

# The Mycielskian of a Graph<sup>1</sup>

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**Summary.** Let  $\omega(G)$  and  $\chi(G)$  be the clique number and the chromatic number of a graph  $G$ . Mycielski [11] presented a construction that for any  $n$  creates a graph  $M_n$  which is triangle-free ( $\omega(G) = 2$ ) with  $\chi(G) > n$ . The starting point is the complete graph of two vertices ( $K_2$ ).  $M_{(n+1)}$  is obtained from  $M_n$  through the operation  $\mu(G)$  called the Mycielskian of a graph  $G$ .

We first define the operation  $\mu(G)$  and then show that  $\omega(\mu(G)) = \omega(G)$  and  $\chi(\mu(G)) = \chi(G) + 1$ . This is done for arbitrary graph  $G$ , see also [10]. Then we define the sequence of graphs  $M_n$  each of exponential size in  $n$  and give their clique and chromatic numbers.

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The notation and terminology used here have been introduced in the following papers: [1], [15], [13], [8], [5], [2], [14], [9], [16], [3], [6], [18], [19], [12], [17], [4], and [7].

## 1. PRELIMINARIES

One can prove the following propositions:

- (1) For all real numbers  $x, y, z$  such that  $0 \leq x$  holds  $x \cdot (y -' z) = x \cdot y -' x \cdot z$ .
- (2) For all natural numbers  $x, y, z$  holds  $x \in y \setminus z$  iff  $z \leq x < y$ .
- (3) For all sets  $A, B, C, D, E, X$  such that  $X \subseteq A$  or  $X \subseteq B$  or  $X \subseteq C$  or  $X \subseteq D$  or  $X \subseteq E$  holds  $X \subseteq A \cup B \cup C \cup D \cup E$ .
- (4) For all sets  $A, B, C, D, E, x$  holds  $x \in A \cup B \cup C \cup D \cup E$  iff  $x \in A$  or  $x \in B$  or  $x \in C$  or  $x \in D$  or  $x \in E$ .

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- (5) Let  $R$  be a symmetric relational structure and  $x, y$  be sets. Suppose  $x \in$  the carrier of  $R$  and  $y \in$  the carrier of  $R$  and  $\langle x, y \rangle \in$  the internal relation of  $R$ . Then  $\langle y, x \rangle \in$  the internal relation of  $R$ .
- (6) For every symmetric relational structure  $R$  and for all elements  $x, y$  of  $R$  such that  $x \leq y$  holds  $y \leq x$ .

## 2. PARTITIONS

One can prove the following proposition

- (7) For every set  $X$  and for every partition  $P$  of  $X$  holds  $\overline{\overline{P}} \subseteq \overline{\overline{X}}$ .

Let  $X$  be a set, let  $P$  be a partition of  $X$ , and let  $S$  be a subset of  $X$ . The functor  $P \upharpoonright S$  yields a partition of  $S$  and is defined by:

- (Def. 1)  $P \upharpoonright S = \{x \cap S; x \text{ ranges over elements of } P: x \text{ meets } S\}$ .

Let  $X$  be a set. Observe that there exists a partition of  $X$  which is finite.

Let  $X$  be a set, let  $P$  be a finite partition of  $X$ , and let  $S$  be a subset of  $X$ . Observe that  $P \upharpoonright S$  is finite.

One can prove the following propositions:

- (8) For every set  $X$  and for every finite partition  $P$  of  $X$  and for every subset  $S$  of  $X$  holds  $\overline{\overline{P \upharpoonright S}} \leq \overline{\overline{P}}$ .
- (9) Let  $X$  be a set,  $P$  be a finite partition of  $X$ , and  $S$  be a subset of  $X$ . Then for every set  $p$  such that  $p \in P$  holds  $p$  meets  $S$  if and only if  $\overline{\overline{P \upharpoonright S}} = \overline{\overline{P}}$ .
- (10) Let  $R$  be a relational structure,  $C$  be a coloring of  $R$ , and  $S$  be a subset of  $R$ . Then  $C \upharpoonright S$  is a coloring of  $\text{sub}(S)$ .

## 3. CHROMATIC NUMBER AND CLIQUE COVER NUMBER

Let  $R$  be a relational structure. We say that  $R$  is finitely colorable if and only if:

- (Def. 2) There exists a coloring of  $R$  which is finite.

One can check that there exists a relational structure which is finitely colorable.

Let us observe that every relational structure which is finite is also finitely colorable.

Let  $R$  be a finitely colorable relational structure. Observe that there exists a coloring of  $R$  which is finite.

Let  $R$  be a finitely colorable relational structure and let  $S$  be a subset of  $R$ . One can verify that  $\text{sub}(S)$  is finitely colorable.

Let  $R$  be a finitely colorable relational structure. The functor  $\chi(R)$  yielding a natural number is defined by:

(Def. 3) There exists a finite coloring  $C$  of  $R$  such that  $\overline{\overline{C}} = \chi(R)$  and for every finite coloring  $C$  of  $R$  holds  $\chi(R) \leq \overline{\overline{C}}$ .

Let  $R$  be an empty relational structure. Observe that  $\chi(R)$  is empty.

Let  $R$  be a non empty finitely colorable relational structure. Observe that  $\chi(R)$  is positive.

Let  $R$  be a relational structure. We say that  $R$  has finite clique cover if and only if:

(Def. 4) There exists a clique-partition of  $R$  which is finite.

One can verify that there exists a relational structure which has finite clique cover.

One can verify that every relational structure which is finite has also finite clique cover.

Let  $R$  be a relational structure with finite clique cover. Observe that there exists a clique-partition of  $R$  which is finite.

Let  $R$  be a relational structure with finite clique cover and let  $S$  be a subset of  $R$ . Observe that  $\text{sub}(S)$  has finite clique cover.

Let  $R$  be a relational structure with finite clique cover. The functor  $\kappa(R)$  yielding a natural number is defined by:

(Def. 5) There exists a finite clique-partition  $C$  of  $R$  such that  $\overline{\overline{C}} = \kappa(R)$  and for every finite clique-partition  $C$  of  $R$  holds  $\kappa(R) \leq \overline{\overline{C}}$ .

Let  $R$  be an empty relational structure. One can check that  $\kappa(R)$  is empty.

Let  $R$  be a non empty relational structure with finite clique cover. One can verify that  $\kappa(R)$  is positive.

We now state several propositions:

- (11) For every finite relational structure  $R$  holds  $\omega(R) \leq \overline{\overline{\text{the carrier of } R}}$ .
- (12) For every finite relational structure  $R$  holds  $\alpha(R) \leq \overline{\overline{\text{the carrier of } R}}$ .
- (13) For every finite relational structure  $R$  holds  $\chi(R) \leq \overline{\overline{\text{the carrier of } R}}$ .
- (14) For every finite relational structure  $R$  holds  $\kappa(R) \leq \overline{\overline{\text{the carrier of } R}}$ .
- (15) For every finitely colorable relational structure  $R$  with finite clique number holds  $\omega(R) \leq \chi(R)$ .
- (16) For every relational structure  $R$  with finite stability number and finite clique cover holds  $\alpha(R) \leq \kappa(R)$ .

#### 4. COMPLEMENT

The following two propositions are true:

- (17) Let  $R$  be a relational structure,  $x, y$  be elements of  $R$ , and  $a, b$  be elements of  $\text{ComplRelStr } R$ . If  $x = a$  and  $y = b$  and  $x \leq y$ , then  $a \not\leq b$ .

- (18) Let  $R$  be a relational structure,  $x, y$  be elements of  $R$ , and  $a, b$  be elements of  $\text{ComplRelStr } R$ . If  $x = a$  and  $y = b$  and  $x \neq y$  and  $x \in$  the carrier of  $R$  and  $a \not\leq b$ , then  $x \leq y$ .

Let  $R$  be a finite relational structure. Note that  $\text{ComplRelStr } R$  is finite.

Next we state four propositions:

- (19) For every symmetric relational structure  $R$  holds every clique of  $R$  is a stable set of  $\text{ComplRelStr } R$ .
- (20) For every symmetric relational structure  $R$  holds every clique of  $\text{ComplRelStr } R$  is a stable set of  $R$ .
- (21) For every relational structure  $R$  holds every stable set of  $R$  is a clique of  $\text{ComplRelStr } R$ .
- (22) For every relational structure  $R$  holds every stable set of  $\text{ComplRelStr } R$  is a clique of  $R$ .

Let  $R$  be a relational structure with finite clique number.

One can verify that  $\text{ComplRelStr } R$  has finite stability number.

Let  $R$  be a symmetric relational structure with finite stability number. Observe that  $\text{ComplRelStr } R$  has finite clique number.

The following propositions are true:

- (23) For every symmetric relational structure  $R$  with finite clique number holds  $\omega(R) = \alpha(\text{ComplRelStr } R)$ .
- (24) For every symmetric relational structure  $R$  with finite stability number holds  $\alpha(R) = \omega(\text{ComplRelStr } R)$ .
- (25) For every relational structure  $R$  holds every coloring of  $R$  is a clique-partition of  $\text{ComplRelStr } R$ .
- (26) For every symmetric relational structure  $R$  holds every clique-partition of  $\text{ComplRelStr } R$  is a coloring of  $R$ .
- (27) For every symmetric relational structure  $R$  holds every clique-partition of  $R$  is a coloring of  $\text{ComplRelStr } R$ .
- (28) For every relational structure  $R$  holds every coloring of  $\text{ComplRelStr } R$  is a clique-partition of  $R$ .

Let  $R$  be a finitely colorable relational structure.

Observe that  $\text{ComplRelStr } R$  has finite clique cover.

Let  $R$  be a symmetric relational structure with finite clique cover. One can check that  $\text{ComplRelStr } R$  is finitely colorable.

The following propositions are true:

- (29) For every finitely colorable symmetric relational structure  $R$  holds  $\chi(R) = \kappa(\text{ComplRelStr } R)$ .
- (30) For every symmetric relational structure  $R$  with finite clique cover holds  $\kappa(R) = \chi(\text{ComplRelStr } R)$ .

## 5. ADJACENT SET

Let  $R$  be a relational structure and let  $v$  be an element of  $R$ . The functor  $\text{Adjacent}(v)$  yields a subset of  $R$  and is defined as follows:

(Def. 6) For every element  $x$  of  $R$  holds  $x \in \text{Adjacent}(v)$  iff  $x < v$  or  $v < x$ .

The following proposition is true

- (31) Let  $R$  be a finitely colorable relational structure,  $C$  be a finite coloring of  $R$ , and  $c$  be a set. Suppose  $c \in C$  and  $\overline{C} = \chi(R)$ . Then there exists an element  $v$  of  $R$  such that  $v \in c$  and for every element  $d$  of  $C$  such that  $d \neq c$  there exists an element  $w$  of  $R$  such that  $w \in \text{Adjacent}(v)$  and  $w \in d$ .

## 6. NATURAL NUMBERS AS VERTICES

Let  $n$  be a natural number. A strict relational structure is said to be a relational structure of  $n$  if:

(Def. 7) The carrier of it =  $n$ .

Let us observe that every relational structure of 0 is empty.

Let  $n$  be a non empty natural number. Note that every relational structure of  $n$  is non empty.

Let  $n$  be a natural number. Note that every relational structure of  $n$  is finite and there exists a relational structure of  $n$  which is irreflexive.

Let  $n$  be a natural number. The functor  $K(n)$  yields a relational structure of  $n$  and is defined as follows:

(Def. 8) The internal relation of  $K(n) = n \times n \setminus \text{id}_n$ .

The following proposition is true

- (32) Let  $n$  be a natural number and  $x, y$  be sets. Suppose  $x, y \in n$ . Then  $\langle x, y \rangle \in$  the internal relation of  $K(n)$  if and only if  $x \neq y$ .

Let  $n$  be a natural number. Note that  $K(n)$  is irreflexive and symmetric.

Let  $n$  be a natural number. Observe that  $\Omega_{K(n)}$  is a clique.

The following propositions are true:

- (33) For every natural number  $n$  holds  $\omega(K(n)) = n$ .  
 (34) For every non empty natural number  $n$  holds  $\alpha(K(n)) = 1$ .  
 (35) For every natural number  $n$  holds  $\chi(K(n)) = n$ .  
 (36) For every non empty natural number  $n$  holds  $\kappa(K(n)) = 1$ .

## 7. MYCIELSKIAN OF A GRAPH

Let  $n$  be a natural number and let  $R$  be a relational structure of  $n$ . The functor Mycielskian  $R$  yields a relational structure of  $2 \cdot n + 1$  and is defined by the condition (Def. 9).

- (Def. 9) The internal relation of Mycielskian  $R = (\text{the internal relation of } R) \cup \{\langle x, y + n \rangle; x \text{ ranges over elements of } \mathbb{N}, y \text{ ranges over elements of } \mathbb{N}: \langle x, y \rangle \in \text{the internal relation of } R\} \cup \{\langle x + n, y \rangle; x \text{ ranges over elements of } \mathbb{N}, y \text{ ranges over elements of } \mathbb{N}: \langle x, y \rangle \in \text{the internal relation of } R\} \cup \{2 \cdot n\} \times (2 \cdot n \setminus n) \cup (2 \cdot n \setminus n) \times \{2 \cdot n\}.$

One can prove the following propositions:

- (37) Let  $n$  be a natural number and  $R$  be a relational structure of  $n$ . Then the carrier of  $R \subseteq$  the carrier of Mycielskian  $R$ .
- (38) Let  $n$  be a natural number,  $R$  be a relational structure of  $n$ , and  $x, y$  be natural numbers. Suppose  $\langle x, y \rangle \in$  the internal relation of Mycielskian  $R$ . Then
  - (i)  $x < n$  and  $y < n$ , or
  - (ii)  $x < n \leq y < 2 \cdot n$ , or
  - (iii)  $n \leq x < 2 \cdot n$  and  $y < n$ , or
  - (iv)  $x = 2 \cdot n$  and  $n \leq y < 2 \cdot n$ , or
  - (v)  $n \leq x < 2 \cdot n$  and  $y = 2 \cdot n$ .
- (39) Let  $n$  be a natural number and  $R$  be a relational structure of  $n$ . Then the internal relation of  $R \subseteq$  the internal relation of Mycielskian  $R$ .
- (40) Let  $n$  be a natural number,  $R$  be a relational structure of  $n$ , and  $x, y$  be sets. Suppose  $x, y \in n$  and  $\langle x, y \rangle \in$  the internal relation of Mycielskian  $R$ . Then  $\langle x, y \rangle \in$  the internal relation of  $R$ .
- (41) Let  $n$  be a natural number,  $R$  be a relational structure of  $n$ , and  $x, y$  be natural numbers. Suppose  $\langle x, y \rangle \in$  the internal relation of  $R$ . Then  $\langle x, y + n \rangle \in$  the internal relation of Mycielskian  $R$  and  $\langle x + n, y \rangle \in$  the internal relation of Mycielskian  $R$ .
- (42) Let  $n$  be a natural number,  $R$  be a relational structure of  $n$ , and  $x, y$  be natural numbers. Suppose  $x \in n$  and  $\langle x, y + n \rangle \in$  the internal relation of Mycielskian  $R$ . Then  $\langle x, y \rangle \in$  the internal relation of  $R$ .
- (43) Let  $n$  be a natural number,  $R$  be a relational structure of  $n$ , and  $x, y$  be natural numbers. Suppose  $y \in n$  and  $\langle x + n, y \rangle \in$  the internal relation of Mycielskian  $R$ . Then  $\langle x, y \rangle \in$  the internal relation of  $R$ .
- (44) Let  $n$  be a natural number,  $R$  be a relational structure of  $n$ , and  $m$  be a natural number. Suppose  $n \leq m < 2 \cdot n$ . Then  $\langle m, 2 \cdot n \rangle \in$  the internal relation of Mycielskian  $R$  and  $\langle 2 \cdot n, m \rangle \in$  the internal relation of Mycielskian  $R$ .

- (45) Let  $n$  be a natural number,  $R$  be a relational structure of  $n$ , and  $S$  be a subset of Mycielskian  $R$ . If  $S = n$ , then  $R = \text{sub}(S)$ .
- (46) For every natural number  $n$  and for every irreflexive relational structure  $R$  of  $n$  such that  $2 \leq \omega(R)$  holds  $\omega(R) = \omega(\text{Mycielskian } R)$ .
- (47) For every finitely colorable relational structure  $R$  and for every subset  $S$  of  $R$  holds  $\chi(R) \geq \chi(\text{sub}(S))$ .
- (48) For every natural number  $n$  and for every irreflexive relational structure  $R$  of  $n$  holds  $\chi(\text{Mycielskian } R) = 1 + \chi(R)$ .

Let  $n$  be a natural number. The functor Mycielskian  $n$  yielding a relational structure of  $3 \cdot 2^n - 1$  is defined by the condition (Def. 10).

(Def. 10) There exists a function  $m_1$  such that

- (i) Mycielskian  $n = m_1(n)$ ,
- (ii)  $\text{dom } m_1 = \mathbb{N}$ ,
- (iii)  $m_1(0) = K(2)$ , and
- (iv) for every natural number  $k$  and for every relational structure  $R$  of  $3 \cdot 2^k - 1$  such that  $R = m_1(k)$  holds  $m_1(k+1) = \text{Mycielskian } R$ .

The following proposition is true

- (49) Mycielskian  $0 = K(2)$  and for every natural number  $k$  holds Mycielskian  $(k+1) = \text{Mycielskian Mycielskian } k$ .

Let  $n$  be a natural number. One can verify that Mycielskian  $n$  is irreflexive.

Let  $n$  be a natural number. Observe that Mycielskian  $n$  is symmetric.

We now state three propositions:

- (50) For every natural number  $n$  holds  $\omega(\text{Mycielskian } n) = 2$  and  $\chi(\text{Mycielskian } n) = n + 2$ .
- (51) For every natural number  $n$  there exists a finite relational structure  $R$  such that  $\omega(R) = 2$  and  $\chi(R) > n$ .
- (52) For every natural number  $n$  there exists a finite relational structure  $R$  such that  $\alpha(R) = 2$  and  $\kappa(R) > n$ .

## REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [4] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(1):81–91, 1997.
- [5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [9] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(5):887–890, 1990.

- [10] M. Larsen, J. Propp, and D. Ullman. The fractional chromatic number of Mycielski's graphs. *Journal of Graph Theory*, 19:411–416, 1995.
- [11] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.
- [12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [13] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(3):441–444, 1990.
- [14] Krzysztof Retel. The class of series – parallel graphs. Part I. *Formalized Mathematics*, 11(1):99–103, 2003.
- [15] Piotr Rudnicki. Dilworth's decomposition theorem for posets. *Formalized Mathematics*, 17(4):223–232, 2009, doi: 10.2478/v10037-009-0028-4.
- [16] Wojciech A. Trybulec and Grzegorz Bancerek. Kuratowski – Zorn lemma. *Formalized Mathematics*, 1(2):387–393, 1990.
- [17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

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