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# Banach Algebra of Bounded Complex-Valued Functionals

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**Summary.** In this article, we describe some basic properties of the Banach algebra which is constructed from all bounded complex-valued functionals.

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The notation and terminology used in this paper are introduced in the following articles: [2], [16], [9], [14], [7], [8], [3], [18], [17], [4], [19], [5], [15], [1], [20], [12], [11], [10], [21], [13], and [6].

Let $V$ be a complex algebra. A complex algebra is called a complex subalgebra of $V$ if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of $V$,

(ii) the addition of it = (the addition of $V$) $\restriction$ (the carrier of it),

(iii) the multiplication of it = (the multiplication of $V$) $\restriction$ (the carrier of it),

(iv) the external multiplication of it = (the external multiplication of $V$)$\restriction$($\mathbb{C} \times$ the carrier of it),

(v) $1_{\rm it} = 1_V$, and

(vi) $0_{\rm it} = 0_V$.

We now state the proposition

(1) Let $X$ be a non empty set, $V$ be a complex algebra, $V_1$ be a non empty subset of $V$, $d_1$, $d_2$ be elements of $X$, $A$ be a binary operation on $X$, $M$ be a function from $X \times X$ into $X$, and $M_1$ be a function from $\mathbb{C} \times X$ into $X$. Suppose that $V_1 = X$ and $d_1 = 0_V$ and $d_2 = 1_V$ and $A$ = (the addition of $V$) $\restriction$ ($V_1$) and $M$ = (the multiplication of $V$) $\restriction$ ($V_1$) and $M_1$ = (the external multiplication of $V$)$\restriction$($\mathbb{C} \times V_1$) and $V_1$ has inverse. Then $\langle X, M, A, M_1, d_2, d_1 \rangle$ is a complex subalgebra of $V$.




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Let $V$ be a complex algebra. One can check that there exists a complex subalgebra of $V$ which is strict.

Let $V$ be a complex algebra and let $V_1$ be a subset of $V$. We say that $V_1$ is $\mathbb{C}$-additively-linearly-closed if and only if:

(Def. 2) $V_1$ is add closed and has inverse and for every complex number $a$ and for every element $v$ of $V$ such that $v \in V_1$ holds $a \cdot v \in V_1$.

Let $V$ be a complex algebra and let $V_1$ be a subset of $V$. Let us assume that $V_1$ is $\mathbb{C}$-additively-linearly-closed and non empty. The functor $\mathrm{Mult}(V_1, V)$ yielding a function from $\mathbb{C} \times V_1$ into $V_1$ is defined as follows:

(Def. 3) $\mathrm{Mult}(V_1, V) = (\text{the external multiplication of } V) \restriction (\mathbb{C} \times V_1)$.

Let $X$ be a non empty set. The functor $\mathbb{C}\text{-BoundedFunctions}\, X$ yielding a non empty subset of $\mathrm{CAlgebra}(X)$ is defined by:

(Def. 4) $\mathbb{C}\text{-BoundedFunctions}\, X = \{f : X \to \mathbb{C}\colon f \restriction X \text{ is bounded}\}$.

Let $X$ be a non empty set. Note that $\mathrm{CAlgebra}(X)$ is scalar unital.

Let $X$ be a non empty set. One can verify that $\mathbb{C}\text{-BoundedFunctions}\, X$ is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let $V$ be a complex algebra. Observe that there exists a non empty subset of $V$ which is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let $V$ be a non empty CLS structure. We say that $V$ is scalar-multiplcation-cancelable if and only if:

(Def. 5) For every complex number $a$ and for every element $v$ of $V$ such that $a \cdot v = 0_V$ holds $a = 0$ or $v = 0_V$.

One can prove the following two propositions:

(2) Let $V$ be a complex algebra and $V_1$ be a $\mathbb{C}$-additively-linearly-closed multiplicatively-closed non empty subset of $V$. Then $\langle V_1, \mathrm{mult}(V_1, V), \mathrm{Add}(V_1, V), \mathrm{Mult}(V_1, V), \mathrm{One}(V_1, V), \mathrm{Zero}(V_1, V)\rangle$ is a complex subalgebra of $V$.

(3) Let $V$ be a complex algebra and $V_1$ be a complex subalgebra of $V$. Then

(i) for all elements $v_1$, $w_1$ of $V_1$ and for all elements $v$, $w$ of $V$ such that $v_1 = v$ and $w_1 = w$ holds $v_1 + w_1 = v + w$,

(ii) for all elements $v_1$, $w_1$ of $V_1$ and for all elements $v$, $w$ of $V$ such that $v_1 = v$ and $w_1 = w$ holds $v_1 \cdot w_1 = v \cdot w$,

(iii) for every element $v_1$ of $V_1$ and for every element $v$ of $V$ and for every complex number $a$ such that $v_1 = v$ holds $a \cdot v_1 = a \cdot v$,

(iv) $\mathbf{1}_{(V_1)} = \mathbf{1}_V$, and

(v) $0_{(V_1)} = 0_V$.

Let $X$ be a non empty set. The $\mathbb{C}$-algebra of bounded functions of $X$ yielding a complex algebra is defined by:

(Def. 6) The $\mathbb{C}$-algebra of bounded functions of $X =$
$\langle \mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{mult}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)),$

Add($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)),
Mult($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)),
One($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)),
Zero($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$))$\rangle$.

One can prove the following proposition

(4) For every non empty set $X$ holds the $\mathbb{C}$-algebra of bounded functions of $X$ is a complex subalgebra of CAlgebra($X$).

Let $X$ be a non empty set. Note that the $\mathbb{C}$-algebra of bounded functions of $X$ is vector distributive and scalar unital.

Next we state several propositions:

(5) Let $X$ be a non empty set, $F$, $G$, $H$ be vectors of the $\mathbb{C}$-algebra of bounded functions of $X$, and $f$, $g$, $h$ be functions from $X$ into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) + g(x)$.

(6) Let $X$ be a non empty set, $a$ be a complex number, $F$, $G$ be vectors of the $\mathbb{C}$-algebra of bounded functions of $X$, and $f$, $g$ be functions from $X$ into $\mathbb{C}$. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element $x$ of $X$ holds $g(x) = a \cdot f(x)$.

(7) Let $X$ be a non empty set, $F$, $G$, $H$ be vectors of the $\mathbb{C}$-algebra of bounded functions of $X$, and $f$, $g$, $h$ be functions from $X$ into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) \cdot g(x)$.

(8) For every non empty set $X$ holds $0_{\text{the } \mathbb{C}\text{-algebra of bounded functions of } X} = X \longmapsto 0$.

(9) For every non empty set $X$ holds $\mathbf{1}_{\text{the } \mathbb{C}\text{-algebra of bounded functions of } X} = X \longmapsto 1_{\mathbb{C}}$.

Let $X$ be a non empty set and let $F$ be a set. Let us assume that $F \in$ $\mathbb{C}$-BoundedFunctions $X$. The functor modetrans($F, X$) yields a function from $X$ into $\mathbb{C}$ and is defined by:

(Def. 7) modetrans($F, X$) $= F$ and modetrans($F, X$)$\restriction X$ is bounded.

Let $X$ be a non empty set and let $f$ be a function from $X$ into $\mathbb{C}$. The functor PreNorms($f$) yields a non empty subset of $\mathbb{R}$ and is defined by:

(Def. 8) PreNorms($f$) $= \{|f(x)| : x$ ranges over elements of $X\}$.

We now state two propositions:

(10) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{C}$ such that $f\restriction X$ is bounded holds PreNorms($f$) is upper bounded.

(11) Let $X$ be a non empty set and $f$ be a function from $X$ into $\mathbb{C}$. Then $f\restriction X$ is bounded if and only if PreNorms($f$) is upper bounded.

Let $X$ be a non empty set. The functor $\mathbb{C}$-BoundedFunctionsNorm $X$ yields a function from $\mathbb{C}$-BoundedFunctions $X$ into $\mathbb{R}$ and is defined by:

(Def. 9) For every set $x$ such that $x \in$ $\mathbb{C}$-BoundedFunctions $X$ holds ($\mathbb{C}$-BoundedFunctionsNorm $X)(x) = \sup$ PreNorms(modetrans($x, X$)).

One can prove the following two propositions:

(13)[1] For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{C}$ such that $f{\restriction}X$ is bounded holds modetrans$(f, X) = f$.

(14) For every non empty set $X$ and for every function $f$ from $X$ into $\mathbb{C}$ such that $f{\restriction}X$ is bounded holds ($\mathbb{C}$-BoundedFunctionsNorm $X)(f) = \sup$ PreNorms$(f)$.

Let $X$ be a non empty set. The $\mathbb{C}$-normed algebra of bounded functions of $X$ yielding a normed complex algebra structure is defined by:

(Def. 10) The $\mathbb{C}$-normed algebra of bounded functions of $X$ =
⟨$\mathbb{C}$-BoundedFunctions $X$, mult($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)),
Add($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)),
Mult($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)),
One($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)),
Zero($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)), $\mathbb{C}$-BoundedFunctionsNorm $X$⟩.

Let $X$ be a non empty set. One can verify that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is non empty.

Let $X$ be a non empty set. One can check that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is unital.

We now state a number of propositions:

(15) Let $W$ be a normed complex algebra structure and $V$ be a complex algebra. Suppose ⟨the carrier of $W$, the multiplication of $W$, the addition of $W$, the external multiplication of $W$, the one of $W$, the zero of $W$⟩ $= V$. Then $W$ is a complex algebra.

(16) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex algebra.

(17) Let $X$ be a non empty set and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$.
Then (Mult($\mathbb{C}$-BoundedFunctions $X$, CAlgebra($X$)))$(1_{\mathbb{C}}, F) = F$.

(18) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex linear space.

(19) For every non empty set $X$ holds
$X \longmapsto 0 = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$.

(20) Let $X$ be a non empty set, $x$ be an element of $X$, $f$ be a function from $X$ into $\mathbb{C}$, and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$. If $f = F$ and $f{\restriction}X$ is bounded, then $|f(x)| \leq \|F\|$.

[1]The proposition (12) has been removed.

(21) For every non empty set $X$ and for every point $F$ of the $\mathbb{C}$-normed algebra of bounded functions of $X$ holds $0 \leq \|F\|$.

(22) Let $X$ be a non empty set and $F$ be a point of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$. Then $0 = \|F\|$.

(23) Let $X$ be a non empty set, $f$, $g$, $h$ be functions from $X$ into $\mathbb{C}$, and $F$, $G$, $H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) + g(x)$.

(24) Let $X$ be a non empty set, $a$ be a complex number, $f$, $g$ be functions from $X$ into $\mathbb{C}$, and $F$, $G$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element $x$ of $X$ holds $g(x) = a \cdot f(x)$.

(25) Let $X$ be a non empty set, $f$, $g$, $h$ be functions from $X$ into $\mathbb{C}$, and $F$, $G$, $H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) \cdot g(x)$.

(26) Let $X$ be a non empty set, $a$ be a complex number, and $F$, $G$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Then

(i) if $\|F\| = 0$, then $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$,

(ii) if $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$, then $\|F\| = 0$,

(iii) $\|a \cdot F\| = |a| \cdot \|F\|$, and

(iv) $\|F + G\| \leq \|F\| + \|G\|$.

Let $X$ be a non empty set. Note that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, and complex normed space-like.

We now state two propositions:

(27) Let $X$ be a non empty set, $f$, $g$, $h$ be functions from $X$ into $\mathbb{C}$, and $F$, $G$, $H$ be points of the $\mathbb{C}$-normed algebra of bounded functions of $X$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element $x$ of $X$ holds $h(x) = f(x) - g(x)$.

(28) Let $X$ be a non empty set and $s_1$ be a sequence of the $\mathbb{C}$-normed algebra of bounded functions of $X$. If $s_1$ is CCauchy, then $s_1$ is convergent.

Let $X$ be a non empty set. Observe that the $\mathbb{C}$-normed algebra of bounded functions of $X$ is complete.

Next we state the proposition

(29) For every non empty set $X$ holds the $\mathbb{C}$-normed algebra of bounded functions of $X$ is a complex Banach algebra.

## References

[1] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[7] Noboru Endou. Banach algebra of bounded complex linear operators. *Formalized Mathematics*, 12(**3**):237–242, 2004.
[8] Noboru Endou. Complex linear space and complex normed space. *Formalized Mathematics*, 12(**2**):93–102, 2004.
[9] Noboru Endou. Complex valued functions space. *Formalized Mathematics*, 12(**3**):231–235, 2004.
[10] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[11] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[12] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[13] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[14] Takashi Mitsuishi, Katsumi Wasaki, and Yasunari Shidama. Property of complex functions. *Formalized Mathematics*, 9(**1**):179–184, 2001.
[15] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[16] Yasunari Shidama, Hikofumi Suzuki, and Noboru Endou. Banach algebra of bounded functionals. *Formalized Mathematics*, 16(**2**):115–122, 2008, doi:10.2478/v10037-008-0017-z.
[17] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[18] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[19] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[20] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

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