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Cartesian Products of Family of Real Linear Spaces

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Summary. In this article we introduced the isomorphism mapping between cartesian products of family of linear spaces [4]. Those products had been formalized by two different ways, i.e., the way using the functor [:X,Y:] and ones using the functor "product". By the same way, the isomorphism mapping was defined between Cartesian products of family of linear normed spaces also.

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The notation and terminology used in this paper are introduced in the following articles: [5], [1], [16], [11], [3], [6], [17], [7], [8], [15], [14], [2], [13], [12], [20], [18], [10], [19], and [9].

1. Preliminaries

One can prove the following propositions:

(1) Let D, E, F, G be non empty sets. Then there exists a function I from $D \times E \times (F \times G)$ into $D \times F \times (E \times G)$ such that

(i) I is one-to-one and onto, and

(ii) for all sets d, e, f, g such that $d \in D$ and $e \in E$ and $f \in F$ and $g \in G$ holds $I(\langle\langle d, e\rangle, \langle f, g\rangle\rangle) = \langle\langle d, f\rangle, \langle e, g\rangle\rangle$.

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(2) Let X be a non empty set and D be a function. Suppose $\operatorname{dom} D = \{1\}$ and $D(1) = X$. Then there exists a function I from X into $\prod D$ such that I is one-to-one and onto and for every set x such that $x \in X$ holds $I(x) = \langle x \rangle$.

(3) Let X, Y be non empty sets and D be a function. Suppose $\operatorname{dom} D = \{1, 2\}$ and $D(1) = X$ and $D(2) = Y$. Then there exists a function I from $X \times Y$ into $\prod D$ such that I is one-to-one and onto and for all sets x, y such that $x \in X$ and $y \in Y$ holds $I(x, y) = \langle x, y \rangle$.

(4) Let X be a non empty set. Then there exists a function I from X into $\prod \langle X \rangle$ such that I is one-to-one and onto and for every set x such that $x \in X$ holds $I(x) = \langle x \rangle$.

Let X, Y be non-empty non empty finite sequences. Observe that $X \frown Y$ is non-empty.

We now state two propositions:

(5) Let X, Y be non empty sets. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that I is one-to-one and onto and for all sets x, y such that $x \in X$ and $y \in Y$ holds $I(x, y) = \langle x, y \rangle$.

(6) Let X, Y be non-empty non empty finite sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod (X \frown Y)$ such that I is one-to-one and onto and for all finite sequences x, y such that $x \in \prod X$ and $y \in \prod Y$ holds $I(x, y) = x \frown y$.

Let G, F be non empty additive loop structures. The functor $\operatorname{prodadd}(G, F)$ yielding a binary operation on (the carrier of G) $\times$ (the carrier of F) is defined by:

(Def. 1) For all points g_1, g_2 of G and for all points f_1, f_2 of F holds $(\operatorname{prodadd}(G, F))(\langle g_1, f_1 \rangle, \langle g_2, f_2 \rangle) = \langle g_1 + g_2, f_1 + f_2 \rangle$.

Let G, F be non empty RLS structures. The functor $\operatorname{prodmlt}(G, F)$ yielding a function from $\mathbb{R} \times$ ((the carrier of G) $\times$ (the carrier of F)) into (the carrier of G) $\times$ (the carrier of F) is defined by:

(Def. 2) For every element r of $\mathbb{R}$ and for every point g of G and for every point f of F holds $(\operatorname{prodmlt}(G, F))(r, \langle g, f \rangle) = \langle r \cdot g, r \cdot f \rangle$.

Let G, F be non empty additive loop structures. The functor $\operatorname{prodzero}(G, F)$ yields an element of (the carrier of G) $\times$ (the carrier of F) and is defined by:

(Def. 3) $\operatorname{prodzero}(G, F) = \langle 0_G, 0_F \rangle$.

Let G, F be non empty additive loop structures. The functor $G \times F$ yielding a strict non empty additive loop structure is defined by:

(Def. 4) $G \times F = \langle$(the carrier of G) $\times$ (the carrier of F), $\operatorname{prodadd}(G, F)$, $\operatorname{prodzero}(G, F)\rangle$.

Let G, F be Abelian non empty additive loop structures. Observe that $G \times F$ is Abelian.

Let G, F be add-associative non empty additive loop structures. Note that $G \times F$ is add-associative.

Let G, F be right zeroed non empty additive loop structures. Note that $G \times F$ is right zeroed.

Let G, F be right complementable non empty additive loop structures. Note that $G \times F$ is right complementable.

Next we state two propositions:

(7) Let G, F be non empty additive loop structures. Then

(i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$, and

(iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$.

(8) Let G, F be add-associative right zeroed right complementable non empty additive loop structures, x be a point of $G \times F$, x_1 be a point of G, and x_2 be a point of F. If $x = \langle x_1, x_2 \rangle$, then $-x = \langle -x_1, -x_2 \rangle$.

Let G, F be Abelian add-associative right zeroed right complementable strict non empty additive loop structures. One can check that $G \times F$ is strict, Abelian, add-associative, right zeroed, and right complementable.

Let G, F be non empty RLS structures. The functor $G \times F$ yields a strict non empty RLS structure and is defined by:

(Def. 5) $G \times F = \langle$(the carrier of G) $\times$ (the carrier of F), $\mathrm{prodzero}(G, F)$, $\mathrm{prodadd}(G, F), \mathrm{prodmlt}(G, F)\rangle$.

Let G, F be Abelian non empty RLS structures. Observe that $G \times F$ is Abelian.

Let G, F be add-associative non empty RLS structures. Note that $G \times F$ is add-associative.

Let G, F be right zeroed non empty RLS structures. Note that $G \times F$ is right zeroed.

Let G, F be right complementable non empty RLS structures. One can check that $G \times F$ is right complementable.

Next we state two propositions:

(9) Let G, F be non empty RLS structures. Then

(i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,

(iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$, and

(iv) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$.

(10) Let G, F be add-associative right zeroed right complementable non empty RLS structures, x be a point of $G \times F$, x_1 be a point of G, and x_2 be a point of F. If $x = \langle x_1, x_2 \rangle$, then $-x = \langle -x_1, -x_2 \rangle$.

Let G, F be vector distributive non empty RLS structures. Note that $G \times F$ is vector distributive.

Let G, F be scalar distributive non empty RLS structures. Note that $G \times F$ is scalar distributive.

Let G, F be scalar associative non empty RLS structures. Observe that $G \times F$ is scalar associative.

Let G, F be scalar unital non empty RLS structures. One can verify that $G \times F$ is scalar unital.

Let G be an Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative scalar unital non empty RLS structure. Note that $\langle G \rangle$ is real-linear-space-yielding.

Let G, F be Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative scalar unital non empty RLS structures. Note that $\langle G, F \rangle$ is real-linear-space-yielding.

2. Cartesian Products of Real Linear Spaces

One can prove the following proposition

(11) Let X be a real linear space. Then there exists a function I from X into $\prod \langle X \rangle$ such that

(i) I is one-to-one and onto,

(ii) for every point x of X holds $I(x) = \langle x \rangle$,

(iii) for all points v, w of X holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of X and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_X) = 0_{\prod \langle X \rangle}$.

Let G, F be non empty real-linear-space-yielding finite sequences. Observe that $G \frown F$ is real-linear-space-yielding.

We now state three propositions:

(12) Let X, Y be real linear spaces. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that

(i) I is one-to-one and onto,

(ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, y \rangle$,

(iii) for all points v, w of $X \times Y$ holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of $X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{X \times Y}) = 0_{\prod\langle X,Y\rangle}$.

(13) Let X, Y be non empty real linear space-sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod(X \frown Y)$ such that

(i) I is one-to-one and onto,

(ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \frown y_1$,

(iii) for all points v, w of $\prod X \times \prod Y$ holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of $\prod X \times \prod Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{\prod X \times \prod Y}) = 0_{\prod(X \frown Y)}$.

(14) Let G, F be real linear spaces. Then

(i) for every set x holds x is a point of $\prod\langle G, F\rangle$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2\rangle$,

(ii) for all points x, y of $\prod\langle G, F\rangle$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2\rangle$,

(iii) $0_{\prod\langle G,F\rangle} = \langle 0_G, 0_F\rangle$,

(iv) for every point x of $\prod\langle G, F\rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2\rangle$ holds $-x = \langle -x_1, -x_2\rangle$, and

(v) for every point x of $\prod\langle G, F\rangle$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2\rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2\rangle$.

3. Cartesian Products of Real Normed Linear Spaces

Let G, F be non empty normed structures. The functor prodnorm(G, F) yields a function from (the carrier of G) $\times$ (the carrier of F) into $\mathbb{R}$ and is defined by:

(Def. 6) For every point g of G and for every point f of F there exists an element v of $\mathcal{R}^2$ such that $v = \langle \|g\|, \|f\|\rangle$ and $(\text{prodnorm}(G, F))(g, f) = |v|$.

Let G, F be non empty normed structures. The functor $G \times F$ yielding a strict non empty normed structure is defined as follows:

(Def. 7) $G \times F = \langle$(the carrier of G) $\times$ (the carrier of F), prodzero(G, F), $\text{prodadd}(G, F), \text{prodmlt}(G, F), \text{prodnorm}(G, F)\rangle$.

Let G, F be real normed spaces. Observe that $G \times F$ is reflexive, discernible, and real normed space-like.

Let G, F be reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty normed structures. One can verify that $G \times F$ is strict, reflexive, discernible, real normed space-like, scalar distributive, vector distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Let G be a reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty normed structure. One can verify that $\langle G \rangle$ is real-norm-space-yielding.

Let G, F be reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty normed structures. Observe that $\langle G, F \rangle$ is real-norm-space-yielding.

One can prove the following propositions:

(15) Let X, Y be real normed spaces. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that

(i) I is one-to-one and onto,

(ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, y \rangle$,

(iii) for all points v, w of $X \times Y$ holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of $X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,

(v) $0_{\prod \langle X, Y \rangle} = I(0_{X \times Y})$, and

(vi) for every point v of $X \times Y$ holds $\|I(v)\| = \|v\|$.

(16) Let X be a real normed space. Then there exists a function I from X into $\prod \langle X \rangle$ such that

(i) I is one-to-one and onto,

(ii) for every point x of X holds $I(x) = \langle x \rangle$,

(iii) for all points v, w of X holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of X and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,

(v) $0_{\prod \langle X \rangle} = I(0_X)$, and

(vi) for every point v of X holds $\|I(v)\| = \|v\|$.

Let G, F be non empty real-norm-space-yielding finite sequences. One can check that $G \frown F$ is non empty and real-norm-space-yielding.

One can prove the following propositions:

(17) Let X, Y be non empty real norm space-sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod (X \frown Y)$ such that

(i) I is one-to-one and onto,

(ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \frown y_1$,

(iii) for all points v, w of $\prod X \times \prod Y$ holds $I(v + w) = I(v) + I(w)$,

(iv) for every point v of $\prod X \times \prod Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,

(v) $I(0_{\prod X \times \prod Y}) = 0_{\prod (X ^\frown Y)}$, and

(vi) for every point v of $\prod X \times \prod Y$ holds $\|I(v)\| = \|v\|$.

(18) Let G, F be real normed spaces. Then

(i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,

(iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$,

(iv) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$,

(v) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$, and

(vi) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ there exists an element w of $\mathcal{R}^2$ such that $w = \langle \|x_1\|, \|x_2\| \rangle$ and $\|x\| = |w|$.

(19) Let G, F be real normed spaces. Then

(i) for every set x holds x is a point of $\prod \langle G, F \rangle$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,

(ii) for all points x, y of $\prod \langle G, F \rangle$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,

(iii) $0_{\prod \langle G, F \rangle} = \langle 0_G, 0_F \rangle$,

(iv) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$,

(v) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$, and

(vi) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ there exists an element w of $\mathcal{R}^2$ such that $w = \langle \|x_1\|, \|x_2\| \rangle$ and $\|x\| = |w|$.

Let X, Y be complete real normed spaces. Observe that $X \times Y$ is complete.

We now state several propositions:

(20) Let X, Y be non empty real norm space-sequences. Then there exists a function I from $\prod \langle \prod X, \prod Y \rangle$ into $\prod (X ^\frown Y)$ such that

(i) I is one-to-one and onto,

(ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(\langle x, y \rangle) = x_1 ^\frown y_1$,

(iii) for all points v, w of $\prod\langle\prod X, \prod Y\rangle$ holds $I(v+w) = I(v) + I(w)$,
(iv) for every point v of $\prod\langle\prod X, \prod Y\rangle$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,
(v) $I(0_{\prod\langle\prod X, \prod Y\rangle}) = 0_{\prod(X^\frown Y)}$, and
(vi) for every point v of $\prod\langle\prod X, \prod Y\rangle$ holds $\|I(v)\| = \|v\|$.

(21) Let X, Y be non empty real linear spaces. Then there exists a function I from $X \times Y$ into $X \times \prod\langle Y\rangle$ such that
(i) I is one-to-one and onto,
(ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, \langle y\rangle\rangle$,
(iii) for all points v, w of $X \times Y$ holds $I(v+w) = I(v) + I(w)$,
(iv) for every point v of $X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and
(v) $I(0_{X \times Y}) = 0_{X \times \prod\langle Y\rangle}$.

(22) Let X be a non empty real linear space-sequence and Y be a real linear space. Then there exists a function I from $\prod X \times Y$ into $\prod(X ^\frown \langle Y\rangle)$ such that
(i) I is one-to-one and onto,
(ii) for every point x of $\prod X$ and for every point y of Y there exist finite sequences x_1, y_1 such that $x = x_1$ and $\langle y\rangle = y_1$ and $I(x, y) = x_1 ^\frown y_1$,
(iii) for all points v, w of $\prod X \times Y$ holds $I(v+w) = I(v) + I(w)$,
(iv) for every point v of $\prod X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$, and
(v) $I(0_{\prod X \times Y}) = 0_{\prod(X^\frown\langle Y\rangle)}$.

(23) Let X, Y be non empty real normed spaces. Then there exists a function I from $X \times Y$ into $X \times \prod\langle Y\rangle$ such that
(i) I is one-to-one and onto,
(ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, \langle y\rangle\rangle$,
(iii) for all points v, w of $X \times Y$ holds $I(v+w) = I(v) + I(w)$,
(iv) for every point v of $X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,
(v) $I(0_{X \times Y}) = 0_{X \times \prod\langle Y\rangle}$, and
(vi) for every point v of $X \times Y$ holds $\|I(v)\| = \|v\|$.

(24) Let X be a non empty real norm space-sequence and Y be a real normed space. Then there exists a function I from $\prod X \times Y$ into $\prod(X ^\frown \langle Y\rangle)$ such that
(i) I is one-to-one and onto,
(ii) for every point x of $\prod X$ and for every point y of Y there exist finite sequences x_1, y_1 such that $x = x_1$ and $\langle y\rangle = y_1$ and $I(x, y) = x_1 ^\frown y_1$,
(iii) for all points v, w of $\prod X \times Y$ holds $I(v+w) = I(v) + I(w)$,

(iv) for every point v of $\prod X \times Y$ and for every element r of $\mathbb{R}$ holds $I(r \cdot v) = r \cdot I(v)$,

(v) $I(0_{\prod X \times Y}) = 0_{\prod (X ^\frown \langle Y \rangle)}$, and

(vi) for every point v of $\prod X \times Y$ holds $\|I(v)\| = \|v\|$.

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