $\mathcal{E}^n_{\mathrm{T}}$, A_3 denotes a subset of $\mathcal{E}^n_{\mathrm{T}}$, A_4 denotes an affinely independent subset of $\mathcal{E}^n_{\mathrm{T}}$, and A_5 denotes a subset of $\mathcal{E}^k_{\mathrm{T}}$.

Next we state three propositions:

(4) $\dim(\mathcal{E}^n_{\mathrm{T}}) = n$.

(5) Let V be a finite dimensional real linear space and A be an affinely independent subset of V. Then $\overline{\overline{A}} \leq 1 + \dim(V)$.

(6) Let V be a finite dimensional real linear space and A be an affinely independent subset of V. Then $\overline{\overline{A}} = \dim(V)+1$ if and only if $\mathrm{Affin}\, A = \Omega_V$.

2. Open and Closed Subsets of a Subspace of the Euclidean Topological Space

One can prove the following propositions:

(7) If $k \leq n$ and $A_3 = \{v \in \mathcal{E}^n_{\mathrm{T}}\colon v{\restriction}k \in A_5\}$, then A_3 is open iff A_5 is open.

(8) Let A be a subset of $\mathcal{E}^{k+n}_{\mathrm{T}}$. Suppose $A = \{v \mathbin{^\frown} (n \mapsto 0) : v$ ranges over elements of $\mathcal{E}^k_{\mathrm{T}}\}$. Let B be a subset of $\mathcal{E}^{k+n}_{\mathrm{T}}{\restriction}A$. Suppose $B = \{v; v$ ranges over points of $\mathcal{E}^{k+n}_{\mathrm{T}}\colon v{\restriction}k \in A_5 \ \wedge\ v \in A\}$. Then A_5 is open if and only if B is open.

(9) For every affinely independent subset A of V and for every subset B of V such that $B \subseteq A$ holds $\mathrm{conv}\, A \cap \mathrm{Affin}\, B = \mathrm{conv}\, B$.

(10) Let V be a non empty RLS structure, A be a non empty set, f be a partial function from A to the carrier of V, and X be a set. Then $(r \cdot f)^\circ X = r \cdot f^\circ X$.

(11) If $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in A_3$, then $\mathrm{Affin}\, A_3 = \Omega_{\mathrm{Lin}(A_3)}$.

Let V be a non empty additive loop structure, let A be a finite subset of V, and let v be an element of V. Note that $v + A$ is finite.

Let V be a non empty RLS structure, let A be a finite subset of V, and let us consider r. Observe that $r \cdot A$ is finite.

Next we state the proposition

(12) For every subset A of V holds $\overline{\overline{A}} = \overline{\overline{r \cdot A}}$ iff $r \neq 0$ or A is trivial.

Let V be a non empty RLS structure, let f be a finite sequence of elements of V, and let us consider r. Note that $r \cdot f$ is finite sequence-like.

3. The Vector of Barycentric Coordinates

Let X be a finite set. A one-to-one finite sequence is said to be an enumeration of X if:

(Def. 1) $\operatorname{rng} \mathrm{it} = X$.

Let X be a 1-sorted structure and let A be a finite subset of X. We see that the enumeration of A is a one-to-one finite sequence of elements of X.

In the sequel E_1 denotes an enumeration of A_2 and E_2 denotes an enumeration of A_4.

One can prove the following three propositions:

(13) Let V be an Abelian add-associative right zeroed right complementable non empty additive loop structure, A be a finite subset of V, E be an enumeration of A, and v be an element of V. Then $E + \overline{\overline{A}} \mapsto v$ is an enumeration of $v + A$.

(14) For every enumeration E of A_1 holds $r \cdot E$ is an enumeration of $r \cdot A_1$ iff $r \neq 0$ or A_1 is trivial.

(15) Let M be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$. Suppose $\mathrm{rk}(M) = n$. Let A be a finite subset of $\mathcal{E}^n_{\mathrm{T}}$ and E be an enumeration of A. Then $\mathrm{Mx2Tran}\, M \cdot E$ is an enumeration of $(\mathrm{Mx2Tran}\, M)^\circ A$.

Let us consider V, A_1, let E be an enumeration of A_1, and let us consider x. The functor $x \rightarrow E$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 2) $x \rightarrow E = (x \rightarrow A_1) \cdot E$.

The following propositions are true:

(16) For every enumeration E of A_1 holds $\mathrm{len}(x \rightarrow E) = \overline{\overline{A_1}}$.

(17) For every enumeration E of $v + A_2$ such that $w \in \operatorname{Affin} A_2$ and $E = E_1 + \overline{\overline{A_2}} \mapsto v$ holds $w \rightarrow E_1 = v + w \rightarrow E$.

(18) For every enumeration r_1 of $r \cdot A_2$ such that $v \in \operatorname{Affin} A_2$ and $r_1 = r \cdot E_1$ and $r \neq 0$ holds $v \rightarrow E_1 = r \cdot v \rightarrow r_1$.

(19) Let M be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$. Suppose $\mathrm{rk}(M) = n$. Let M_1 be an enumeration of $(\mathrm{Mx2Tran}\, M)^\circ A_4$. If $M_1 = \mathrm{Mx2Tran}\, M \cdot E_2$, then for every p_3 such that $p_3 \in \operatorname{Affin} A_4$ holds $p_3 \rightarrow E_2 = (\mathrm{Mx2Tran}\, M)(p_3) \rightarrow M_1$.

(20) Let A be a subset of V. Suppose $A \subseteq A_2$ and $x \in \operatorname{Affin} A_2$. Then $x \in \operatorname{Affin} A$ if and only if for every set y such that $y \in \mathrm{dom}(x \rightarrow E_1)$ and $E_1(y) \notin A$ holds $(x \rightarrow E_1)(y) = 0$.

(21) For every E_1 such that $x \in \operatorname{Affin} A_2$ holds $x \in \operatorname{Affin}(E_1{}^\circ \operatorname{Seg} k)$ iff $x \rightarrow E_1 = ((x \rightarrow E_1) {\restriction} k) ^\frown ((\overline{\overline{A_2}} -' k) \mapsto 0)$.

(22) For every E_1 such that $k \leq \overline{\overline{A_2}}$ and $x \in \operatorname{Affin} A_2$ holds $x \in \operatorname{Affin}(A_2 \setminus E_1{}^\circ \operatorname{Seg} k)$ iff $x \rightarrow E_1 = (k \mapsto 0) ^\frown ((x \rightarrow E_1)_{\lfloor k})$.

(23) Suppose $\langle\underbrace{0,\ldots,0}_{n}\rangle \in A_4$ and $E_2(\operatorname{len} E_2) = \langle\underbrace{0,\ldots,0}_{n}\rangle$. Then

(i) $\operatorname{rng}(E_2 \restriction (\overline{\overline{A_4}} -' 1)) = A_4 \setminus \{\langle\underbrace{0,\ldots,0}_{n}\rangle\}$, and

(ii) for every subset A of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ such that $A_4 = A$ holds $E_2 \restriction (\overline{\overline{A_4}} -' 1)$ is an ordered basis of $\operatorname{Lin}(A)$.

(24) Let A be a subset of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $A_4 = A$ and $\langle\underbrace{0,\ldots,0}_{n}\rangle \in A_4$ and $E_2(\operatorname{len} E_2) = \langle\underbrace{0,\ldots,0}_{n}\rangle$. Let B be an ordered basis of $\operatorname{Lin}(A)$. If $B = E_2 \restriction (\overline{\overline{A_4}} -' 1)$, then for every element v of $\operatorname{Lin}(A)$ holds $v \to B = (v \to E_2) \restriction (\overline{\overline{A_4}} -' 1)$.

(25) For all E_2, A_3 such that $k \leq n$ and $\overline{\overline{A_4}} = n + 1$ and $A_3 = \{p_3 : (p_3 \to E_2) \restriction k \in A_5\}$ holds A_5 is open iff A_3 is open.

(26) For every E_2 such that $k \leq n$ and $\overline{\overline{A_4}} = n + 1$ and $A_3 = \{p_3 : (p_3 \to E_2) \restriction k \in A_5\}$ holds A_5 is closed iff A_3 is closed.

Let us consider n. One can verify that every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is affine is also closed.

In the sequel p_4 denotes an element of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$.

Next we state two propositions:

(27) For every E_2 and for every subset B of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$ such that $k < \overline{\overline{A_4}}$ and $B = \{p_4 : (p_4 \to E_2) \restriction k \in A_5\}$ holds A_5 is open iff B is open.

(28) Let given E_2 and B be a subset of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$. Suppose $k < \overline{\overline{A_4}}$ and $B = \{p_4 : (p_4 \to E_2) \restriction k \in A_5\}$. Then A_5 is closed if and only if B is closed.

Let us consider n and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $\operatorname{halfline}(p, q)$ is closed.

4. Continuity of Barycentric Coordinates

Let us consider V, let A be a subset of V, and let us consider x. The functor $\vdash (A, x)$ yielding a function from V into $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 3) $(\vdash (A, x))(v) = (v \to A)(x)$.

One can prove the following four propositions:

(29) For every subset A of V such that $x \notin A$ holds $\vdash (A, x) = \Omega_V \longmapsto 0$.

(30) For every affinely independent subset A of V such that $\vdash (A, x) = \Omega_V \longmapsto 0$ holds $x \notin A$.

(31) $\vdash (A_4, x) \restriction \operatorname{Affin} A_4$ is a continuous function from $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$ into $\mathbb{R}^{\mathbf{1}}$.

(32) If $\overline{\overline{A_4}} = n + 1$, then $\vdash (A_4, x)$ is continuous.

Let us consider n, A_4. Note that $\operatorname{conv} A_4$ is closed.

We now state the proposition

(33) If $\overline{\overline{A_4}} = n + 1$, then Int A_4 is open.

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