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Riemann Integral of Functions from $\mathbb{R}$ into Real Normed Space

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Summary. In this article, we define the Riemann integral on functions from $\mathbb{R}$ into real normed space and prove the linearity of this operator. As a result, the Riemann integration can be applied to a wider range of functions. The proof method follows the [16].

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The terminology and notation used here have been introduced in the following articles: [2], [3], [4], [5], [7], [10], [8], [9], [1], [14], [6], [13], [15], [11], [19], [17], [12], [18], and [20].

1. Preliminaries

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, and let D be a Division of A. A finite sequence of elements of X is said to be a middle volume of f and D if it satisfies the conditions (Def. 1).

(Def. 1)(i) $\operatorname{len} \mathrm{it} = \operatorname{len} D$, and

(ii) for every natural number i such that $i \in \operatorname{dom} D$ there exists a point c of X such that $c \in \operatorname{rng}(f \restriction \operatorname{divset}(D, i))$ and $\mathrm{it}(i) = \operatorname{vol}(\operatorname{divset}(D, i)) \cdot c$.

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Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let D be a Division of A, and let F be a middle volume of f and D. The functor middle sum(f, F) yielding a point of X is defined by:

(Def. 2) middle sum$(f, F) = \sum F$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, and let T be a division sequence of A. A function from $\mathbb{N}$ into (the carrier of $X)^*$ is said to be a middle volume sequence of f and T if:

(Def. 3) For every element k of $\mathbb{N}$ holds it(k) is a middle volume of f and $T(k)$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let T be a division sequence of A, let S be a middle volume sequence of f and T, and let k be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of f and $T(k)$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into the carrier of X, let T be a division sequence of A, and let S be a middle volume sequence of f and T. The functor middle sum(f, S) yielding a sequence of X is defined as follows:

(Def. 4) For every element i of $\mathbb{N}$ holds (middle sum$(f, S))(i) =$ middle sum$(f, S(i))$.

2. Definition of Riemann Integral on Functions from $\mathbb{R}$ into Real Normed Space

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into the carrier of X. We say that f is integrable if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a point I of X such that for every division sequence T of A and for every middle volume sequence S of f and T if δ_T is convergent and $\lim(\delta_T) = 0$, then middle sum(f, S) is convergent and lim middle sum$(f, S) = I$.

We now state three propositions:

(1) Let X be a real normed space and R_1, R_2, R_3 be finite sequences of elements of X. If $\operatorname{len} R_1 = \operatorname{len} R_2$ and $R_3 = R_1 + R_2$, then $\sum R_3 = \sum R_1 + \sum R_2$.

(2) Let X be a real normed space and R_1, R_2, R_3 be finite sequences of elements of X. If $\operatorname{len} R_1 = \operatorname{len} R_2$ and $R_3 = R_1 - R_2$, then $\sum R_3 = \sum R_1 - \sum R_2$.

(3) Let X be a real normed space, R_1, R_2 be finite sequences of elements of X, and a be an element of $\mathbb{R}$. If $R_2 = a\, R_1$, then $\sum R_2 = a \cdot \sum R_1$.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into the carrier of X. Let us assume that f is integrable. The functor integral f yields a point of X and is defined by the condition (Def. 6).

(Def. 6) Let T be a division sequence of A and S be a middle volume sequence of f and T. If δ_T is convergent and $\lim(\delta_T) = 0$, then middle sum(f, S) is convergent and lim middle sum$(f, S) =$ integral f.

We now state four propositions:

(4) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, r be a real number, and f, h be functions from A into the carrier of X. If $h = r\, f$ and f is integrable, then h is integrable and integral $h = r \cdot$ integral f.

(5) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, h be functions from A into the carrier of X. If $h = -f$ and f is integrable, then h is integrable and integral $h = -$integral f.

(6) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, g, h be functions from A into the carrier of X. Suppose $h = f + g$ and f is integrable and g is integrable. Then h is integrable and integral $h =$ integral $f +$ integral g.

(7) Let X be a real normed space, A be a closed-interval subset of $\mathbb{R}$, and f, g, h be functions from A into the carrier of X. Suppose $h = f - g$ and f is integrable and g is integrable. Then h is integrable and integral $h =$ integral $f -$ integral g.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{R}$ to the carrier of X. We say that f is integrable on A if and only if:

(Def. 7) There exists a function g from A into the carrier of X such that $g = f\restriction A$ and g is integrable.

Let X be a real normed space, let A be a closed-interval subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{R}$ to the carrier of X. Let us assume that $A \subseteq \operatorname{dom} f$. The functor $\int\limits_A f(x)dx$ yields an element of X and is defined as follows:

(Def. 8) There exists a function g from A into the carrier of X such that $g = f\restriction A$ and $\int\limits_A f(x)dx =$ integral g.

We now state several propositions:

(8) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to the carrier of X, and g be a function from A into the carrier of X. Suppose $f\restriction A = g$. Then f is integrable on A if and only if g is integrable.

(9) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to the carrier of X, and g be a function from A into the carrier of X. If

$A \subseteq \operatorname{dom} f$ and $f \restriction A = g$, then $\int\limits_A f(x)dx = \operatorname{integral} g$.

(10) Let X, Y be non empty sets, V be a real normed space, g, f be partial functions from X to the carrier of V, and g_1, f_1 be partial functions from Y to the carrier of V. If $g = g_1$ and $f = f_1$, then $g_1 + f_1 = g + f$.

(11) Let X, Y be non empty sets, V be a real normed space, g, f be partial functions from X to the carrier of V, and g_1, f_1 be partial functions from Y to the carrier of V. If $g = g_1$ and $f = f_1$, then $g_1 - f_1 = g - f$.

(12) Let r be a real number, X, Y be non empty sets, V be a real normed space, g be a partial function from X to the carrier of V, and g_1 be a partial function from Y to the carrier of V. If $g = g_1$, then $r\, g_1 = r\, g$.

3. Linearity of the Integration Operator

Next we state three propositions:

(13) Let r be a real number, A be a closed-interval subset of $\mathbb{R}$, and f be a partial function from $\mathbb{R}$ to the carrier of X. Suppose $A \subseteq \operatorname{dom} f$ and f is integrable on A. Then $r\, f$ is integrable on A and $\int\limits_A (r\, f)(x)dx = r \cdot \int\limits_A f(x)dx$.

(14) Let A be a closed-interval subset of $\mathbb{R}$ and f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of X. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \operatorname{dom} f_1$ and $A \subseteq \operatorname{dom} f_2$. Then $f_1 + f_2$ is integrable on A and $\int\limits_A (f_1 + f_2)(x)dx = \int\limits_A f_1(x)dx + \int\limits_A f_2(x)dx$.

(15) Let A be a closed-interval subset of $\mathbb{R}$ and f_1, f_2 be partial functions from $\mathbb{R}$ to the carrier of X. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \operatorname{dom} f_1$ and $A \subseteq \operatorname{dom} f_2$. Then $f_1 - f_2$ is integrable on A and $\int\limits_A (f_1 - f_2)(x)dx = \int\limits_A f_1(x)dx - \int\limits_A f_2(x)dx$.

Let X be a real normed space, let f be a partial function from $\mathbb{R}$ to the carrier of X, and let a, b be real numbers. The functor $\int\limits_a^b f(x)dx$ yielding an element of X is defined as follows:

(Def. 9) $$\int\limits_a^b f(x)dx = \begin{cases} \int\limits_{[a,b]} f(x)dx, & \text{if } a \leq b, \\ -\int\limits_{[b,a]} f(x)dx, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

(16) Let f be a partial function from $\mathbb{R}$ to the carrier of X, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [a, b]$, then $\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

(17) Let f be a partial function from $\mathbb{R}$ to the carrier of X and A be a closed-interval subset of $\mathbb{R}$. If $\mathrm{vol}(A) = 0$ and $A \subseteq \mathrm{dom}\, f$, then f is integrable on A and $\int\limits_{A} f(x)dx = 0_X$.

(18) Let f be a partial function from $\mathbb{R}$ to the carrier of X, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [b, a]$ and $A \subseteq \mathrm{dom}\, f$, then $-\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

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